Sampling and Aliasing, and The Discrete Fourier Transform

Image Processing

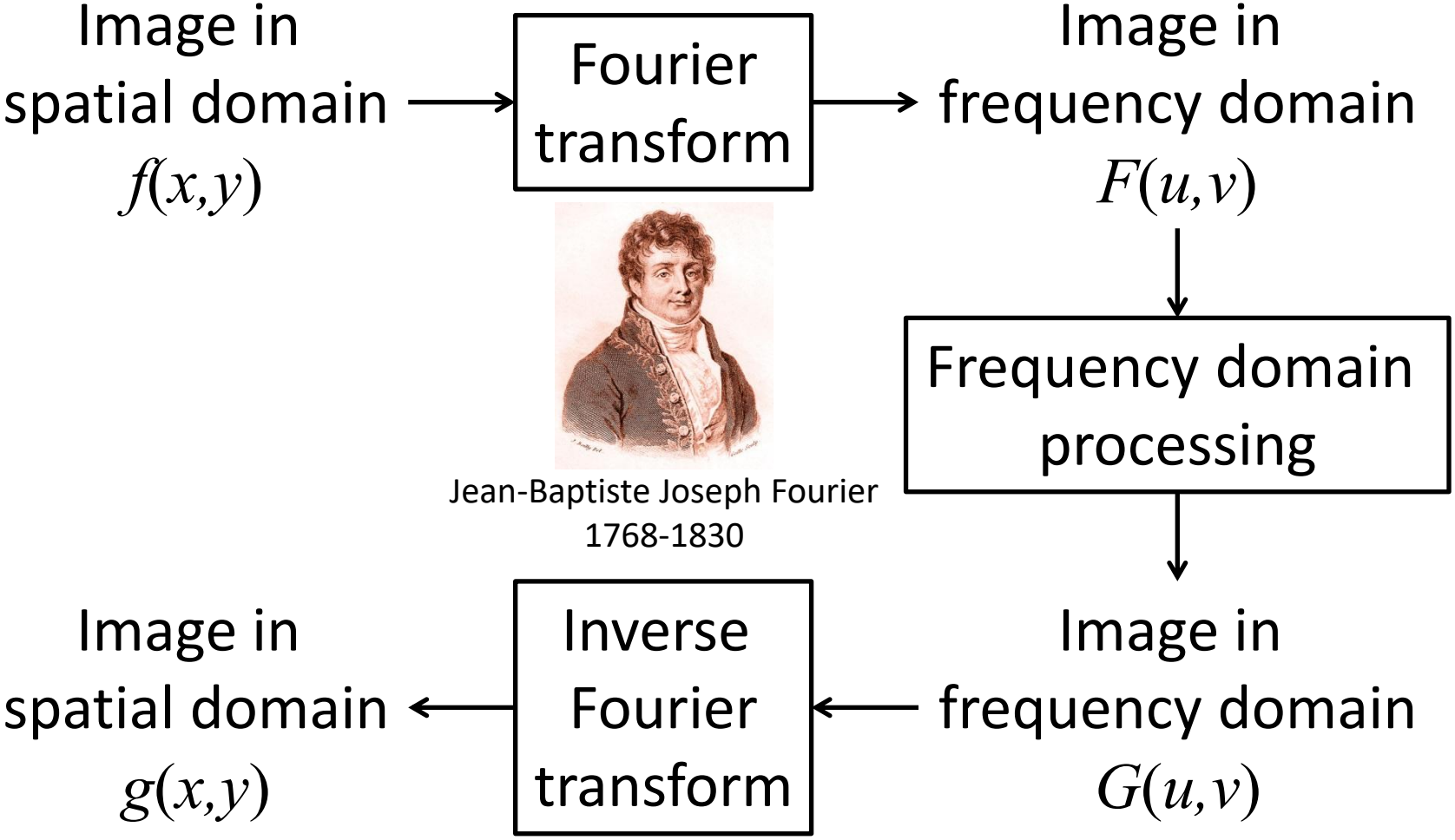
CSE 166

Lecture 6

Announcements

- Assignment 2 is due today, 11:59 PM
- Assignment 3 will be released Apr 20
  - Due Apr 27, 11:59 PM
- Reading
  - Chapter 4: Filtering in the Frequency Domain
    - Sections 4.2, 4.3, and 4.4

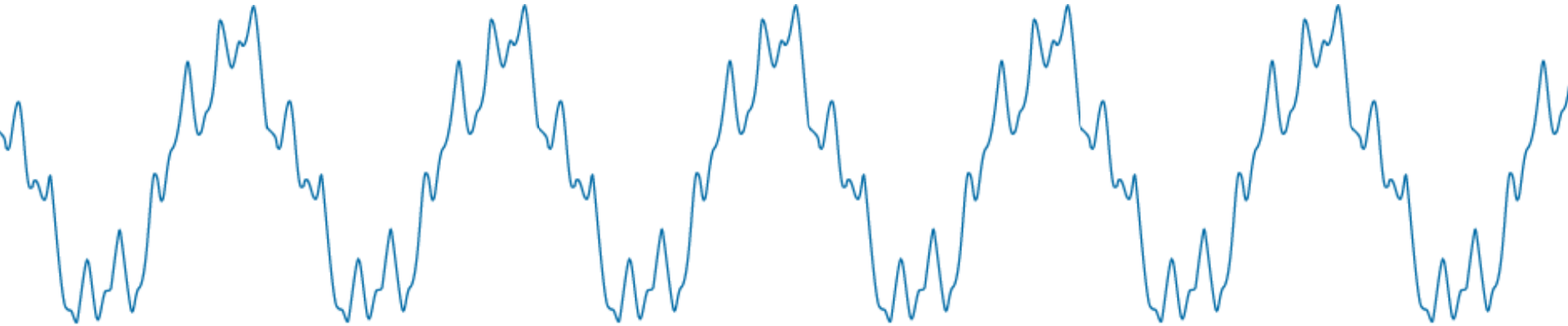
Overview: Image processing in the frequency domain



Jean-Baptiste Joseph Fourier
1768-1830

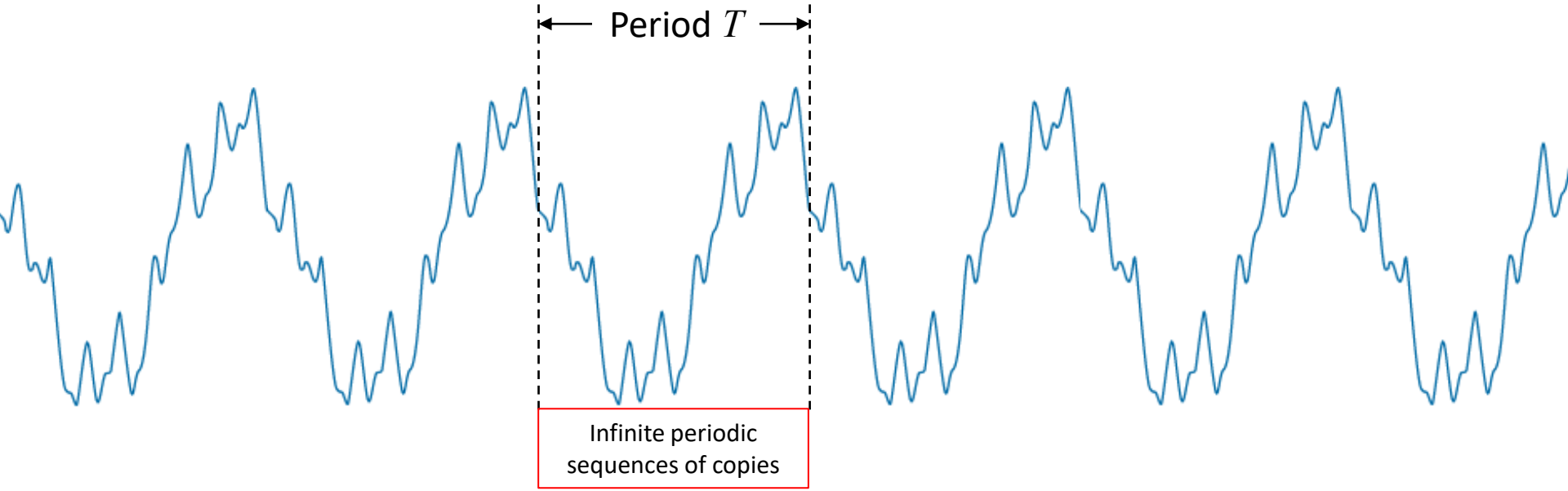
Periodic functions

- Example periodic function



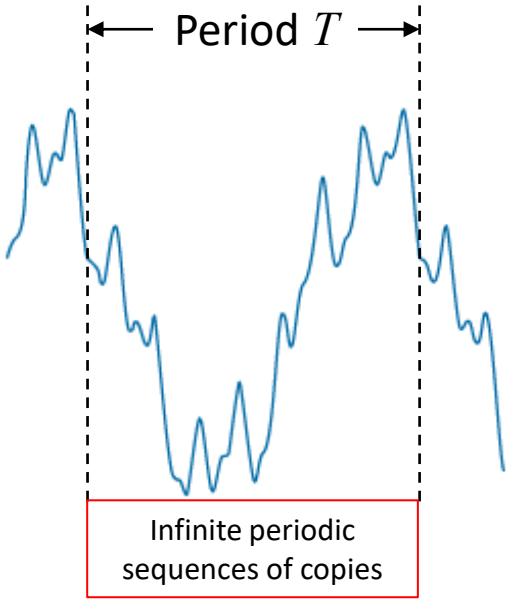
Periodic functions

- Example periodic function



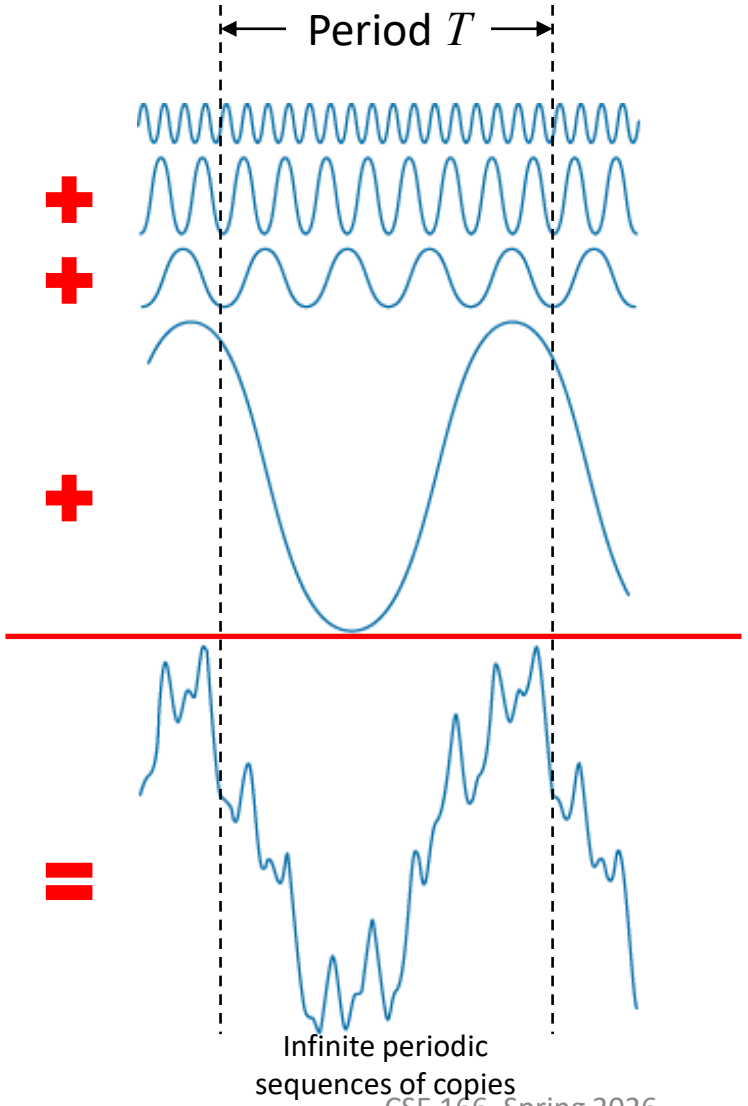
Periodic functions

- Example periodic function



1D Fourier series

Sines and cosines



Weighted by magnitude

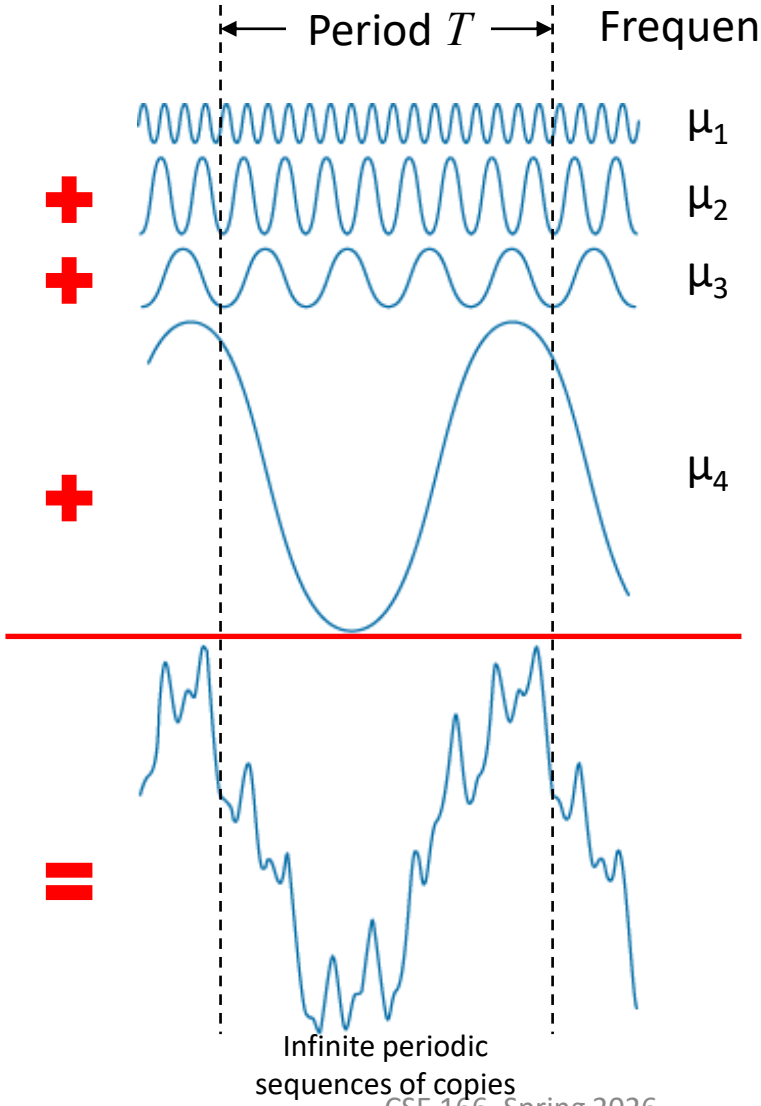
Shifted by phase

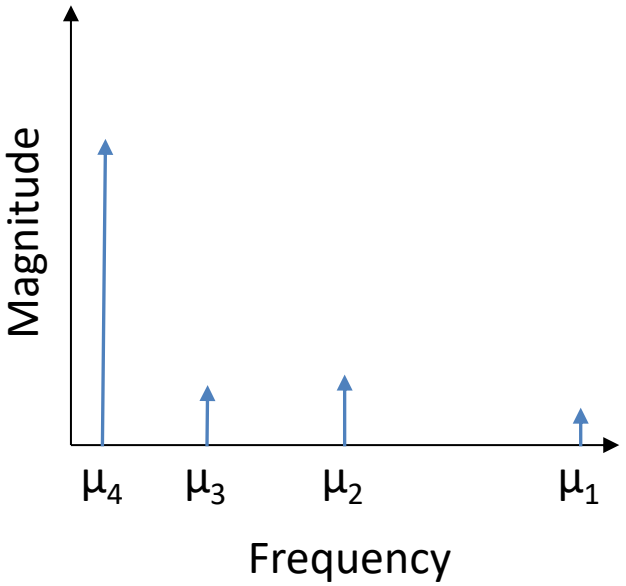
Periodic function

1D Fourier transform

Sines and cosines

Periodic function





1D continuous Fourier transform

- (Forward) Fourier transform

The continuous Fourier transform of a continuous function $f(t)$ is

$$\mathfrak{F}\{f(t)\} = F(\mu) = \int_{-\infty}^{\infty} f(t)e^{-j2\pi\mu t}\,\mathrm{d}t \text{ where } \mu \text{ is a continuous variable}$$

Since t is integrated out, $\mathfrak{F}\{f(t)\}$ is only a function of μ.

- Inverse Fourier transform

The inverse Fourer transform

$$\mathfrak{F}^{-1}\{F(\mu)\} = f(t) = \int_{-\infty}^{\infty} F(\mu)e^{j2\pi\mu t}\,\mathrm{d}\mu \text{ where } t \text{ is a continuous variable}$$

μ is integrated out, so $\mathfrak{F}^{-1}\{f(\mu)\}$ is only a function of t.

1D continuous Fourier transform

- Fourier transform pair

$$\mathfrak{F}\{f(t)\} = F(\mu) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t}\,\mathrm{d}t$$

$$\mathfrak{F}^{-1}\{F(\mu)\} = f(t) = \int_{-\infty}^{\infty} F(\mu) e^{j2\pi\mu t}\,\mathrm{d}\mu$$

1D continuous Fourier transform

- Using Euler's formula

$$F(\mu) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t}\, \mathrm{d}t$$

$$F(\mu) = \int_{-\infty}^{\infty} f(t) \left[\cos(2\pi\mu t) - j \sin(2\pi\mu t)\right]\, \mathrm{d}t$$

Fourier transform is in frequency domain
Frequency μ is in units of cycles per unit of intput variable (e.g., cycles/seconds, cycles/meters).
If $f(t)$ is real, then (in general) $F(\mu)$ is complex. As such, it is customary to display the magnitude of $F(\mu)$, which is real.

From continuous to discrete

1D impulse function

- Continuous

Continuous variable t (at $t = 0$)

$$\delta(t) = \begin{cases} \infty & \text{if } t = 0 \\ 0 & \text{otherwise} \end{cases}, \text{ where } \int_{-\infty}^{\infty} \delta(t)\,\mathrm{d}t = 1$$

Infinite amplitude, zero duration

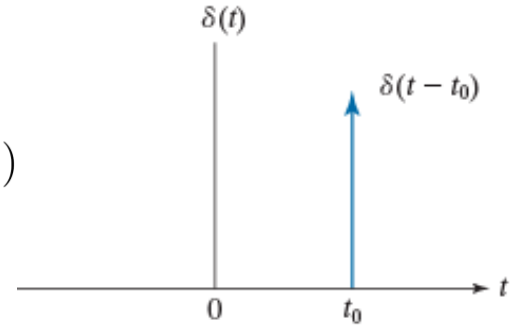
If $f(t)$ is continuous at $t = 0$, then

$$\int_{-\infty}^{\infty} f(t)\delta(t)\,\mathrm{d}t = f(0)$$

Generalizes to an arbitrary point t_0

Shifting property $$\int_{-\infty}^{\infty} f(t)\delta(t - t_0)\,\mathrm{d}t = f(t_0)$$

The value of the function f at the location of the impulse (i.e., sample of the function)



1D impulse function

- Discrete

Discrete variable x (at $x = 0$)

$$\delta(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases} \qquad \left(\text{note } \sum_{x=-\infty}^{\infty} \delta(x) = 1 \right)$$

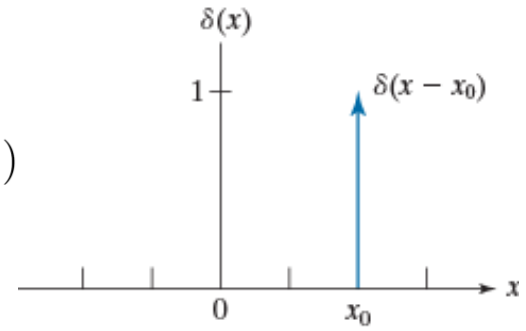
Unit impulse

If $f(x)$ is discrete at $x = 0$, then

$$\sum_{x=-\infty}^{\infty} f(x)\delta(x) = f(0)$$

Generalizes to an arbitrary point x_0

Shifting property $$\sum_{x=-\infty}^{\infty} f(x)\delta(x - x_0) = f(x_0)$$



The value of the function f at the location of the impulse (i.e., sample of the function)

1D impulse function

- Continuous, shifting property

$$\int_{-\infty}^{\infty} f(t)\delta(t-t_0)\,\mathrm{d}t = f(t_0)$$

Sample of the function f at t_0

- Fourier transform

$$\mathfrak{F}\{\delta(t-t_0)\} = \int_{-\infty}^{\infty} \delta(t-t_0)e^{-j2\pi\mu t}\,\mathrm{d}t$$

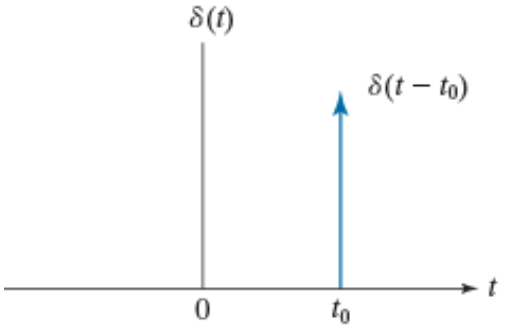
$$\mathfrak{F}\{\delta(t-t_0)\} = e^{-j2\pi\mu t_0}$$

Sample of $e^{-j2\pi\mu t}$ at t_0

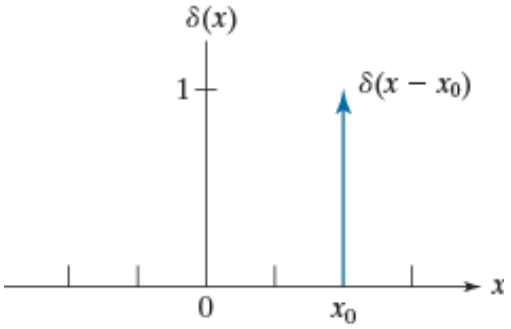
1D impulse function

Impulse function

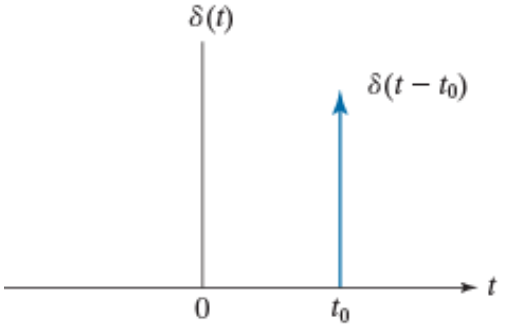
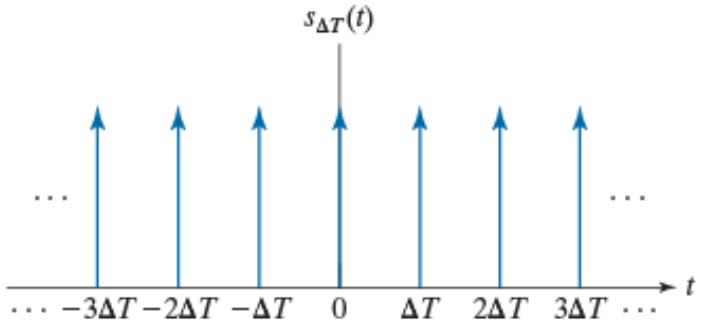
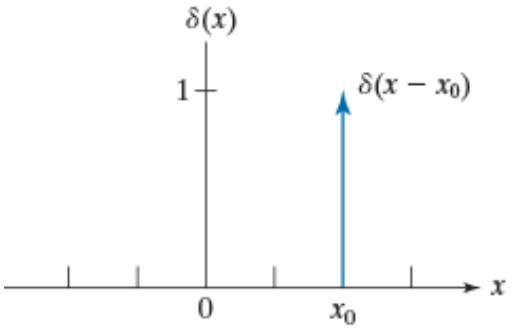
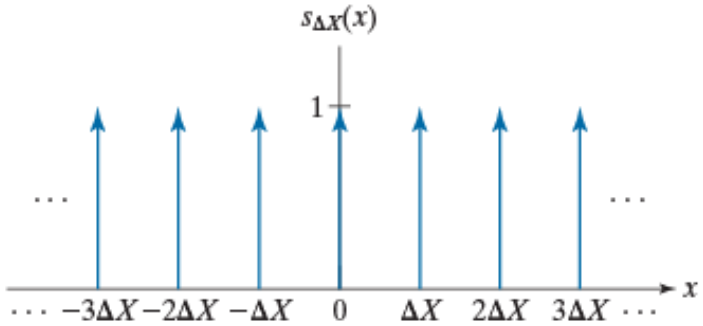
Continuous



Discrete



1D impulse function and impulse train

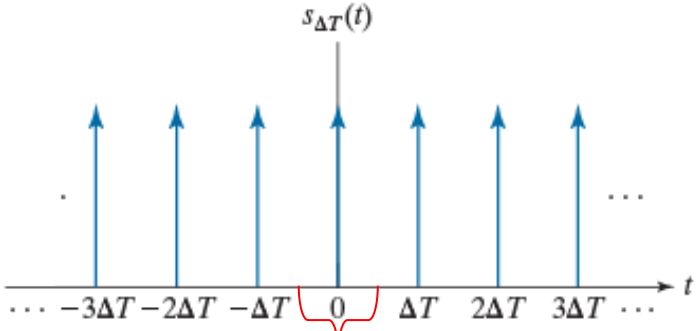
	Impulse function	Impulse train
Continuous		
Discrete		

Sum of infinitely many periodic impulses ΔT or ΔX units apart

1D impulse train

Sum of infinitely many periodic impulses ΔT units apart

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta T)$$



Impulses can be continuous or discrete
As a Fourier series

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{\Delta T}t}, \text{ where } c_n = \frac{1}{\Delta T}\int_{-\Delta T/2}^{\Delta T/2} s_{\Delta T}(t) e^{-j\frac{2\pi n}{\Delta T}t}\,\mathrm{d}t$$

Only contains impulse at origin ($t = 0$)

$$c_n = \frac{1}{\Delta T}\int_{-\Delta T/2}^{\Delta T/2} \delta(t) e^{-j\frac{2\pi n}{\Delta T}t}\,\mathrm{d}t$$

$$c_n = \frac{1}{\Delta T} e^{0}$$

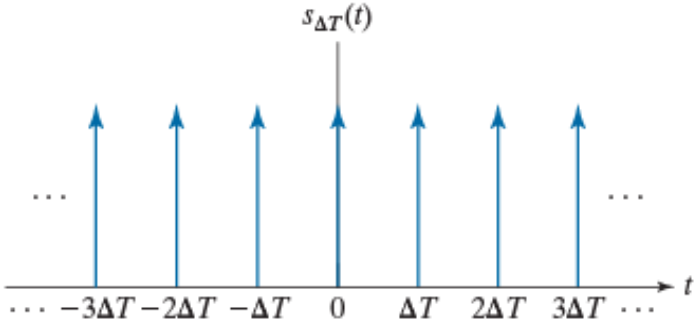
$$c_n = \frac{1}{\Delta T}$$

$$s_{\Delta T}(t) = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi n}{\Delta T}t}$$

1D impulse train

- Continuous

$$s_{\Delta T}(t) = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} e^{j\frac{2\pi n}{\Delta T}t}$$



- Fourier transform

$$S(\mu) = \mathfrak{F}\{s_{\Delta T}(t)\}$$

$$S(\mu) = \mathfrak{F}\left\{\frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} e^{j\frac{2\pi n}{\Delta T}t}\right\}$$

$$S(\mu) = \frac{1}{\Delta T}\mathfrak{F}\left\{\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi n}{\Delta T}t}\right\}$$

$$S(\mu) = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} \delta\left(\mu - \frac{n}{\Delta T}\right), \text{ where } \mathfrak{F}\left\{e^{j\frac{2\pi n}{\Delta T}t}\right\} = \delta\left(\mu - \frac{n}{\Delta T}\right)$$

This is an infinite, periodic sequence of copies of $\delta(\mu)$

Sampling

- Sampled function

Continuous function $f(t)$
Sampled function

$$\tilde{f}(t) = f(t)s_{\Delta T}(t)$$

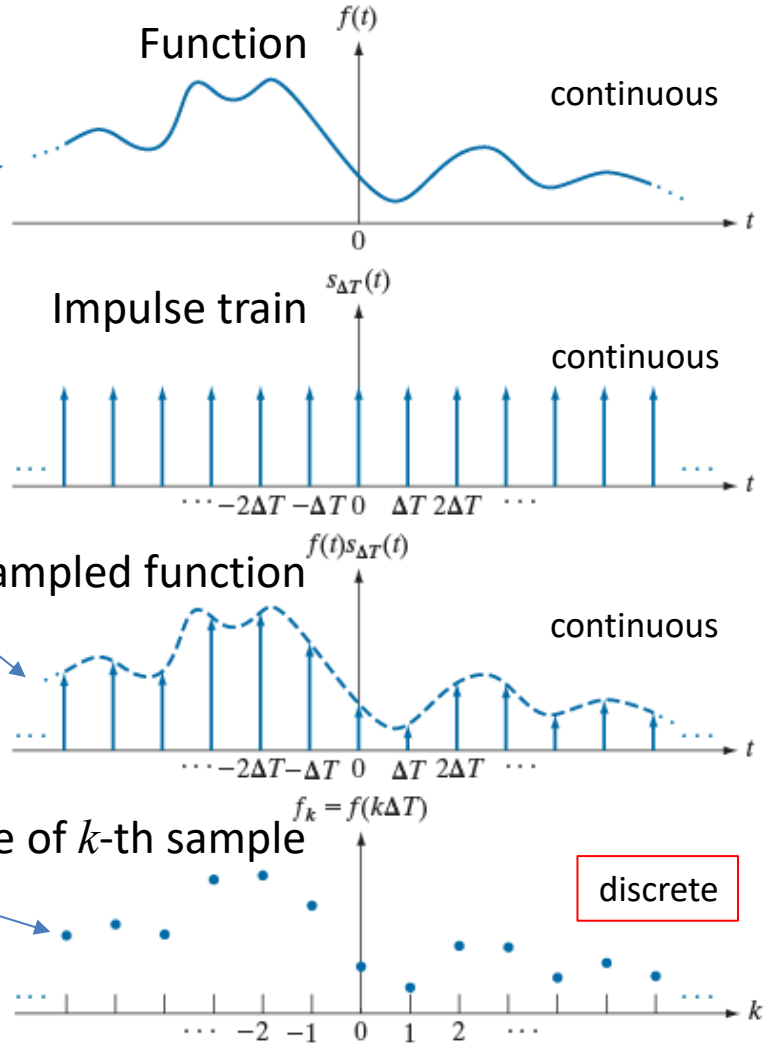
$$\tilde{f}(t) = \sum_{n=-\infty}^{\infty} f(t)\delta(t - n\Delta T)$$

Value of the k-th sample

$$f_k = \int_{-\infty}^{\infty} f(t)\delta(t - k\Delta T)\,dt$$

$$f_k = f(k\Delta T)$$

where $k = \ldots, -2, -1, 0, 1, 2, \ldots$



Sampling

- Fourier transform of sampled function

$$\tilde{F}(\mu) = \mathfrak{F}\left\{\tilde{f}(t)\right\}$$

$$\tilde{F}(\mu) = \mathfrak{F}\left\{f(t)s_{\Delta T}(t)\right\}$$

$$\tilde{F}(\mu) = F(\mu) \bigstar S(\mu)$$

Convolution theorem

$$\tilde{F}(\mu) = \int_{-\infty}^{\infty} F(\tau)S(\mu - \tau)\,\mathrm{d}\tau$$

$$\tilde{F}(\mu) = \frac{1}{\Delta T}\int_{-\infty}^{\infty} F(\tau)\sum_{n=-\infty}^{\infty}\delta\left(\mu - \tau - \frac{n}{\Delta T}\right)\mathrm{d}\tau$$

$$\tilde{F}(\mu) = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} F(\tau)\delta\left(\mu - \tau - \frac{n}{\Delta T}\right)\mathrm{d}\tau$$

Shifting property

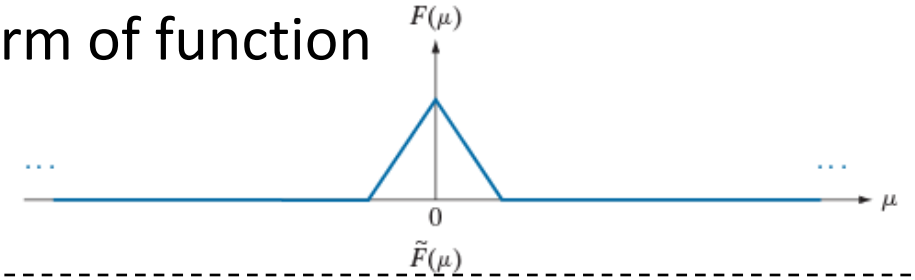
$$\tilde{F}(\mu) = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} F\left(\mu - \frac{n}{\Delta t}\right)$$

This is an infinite, periodic sequence of copies of $F(\mu)$

Since $F(\mu)$ is continuous, $\tilde{F}(\mu)$ is continuous

Sampling

Fourier transform of function



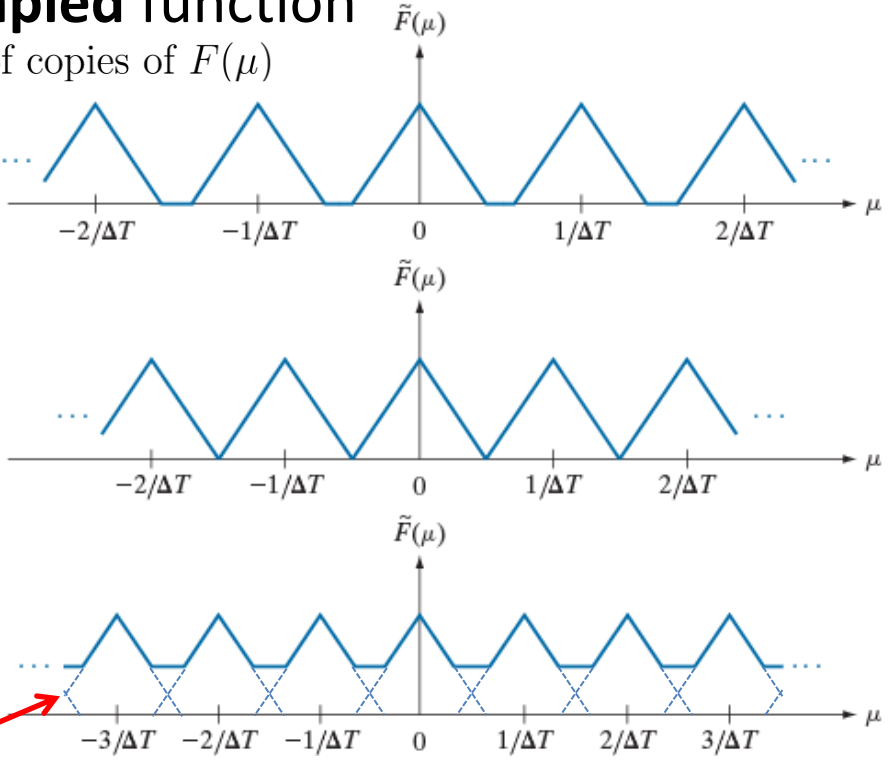
Fourier transforms of **sampled** function

This is an infinite, periodic sequence of copies of $F(\mu)$

Over-sampled

Critically-sampled

Under-sampled



Sampling rate $\frac{1}{\Delta T}$

Interference

The sampling theorem

- Sufficient separation between copies is guaranteed if

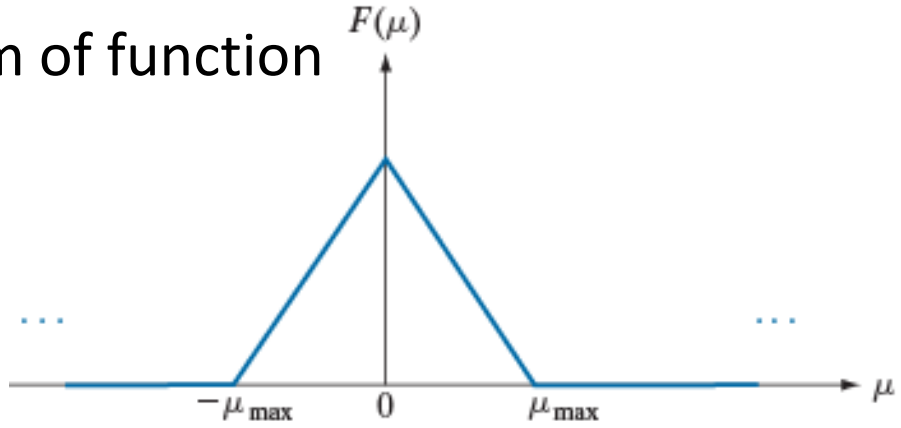
$$\frac{1}{2\Delta T} > \mu_{\max}$$

$$\frac{1}{\Delta T} > 2\mu_{\max}$$

- A sampling rate equal to exactly twice the highest frequency is called the Nyquist rate
- **Sampling must exceed the Nyquist rate**

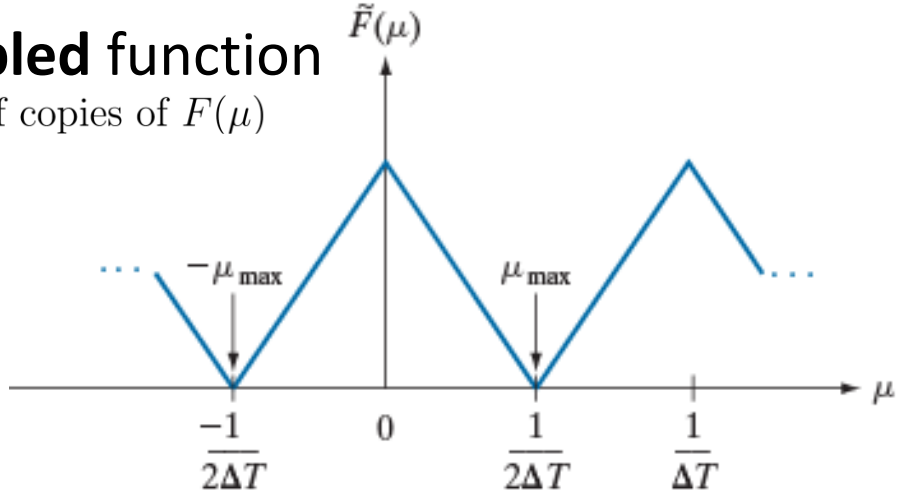
The sampling theorem

Fourier transform of function



Fourier transform of **sampled** function

This is an infinite, periodic sequence of copies of $F(\mu)$



Critically-sampled
(at Nyquist rate)

Sampling

Recovery of $F(\mu)$ from $\tilde{F}(\mu)$

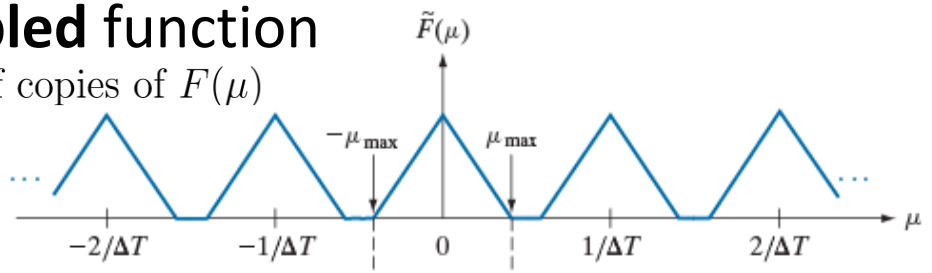
$$F(\mu) = H(\mu)\tilde{F}(\mu), \text{ where } H(\mu) = \begin{cases} \Delta T & \text{if } -\mu_{\max} \le \mu \le \mu_{\max} \\ 0 & \text{otherwise} \end{cases}$$

Ideal lowpass filter
(covered next lecture)

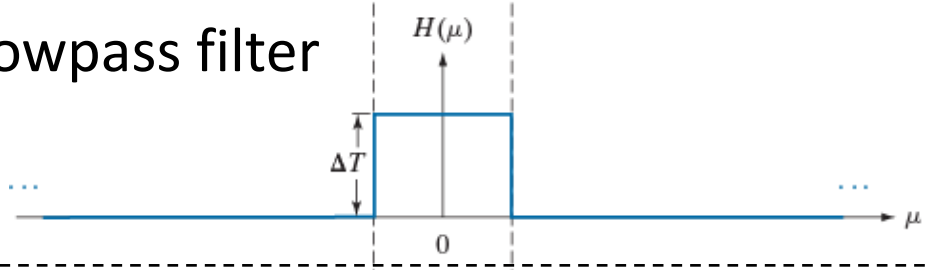
Fourier transform of **sampled** function

This is an infinite, periodic sequence of copies of $F(\mu)$

Over-sampled

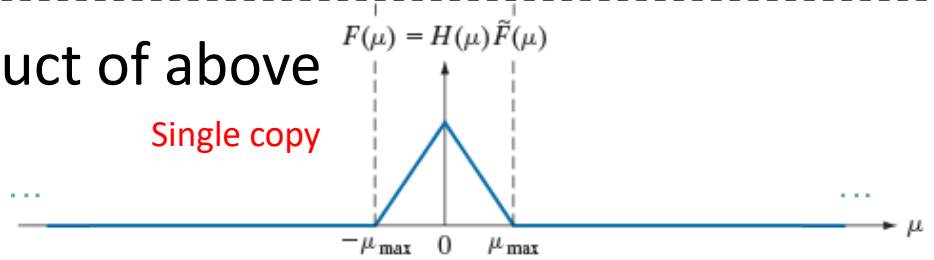


Ideal lowpass filter



Product of above

Single copy



Recovered without error

$$f(t) = \int_{-\infty}^{\infty} F(\mu) e^{j2\pi\mu t}\, \mathrm{d}\mu$$

Sampling

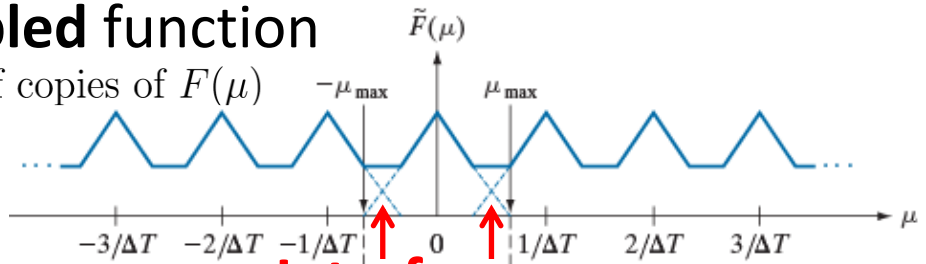
Recovery of $F(\mu)$ from $\tilde{F}(\mu)$

$$F(\mu) = H(\mu)\tilde{F}(\mu), \text{ where } H(\mu) = \begin{cases} \Delta T & \text{if } -\mu_{\max} \le \mu \le \mu_{\max} \\ 0 & \text{otherwise} \end{cases}$$

Ideal lowpass filter
(covered next lecture)

Fourier transform of **sampled** function

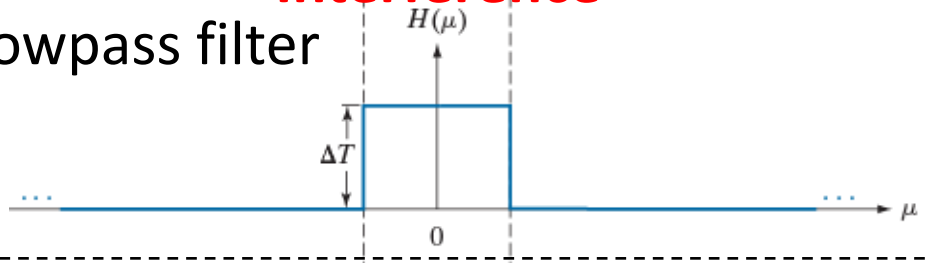
This is an infinite, periodic sequence of copies of $F(\mu)$



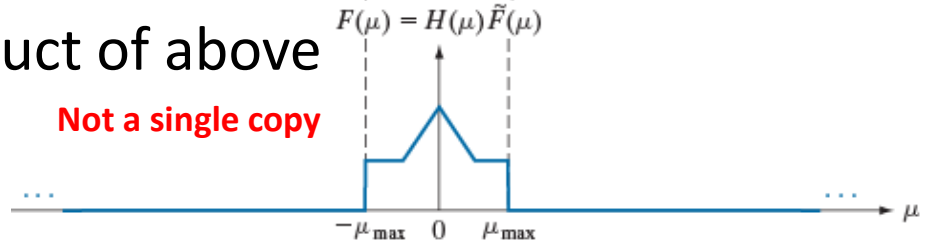
Under-sampled

Interference

Ideal lowpass filter



Product of above

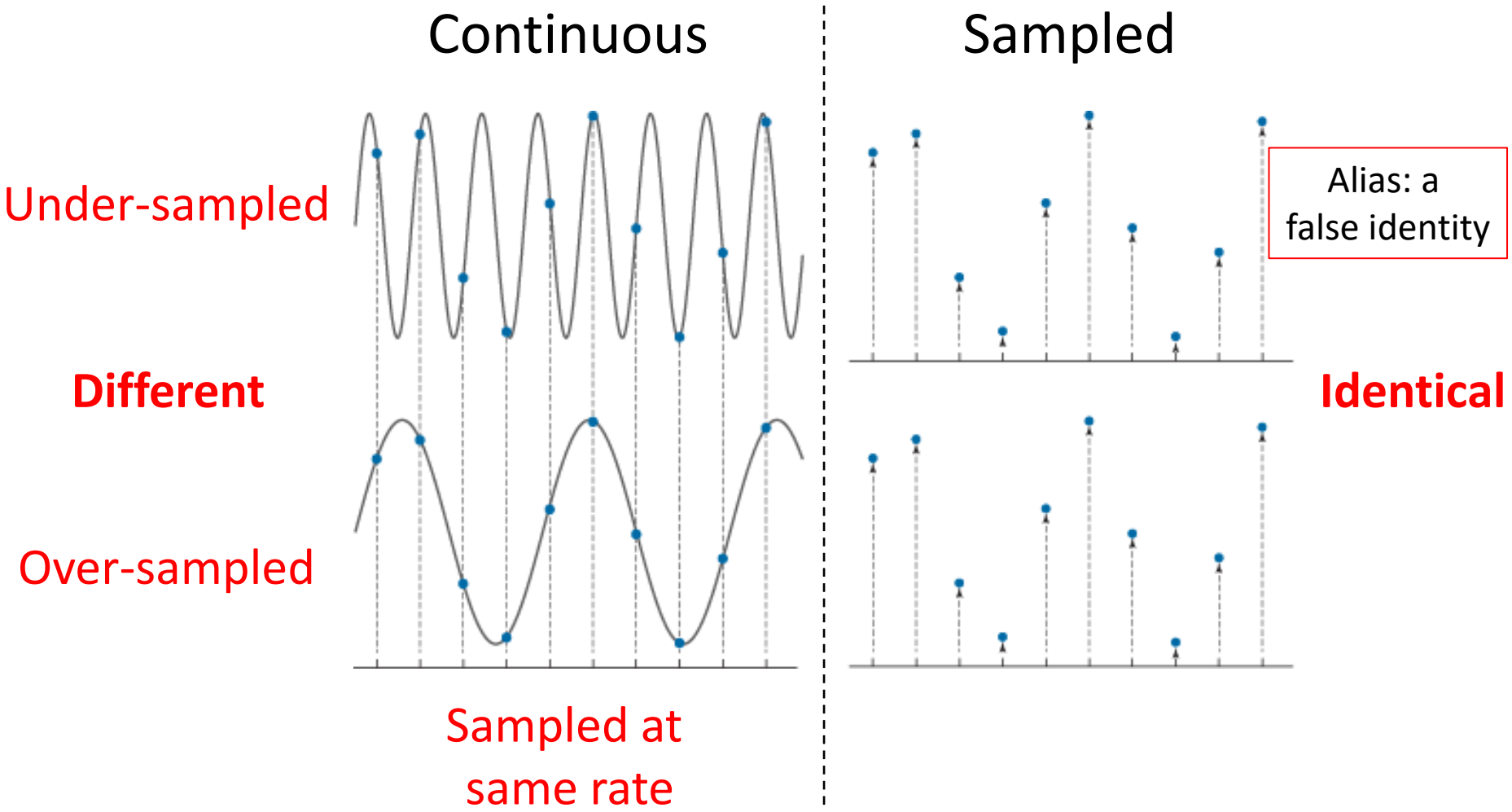


Not a single copy

Recovered with error

Corrupted $f(t) = \int_{-\infty}^{\infty} F(\mu) e^{j2\pi\mu t}\, \mathrm{d}\mu$

Sampling


Continuous
Sampled
Under-sampled
Different
Over-sampled
Alias: a false identity
Identical
Sampled at same rate

Sampling

- Aliasing

Under-sampled function looks like a lower frequency sine wave

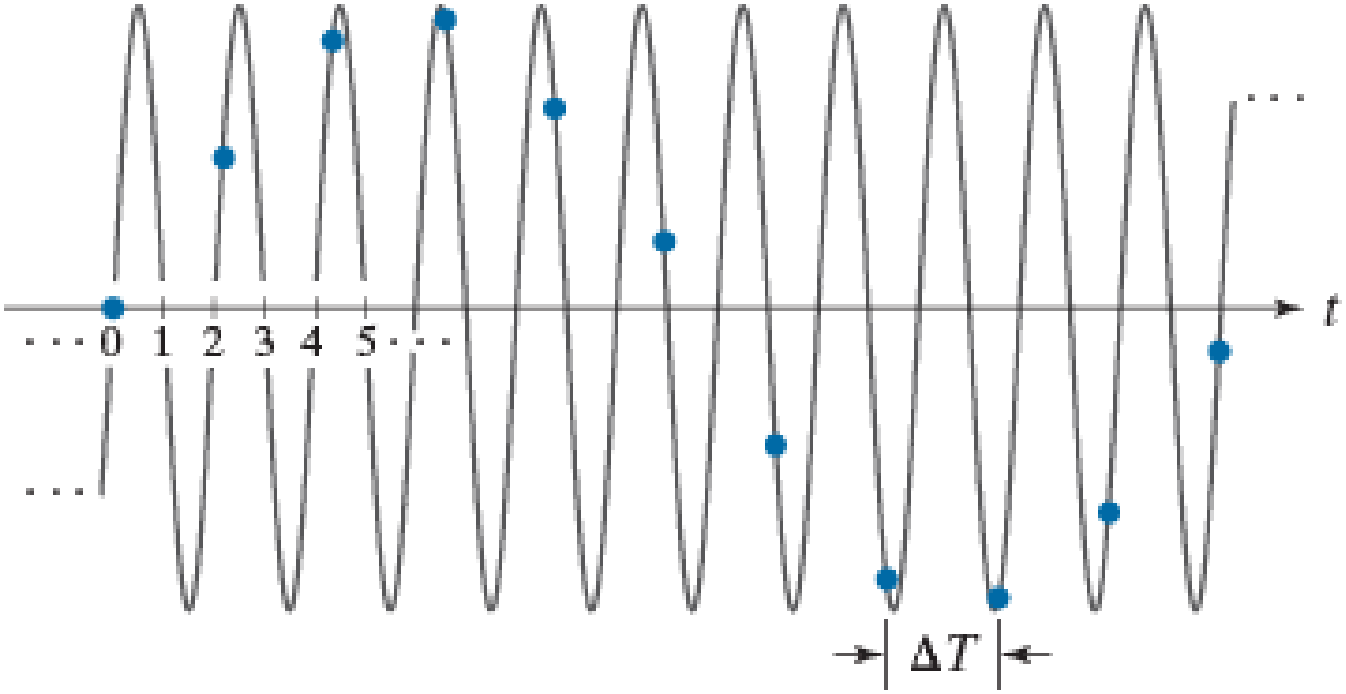


FIGURE 4.11 Illustration of aliasing. The under-sampled function (dots) looks like a sine wave having a frequency much lower than the frequency of the continuous signal. The period of the sine wave is 2 s, so the zero crossings of the horizontal axis occur every second. ΔT is the separation between samples.

1D discrete Fourier transform (DFT)

Sampled function $\tilde{f}(t)$ with Fourier transform

$$\tilde{F}(\mu) = \int_{-\infty}^{\infty} \tilde{f}(t) e^{-j2\pi\mu t}\, \mathrm{d}t$$

$$\tilde{F}(\mu) = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f(t)\delta(t - n\Delta T) e^{-j2\pi\mu t}\, \mathrm{d}t$$

$$\tilde{F}(\mu) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\delta(t - n\Delta T) e^{-j2\pi\mu t}\, \mathrm{d}t$$

$$\tilde{F}(\mu) = \sum_{n=-\infty}^{\infty} f_n e^{-j2\pi\mu n\Delta T}, \text{ where } f_n = \int_{-\infty}^{\infty} f(t)\delta(t - n\Delta T)\, \mathrm{d}t \text{ for discrete values of } n$$

To obtain M equally spaced samples of $\tilde{F}(\mu)$ taken over the period $\mu = 0$ to $\mu = \frac{1}{\Delta T}$, sample at the following frequencies

$$\mu = \frac{m}{M\Delta T} \quad m = 0, 1, 2, \ldots, M-1$$

Resulting in

$$F_m = \sum_{n=0}^{M-1} f_n e^{-j2\pi mn/M} \quad m = 0, 1, 2, \ldots, M-1$$

Units of cycles/sample

1D discrete Fourier transform (DFT)

- (Forward) Fourier transform

$$F(u) = \sum_{x=0}^{M-1} f(x) e^{-j2\pi ux/M} \quad u = 0, 1, 2, \ldots, M-1$$

- Inverse Fourier transform

$$f(x) = \frac{1}{M} \sum_{u=0}^{M-1} F(u) e^{j2\pi ux/M} \quad x = 0, 1, 2, \ldots, M-1$$

1D discrete Fourier transform (DFT)

- Fourier transform pairs

$$F(u) = \sum_{x=0}^{M-1} f(x) e^{-j2\pi ux/M} \qquad u = 0, 1, 2, \ldots, M-1$$

$$f(x) = \frac{1}{M} \sum_{u=0}^{M-1} F(u) e^{j2\pi ux/M} \qquad x = 0, 1, 2, \ldots, M-1$$

$$F(u) = \frac{1}{M} \sum_{x=0}^{M-1} f(x) e^{-j2\pi ux/M} \qquad u = 0, 1, 2, \ldots, M-1$$

$$f(x) = \sum_{u=0}^{M-1} F(u) e^{j2\pi ux/M} \qquad x = 0, 1, 2, \ldots, M-1$$

$$F(u) = \frac{1}{\sqrt{M}} \sum_{x=0}^{M-1} f(x) e^{-j2\pi ux/M} \qquad u = 0, 1, 2, \ldots, M-1$$

$$f(x) = \frac{1}{\sqrt{M}} \sum_{u=0}^{M-1} F(u) e^{j2\pi ux/M} \qquad x = 0, 1, 2, \ldots, M-1$$

Next Lecture

- Filtering in the frequency domain
- Reading
  - Chapter 4: Filtering in the Frequency Domain
    - Sections 4.5, 4.6, 4.7, 4.8, 4.9, and 4.10