The Continuous Fourier Transform

Image Processing

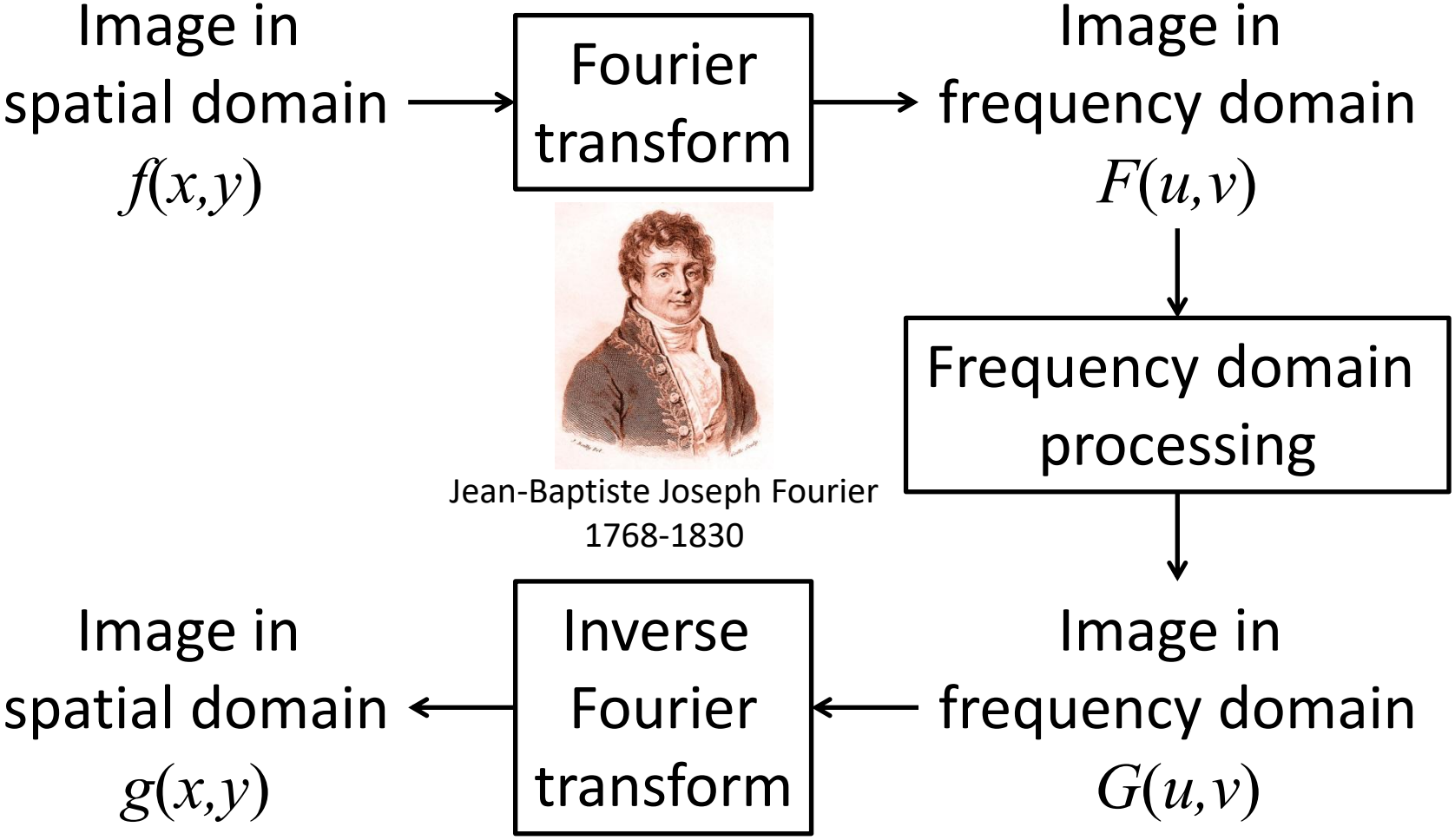
CSE 166

Lecture 5

Announcements

- Assignment 2 is due Apr 15, 11:59 PM
- Assignment 3 will be released Apr 20
  - Due Apr 27, 11:59 PM
- Reading
  - Chapter 4: Filtering in the Frequency Domain
    - Sections 4.1 and 4.2

Overview: Image processing in the frequency domain



Jean-Baptiste Joseph Fourier
1768-1830

Review

- Complex numbers
- Complex functions

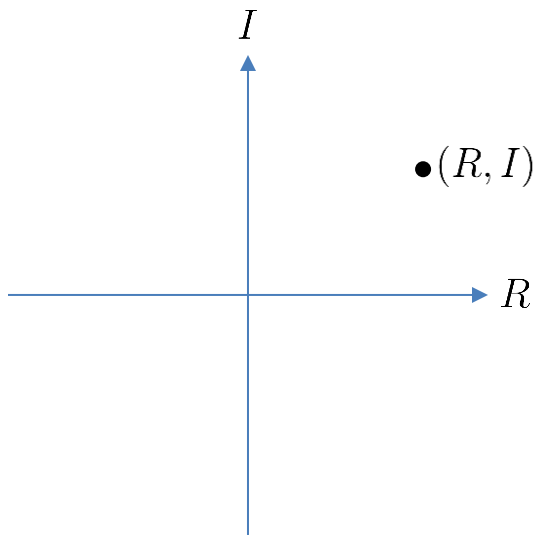
Complex numbers

$$C = R + jI, \text{ where } j = \sqrt{-1}$$

Real numbers are a subset of complex numbers where $I = 0$

Complex conjugate

$$C^* = R - jI$$



Complex plane

Complex numbers

Magnitude (or complex modulus)

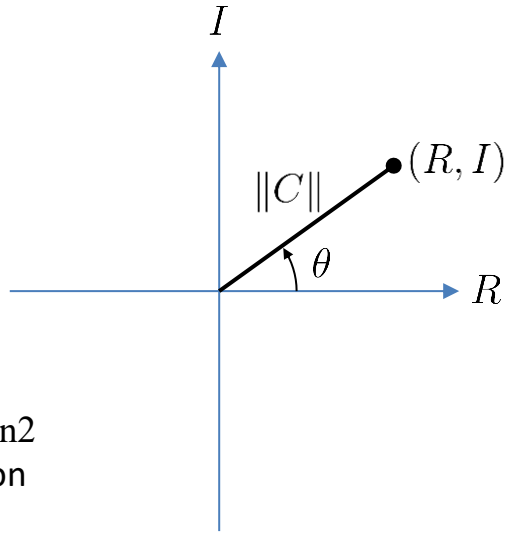
$$\|C\| = \sqrt{R^2 + I^2}$$

Angle (or phase or complex argument)

$$\tan(\theta) = \frac{I}{R} = \frac{\sin(\theta)}{\cos(\theta)}$$

$$\theta = \tan^{-1}\left(\frac{I}{R}\right)$$

Use atan2 function



Complex plane

Use these to convert from C to $\|C\|$ and θ

$$C = \|C\|(\cos(\theta) + j\sin(\theta))$$

$$C = \|C\|e^{j\theta}, \text{ where } e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

Euler's formula

Use this to convert from $\|C\|$ and θ to C

Complex functions

$$F(u) = R(u) + jI(u)$$

$$F^*(u) = R(u) - jI(u) \qquad \text{Complex conjugate}$$

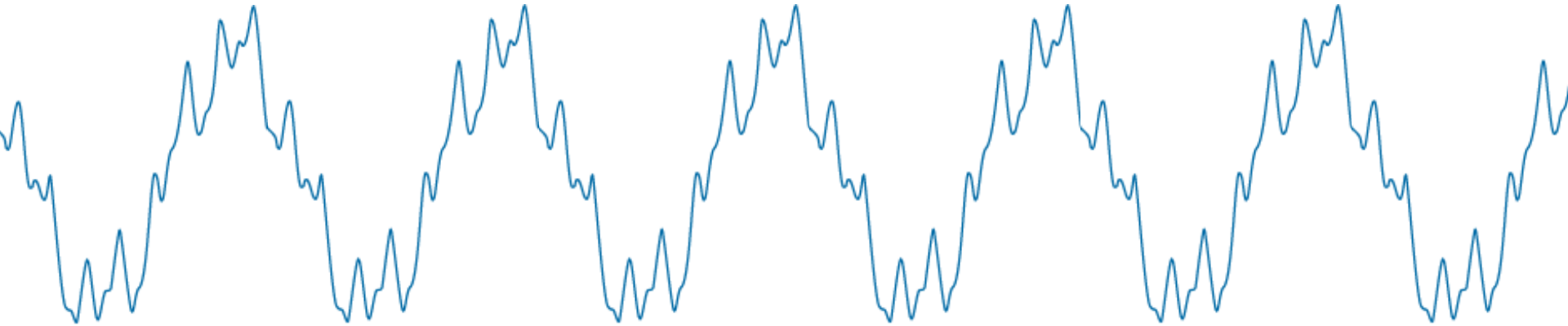
$$\|F(u)\| = \sqrt{R(u)^2 + I(u)^2} \qquad \text{Magnitude}$$

$$\theta(u) = \tan^{-1}\left(\frac{I(u)}{R(u)}\right) \qquad \text{Angle (or phase)}$$

Use atan2 function

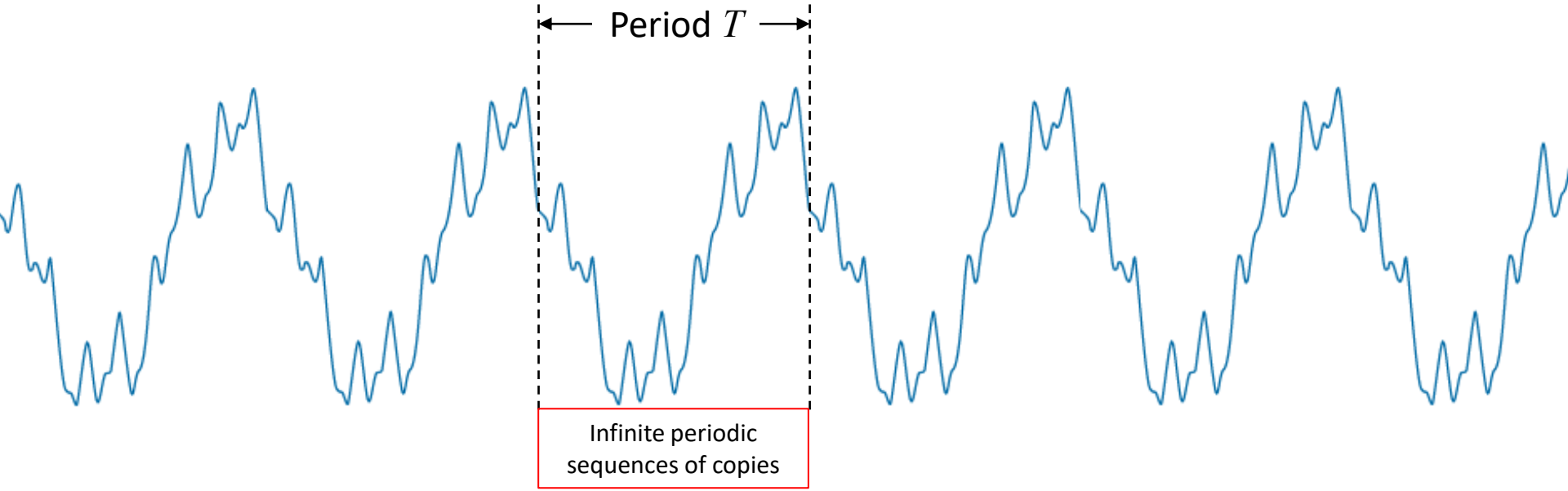
Periodic functions

- Example periodic function



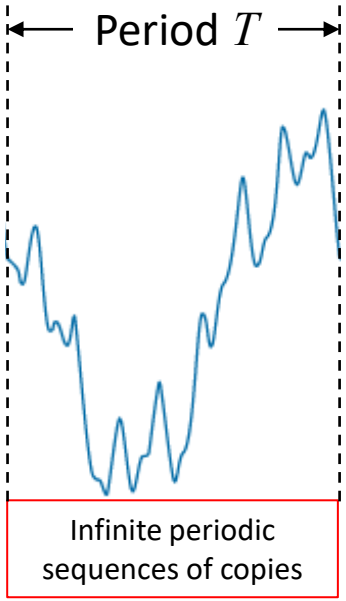
Periodic functions

- Example periodic function



Periodic functions

- Example periodic function



1D Fourier series

A function $f(t)$ of a continuous variable t that is periodic with period T can be expressed as a sum of sines and cosines multiplied by appropriate coefficients

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{T}t}$$

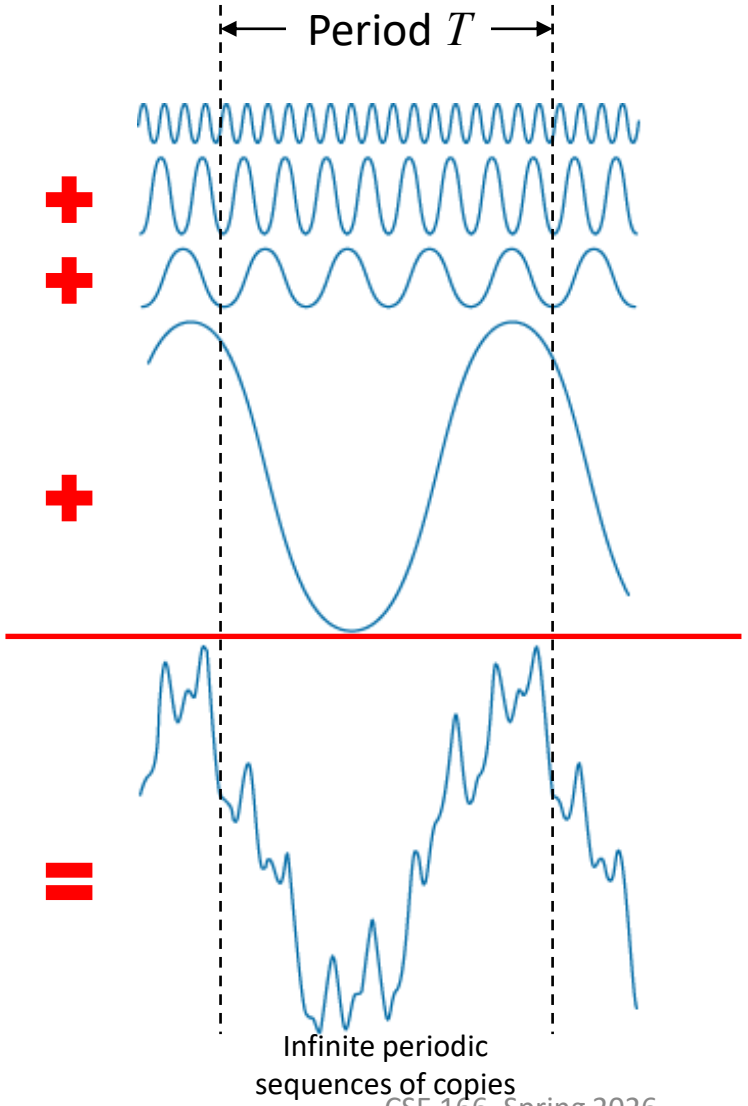
where

$$c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-j\frac{2\pi n}{T}t}\,\mathrm{d}t \text{ are the coefficients}$$

for $n = \ldots, -2, -1, 0, 1, 2, \ldots$

1D Fourier series

Sines and cosines



Weighted by magnitude

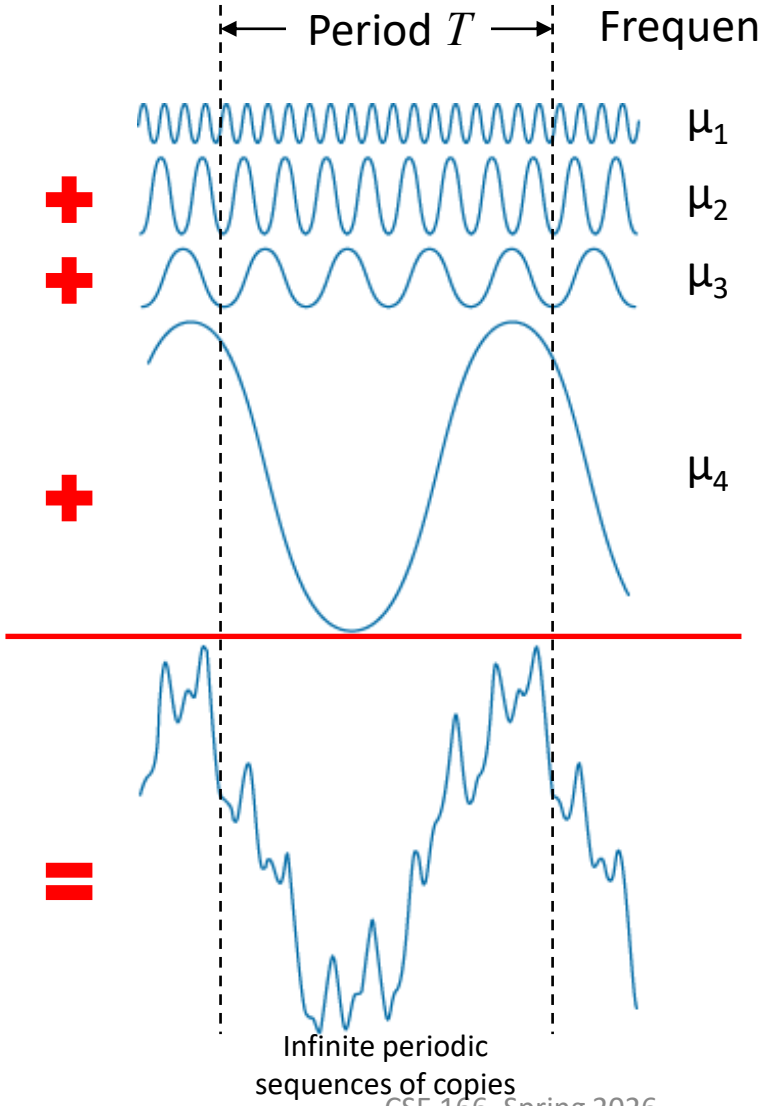
Shifted by phase

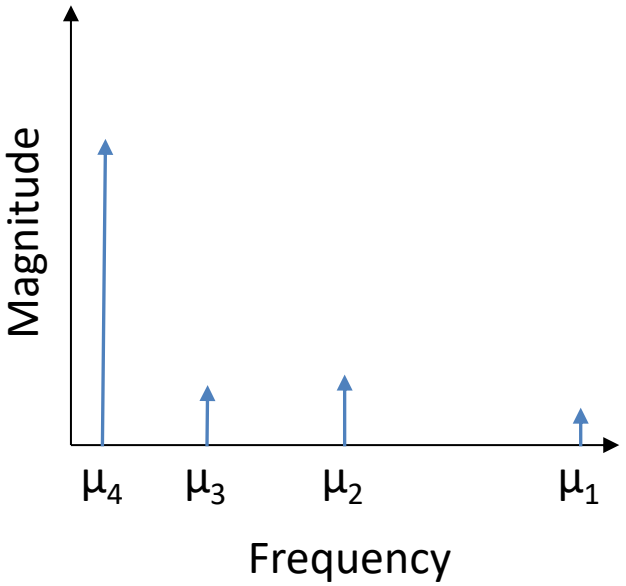
Periodic function

1D Fourier transform

Sines and cosines

Periodic function





1D continuous Fourier transform

- (Forward) Fourier transform

The continuous Fourier transform of a continuous function $f(t)$ is

$$\mathfrak{F}\{f(t)\} = F(\mu) = \int_{-\infty}^{\infty} f(t)e^{-j2\pi\mu t}\,\mathrm{d}t \text{ where } \mu \text{ is a continuous variable}$$

Since t is integrated out, $\mathfrak{F}\{f(t)\}$ is only a function of μ.

- Inverse Fourier transform

The inverse Fourer transform

$$\mathfrak{F}^{-1}\{F(\mu)\} = f(t) = \int_{-\infty}^{\infty} F(\mu)e^{j2\pi\mu t}\,\mathrm{d}\mu \text{ where } t \text{ is a continuous variable}$$

μ is integrated out, so $\mathfrak{F}^{-1}\{f(\mu)\}$ is only a function of t.

1D continuous Fourier transform

- Fourier transform pair

$$\mathfrak{F}\{f(t)\} = F(\mu) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t}\,\mathrm{d}t$$

$$\mathfrak{F}^{-1}\{F(\mu)\} = f(t) = \int_{-\infty}^{\infty} F(\mu) e^{j2\pi\mu t}\,\mathrm{d}\mu$$

1D continuous Fourier transform

- Using Euler's formula

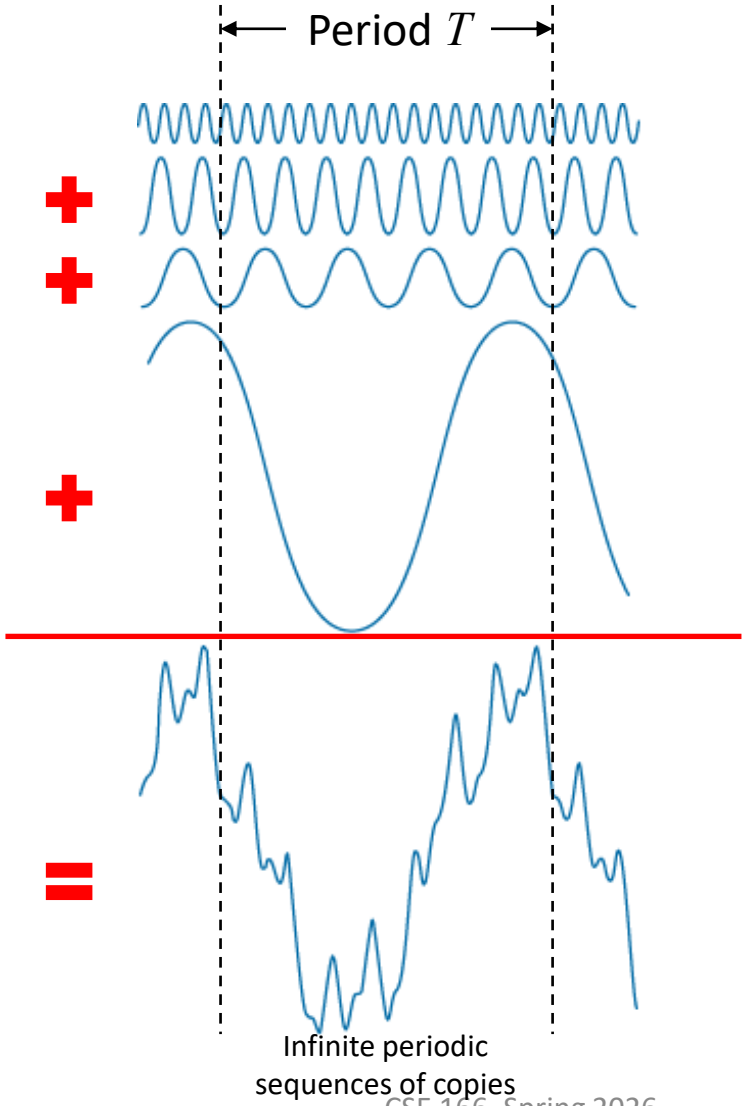
$$F(\mu) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t}\, \mathrm{d}t$$

$$F(\mu) = \int_{-\infty}^{\infty} f(t) \left[\cos(2\pi\mu t) - j \sin(2\pi\mu t)\right]\, \mathrm{d}t$$

Fourier transform is in frequency domain
Frequency μ is in units of cycles per unit of intput variable (e.g., cycles/seconds, cycles/meters).
If $f(t)$ is real, then (in general) $F(\mu)$ is complex. As such, it is customary to display the magnitude of $F(\mu)$, which is real.

1D Fourier series

Sines and cosines



Periodic function

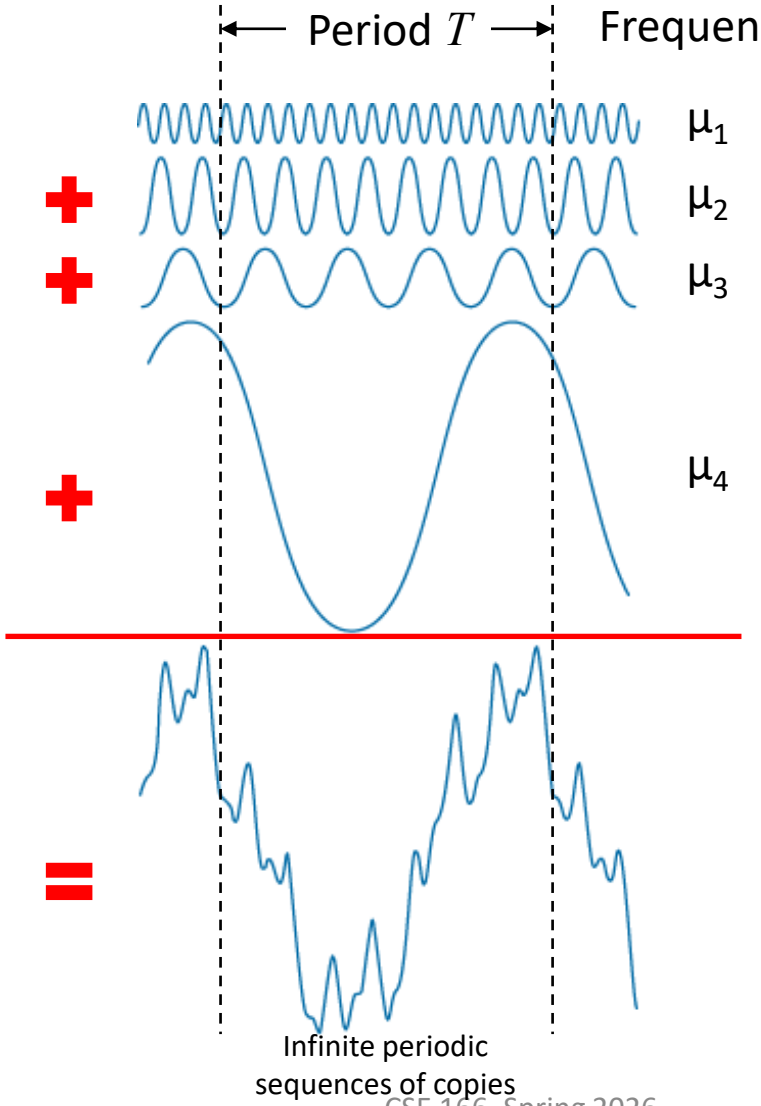
Weighted by magnitude

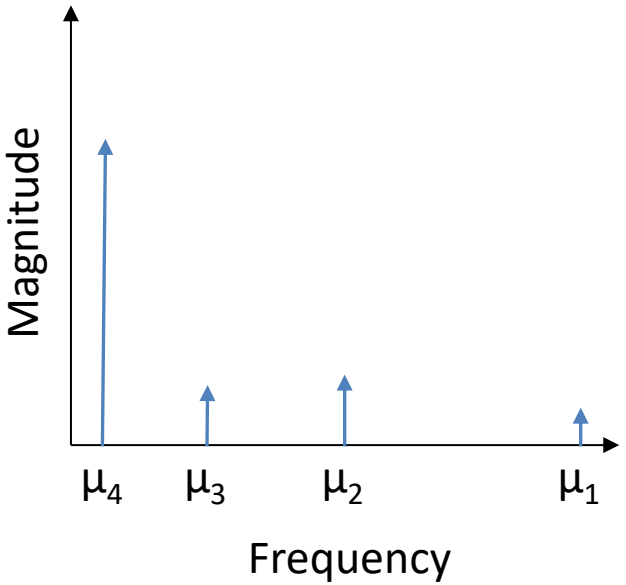
Shifted by phase

1D Fourier transform

Sines and cosines

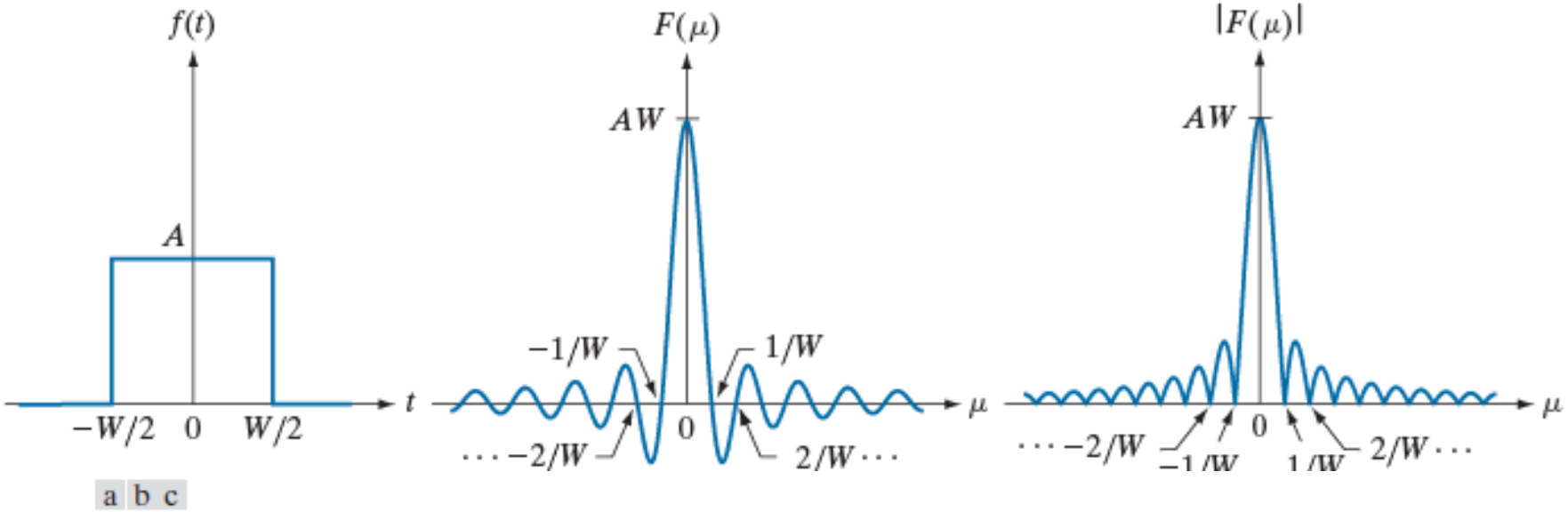
Periodic function





1D continuous Fourier transform

- Example: box function
  - The Fourier transform of a box function does not have an imaginary component (i.e., it is real)



a b c

FIGURE 4.4 (a) A box function, (b) its Fourier transform, and (c) its spectrum. All functions extend to infinity in both directions. Note the inverse relationship between the width, W, of the function and the zeros of the transform.

1D continuous Fourier transform

- 1D continuous convolution

$$f(t)\bigstar h(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)\,\mathrm{d}\tau$$

$$\mathfrak{F}\{f(t)\bigstar h(t)\} = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\tau)h(t-\tau)\,\mathrm{d}\tau\right]e^{-j2\pi\mu t}\,\mathrm{d}t$$

$$\mathfrak{F}\{f(t)\bigstar h(t)\} = \int_{-\infty}^{\infty} f(\tau)\left[\int_{-\infty}^{\infty} h(t-\tau)e^{-j2\pi\mu t}\,\mathrm{d}t\right]\,\mathrm{d}\tau$$

$$\mathfrak{F}\{f(t)\bigstar h(t)\} = \int_{-\infty}^{\infty} f(\tau)\left[H(\mu)e^{-j2\pi\mu\tau}\right]\,\mathrm{d}\tau$$

$$\mathfrak{F}\{f(t)\bigstar h(t)\} = H(\mu)\int_{-\infty}^{\infty} f(\tau)e^{-j2\pi\mu\tau}\,\mathrm{d}\tau$$

$$\mathfrak{F}\{f(t)\bigstar h(t)\} = H(\mu)F(\mu)$$

where

$$\mathfrak{F}\{h(t-\tau)\} = \int_{-\infty}^{\infty} h(t-\tau)e^{-j2\pi\mu t}\,\mathrm{d}t$$

$$\mathfrak{F}\{h(t-\tau)\} = H(\mu)e^{-j2\pi\mu\tau}$$

(Shown in future lecture)

1D continuous Fourier transform

- 1D convolution theorem

$$f(t) \bigstar h(t) \Longleftrightarrow H(\mu)F(\mu)$$
$$f(t)h(t) \Longleftrightarrow H(\mu) \bigstar F(\mu)$$

Key to filtering in the frequency domain

Next Lecture

- Sampling and aliasing, and the discrete Fourier transform
- Reading
  - Chapter 4: Filtering in the Frequency Domain
    - Sections 4.2, 4.3, and 4.4