Basis Vectors, Matrix-Based Transforms, and Basis Images

Image Processing

CSE 166

Lecture 10

Announcements

- Assignment 4 is due May 11, 11:59 PM
- Assignment 5 will be released May 13
  - Due May 20, 11:59 PM
- Reading
  - Chapter 6: Wavelet and Other Image Transforms
    - Sections 6.1, 6.2, and 6.5

Review

- Complex vectors
- Basis vectors

Complex vectors

Complex conjugate transpose

$$\mathbf{v}^H = (\mathbf{v}^*)^T = \left(\mathbf{v}^T\right)^*$$

where the elements of $\mathbf{v}$ are complex numbers

$$C = R + jI, \text{ where } j = \sqrt{-1}$$

Real numbers are a subset of complex numbers where $I = 0$

Complex conjugate

$$C^* = R - jI$$

Complex vectors

- Inner product

Real vectors

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T \mathbf{v} = \sum_{i=0}^{N-1} u_i v_i = \mathbf{v}^T \mathbf{u} = \langle \mathbf{v}, \mathbf{u} \rangle$$

Complex vectors

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^H \mathbf{v} = (\mathbf{u}^*)^T \mathbf{v} = \sum_{i=0}^{N-1} u_i^* v_i = \left(\mathbf{v}^H \mathbf{u}\right)^* = \langle \mathbf{v}, \mathbf{u} \rangle^*$$

If the vectors are continuous functions on the interval $a \leq x \leq b$, then

$$\langle f(x), g(x) \rangle = \int_a^b f^*(x) g(x) \, \mathrm{d}x$$

Complex multiplication

$$(a + jb)(c + jd) = (ac - db) + j(ad + bc)$$

Complex vectors

- Properties

$$\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$$
$$\langle \mathbf{u}, \alpha\mathbf{v} \rangle = \alpha \langle \mathbf{u}, \mathbf{v} \rangle, \text{ where } \alpha \text{ is a scalar}$$
$$\langle \mathbf{v}, \mathbf{v} \rangle \geq 0, \text{ and } \langle \mathbf{v}, \mathbf{v} \rangle = 0 \text{ if and only if } \mathbf{v} = \mathbf{0}$$

The norm of a vector $\mathbf{z}$ is $\|\mathbf{z}\| = \sqrt{\langle \mathbf{z}, \mathbf{z} \rangle}$
If $\|\mathbf{z}\| = 1$, then $\mathbf{z}$ is called a unit vector
The angle between two nonzero vectors is

$$\theta = \cos^{-1} \frac{\langle \mathbf{z}, \mathbf{w} \rangle}{\|\mathbf{z}\| \|\mathbf{w}\|}$$

If $\theta = 90°$, then $\mathbf{z}$ and $\mathbf{w}$ are called orthogonal (i.e., $\langle \mathbf{z}, \mathbf{w} \rangle = 0$)

Basis vectors

A set of nonzero vectors $\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2, \ldots$ such that

$$\langle \mathbf{w}_k, \mathbf{w}_l \rangle = 0 \text{ for } k \neq l \quad \text{(mutually orthogonal)}$$

are an **orthogonal** basis of the inner product space they are said to span. Further, if these basis vectors are unit vectors, then they are an **orthonormal** basis and

$$\langle \mathbf{w}_k, \mathbf{w}_l \rangle = \delta_{kl} = \begin{cases} 0 & \text{for } k \neq l \\ 1 & \text{for } k = l \end{cases}$$

Similarly, a set of vectors $\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2, \ldots$ and a complementary set of dual vectors $\tilde{\mathbf{w}}_0, \tilde{\mathbf{w}}_1, \tilde{\mathbf{w}}_2, \ldots$ such that

$$\langle \tilde{\mathbf{w}}_k, \mathbf{w}_l \rangle = 0 \text{ for } k \neq l \quad \text{(mutually biorthogonal)}$$

are a **biorthogonal** basis. They are a **biorthonormal** basis if and only if

$$\langle \tilde{\mathbf{w}}_k, \mathbf{w}_l \rangle = \delta_{kl} = \begin{cases} 0 & \text{for } k \neq l \\ 1 & \text{for } k = l \end{cases}$$

Basis vectors

Let $W = \{\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2, \dots\}$ be an orthogonal basis of inner product space V, and let $\mathbf{z} \in V$. Vector $\mathbf{z}$ can be expressed as a linear combination of basis vectors

$$\mathbf{z} = \alpha_0\mathbf{w}_0 + \alpha_1\mathbf{w}_1 + \alpha_2\mathbf{w}_2 + \cdots$$

The inner products of $\mathbf{z}$ and the i-th basis vector $\mathbf{w}_i$ is

$$\begin{aligned}
\langle \mathbf{w}_i, \mathbf{z} \rangle &= \langle \mathbf{w}_i, \alpha_0\mathbf{w}_0 + \alpha_1\mathbf{w}_1 + \alpha_2\mathbf{w}_2 + \cdots \rangle \\
\langle \mathbf{w}_i, \mathbf{z} \rangle &= \alpha_0\langle \mathbf{w}_i, \mathbf{w}_0 \rangle + \alpha_1\langle \mathbf{w}_i, \mathbf{w}_1 \rangle + \cdots + \alpha_i\langle \mathbf{w}_i, \mathbf{w}_i \rangle + \cdots \\
\langle \mathbf{w}_i, \mathbf{z} \rangle &= \alpha_i\langle \mathbf{w}_i, \mathbf{w}_i \rangle, \text{ solve for } \alpha_i \\
\frac{\langle \mathbf{w}_i, \mathbf{z} \rangle}{\langle \mathbf{w}_i, \mathbf{w}_i \rangle} &= \alpha_i
\end{aligned}$$

If W is an **orthonormal** basis, then $\alpha_i = \langle \mathbf{w}_i, \mathbf{z} \rangle$ (because $\langle \mathbf{w}_i, \mathbf{w}_i \rangle = 1$)
Similarly, let $W = \{\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2, \dots\}$ and $\tilde{W} = \{\tilde{\mathbf{w}}_0, \tilde{\mathbf{w}}_1, \tilde{\mathbf{w}}_2, \dots\}$ be a biorthogonal basis.

$$\begin{aligned}
\langle \tilde{\mathbf{w}}_i, \mathbf{z} \rangle &= \langle \tilde{\mathbf{w}}_i, \alpha_0\mathbf{w}_0 + \alpha_1\mathbf{w}_1 + \alpha_2\mathbf{w}_2 + \cdots \rangle \\
\langle \tilde{\mathbf{w}}_i, \mathbf{z} \rangle &= \alpha_i\langle \tilde{\mathbf{w}}_i, \mathbf{w}_i \rangle, \text{ solve for } \alpha_i \\
\frac{\langle \tilde{\mathbf{w}}_i, \mathbf{z} \rangle}{\langle \tilde{\mathbf{w}}_i, \mathbf{w}_i \rangle} &= \alpha_i
\end{aligned}$$

If W and $\tilde{W}$ are a **biorthonormal** basis, then $\alpha_i = \langle \tilde{\mathbf{w}}_i, \mathbf{z} \rangle$ (because $\langle \tilde{\mathbf{w}}_i, \mathbf{w}_i \rangle = 1$)

Matrix-based transforms

General forward transform using basis vectors

- 1D forward transform

$$T(u) = \sum_{x=0}^{N-1} f(x) r(x, u)$$

where

$T(u)$ is the transform of $f(x)$

x is a spatial variable

u is a transform variable

$r(x, u)$ is a forward transformation kernel

General inverse transform using basis vectors

- 1D inverse transform

$$f(x) = \sum_{u=0}^{N-1} T(u)s(x,u)$$

where

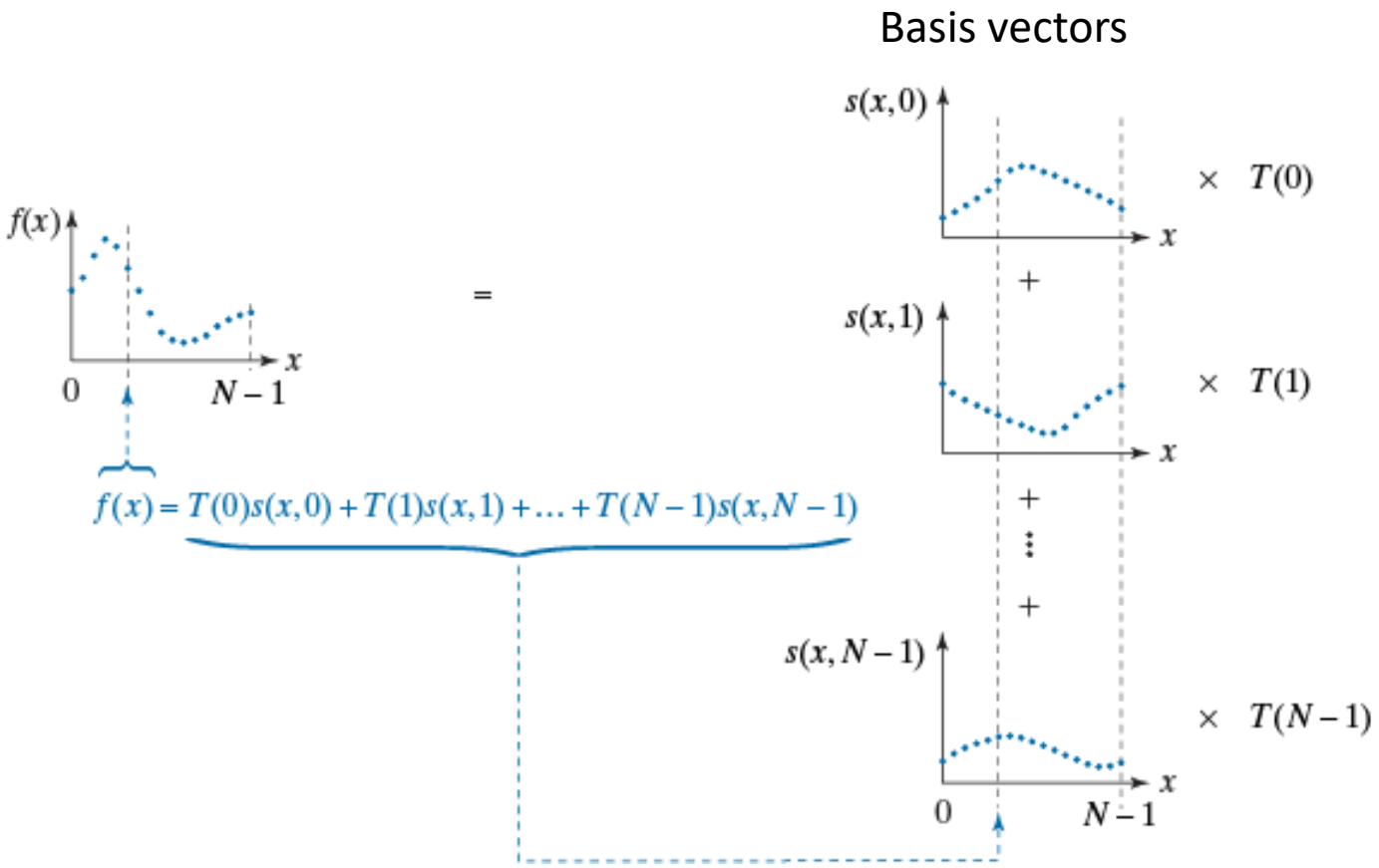
$f(x)$ is the inverse transform of $T(u)$

u is a transform variable

x is a spatial variable

$s(x,u)$ is an inverse transformation kernel

General inverse transform using basis vectors



Inverse transform using basis vectors

- Example: discrete cosine transform (DCT) basis vectors

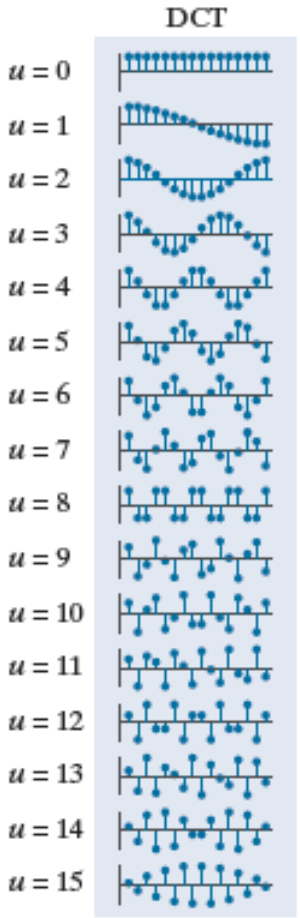
Inverse transformation kernel

$$s(x,u) = \alpha(u)\cos\left(\frac{(2x+1)u\pi}{2N}\right)$$

$N = 16$

where

$$\alpha(u) = \begin{cases} \sqrt{\frac{1}{N}} & \text{for } u = 0 \\ \sqrt{\frac{2}{N}} & \text{otherwise} \end{cases}$$



Inverse transform using basis vectors

- Example: Haar transform (HAAR) basis vectors

Inverse transformation kernel

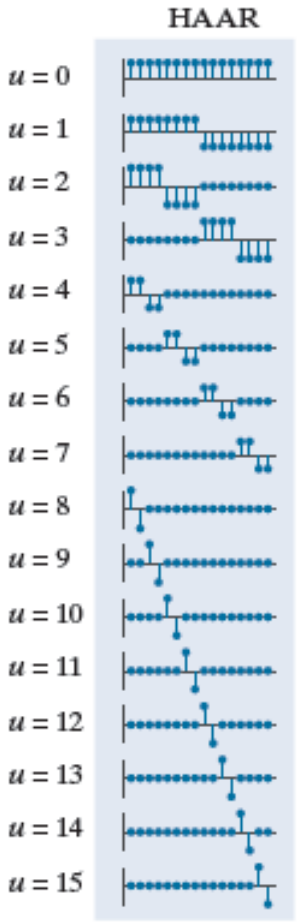
$$s(x,u) = \frac{1}{\sqrt{N}} h_u(x/N) \quad \text{for } x = 0, 1, \ldots, N-1$$

where

$$h_u(x) = \begin{cases} 1 & \text{if } u = 0 \text{ and } 0 \le x < 1 \\ 2^{p/2} & \text{if } u > 0 \text{ and } q/2^p \le x < (q+0.5)/2^p \\ -2^{p/2} & \text{if } u > 0 \text{ and } (q+0.5)/2^p \le x < (q+1)/2^p \\ 0 & \text{otherwise} \end{cases}$$

where $p = \lceil \log_2 u \rceil$ and $q = 2^p - u$

$N = 16$



Transforms using basis vectors

- Transform pair

$$T(u) = \sum_{x=0}^{N-1} f(x) r(x,u)$$

$$f(x) = \sum_{u=0}^{N-1} T(u) s(x,u)$$

where

x is a spatial variable
u is a transform variable
$T(u)$ is the transform of $f(x)$
$f(x)$ is the inverse transform of $T(u)$
$r(x,u)$ is a forward transformation kernel
$s(x,u)$ is an inverse transformation kernel

Note the transformation kernels $r(x,u)$ and $s(x,u)$ depend only on x and u, not values of $f(x)$ and $T(u)$

Matrix-based transforms using orthonormal basis vectors

- 1D inverse transform

$$f(x) = \sum_{u=0}^{N-1} T(u)s(x,u)$$

$$f(x) = T(0)s(x,0) + T(1)s(x,1) + \cdots + T(N-1)s(x,N-1)$$

$$\text{(which looks like } \mathbf{z} = \alpha_0\mathbf{w}_0 + \alpha_1\mathbf{w}_1 + \cdots + \alpha_{N-1}\mathbf{w}_{N-1}\text{)}$$

If $s(x,0), s(x,1), s(x,2), \ldots$ are orthonormal basis vectors, then

$$\langle s(x,u), f(x)\rangle = T(0)\langle s(x,u), s(x,0)\rangle + T(1)\langle s(x,u), s(x,1)\rangle + \cdots + T(u)\langle s(x,u), s(x,u)\rangle + \cdots$$

$$\langle s(x,u), f(x)\rangle = T(u)$$

Matrix-based transforms using orthonormal basis vectors

$$\langle s(x,u), f(x)\rangle = T(0)\langle s(x,u), s(x,0)\rangle + T(1)\langle s(x,u), s(x,1)\rangle + \cdots + T(u)\langle s(x,u), s(x,u)\rangle + \cdots$$

$$\langle s(x,u), f(x)\rangle = T(u)\langle s(x,u), s(x,u)\rangle$$

$$\frac{\langle s(x,u), f(x)\rangle}{\langle s(x,u), s(x,u)\rangle} = T(u)$$

If $s(x,0), s(x,1), s(x,2), \ldots$ are orthonormal basis vectors, then

$$\langle s(x,u), f(x)\rangle = T(u)$$

- In vector form

$$T(u) = \langle s(x,u), f(x)\rangle$$

$$T(u) = \langle \mathbf{s}_u, \mathbf{f}\rangle$$

$$\text{where } \mathbf{s}_u = \begin{bmatrix} s(0,u) \\ s(1,u) \\ \vdots \\ s(N-1,u) \end{bmatrix} \text{ and } \mathbf{f} = \begin{bmatrix} f(0) \\ f(1) \\ \vdots \\ f(N-1) \end{bmatrix}$$

(x indexes the rows of $\mathbf{s}_u$ and $\mathbf{f}$)

$$T(u) = \mathbf{s}_u^H \mathbf{f} \text{ for complex vectors}$$

$$T(u) = \mathbf{s}_u^T \mathbf{f} \text{ for real vectors}$$

Matrix-based transforms using orthonormal basis vectors

- In matrix form

$$\underset{u}{\begin{bmatrix} T(0) \\ T(1) \\ \vdots \\ T(N-1) \end{bmatrix}} = \underset{u}{\overset{x}{\begin{bmatrix} \mathbf{s}_0^H \\ \mathbf{s}_1^H \\ \vdots \\ \mathbf{s}_{N-1}^H \end{bmatrix}}} \underset{x}{\begin{bmatrix} f(0) \\ f(1) \\ \vdots \\ f(N-1) \end{bmatrix}} \quad \text{for complex vectors}$$

$$\begin{bmatrix} T(0) \\ T(1) \\ \vdots \\ T(N-1) \end{bmatrix} = \begin{bmatrix} \mathbf{s}_0^T \\ \mathbf{s}_1^T \\ \vdots \\ \mathbf{s}_{N-1}^T \end{bmatrix} \begin{bmatrix} f(0) \\ f(1) \\ \vdots \\ f(N-1) \end{bmatrix} \quad \text{for real vectors}$$

$$\mathbf{t} = \mathbf{A}\mathbf{f}$$

where transformation matrix

$$\mathbf{A} = \underset{u}{\overset{x}{\begin{bmatrix} \mathbf{s}_0^H \\ \mathbf{s}_1^H \\ \vdots \\ \mathbf{s}_{N-1}^H \end{bmatrix}}} \quad \text{for complex vectors} \qquad N \times N$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{s}_0^T \\ \mathbf{s}_1^T \\ \vdots \\ \mathbf{s}_{N-1}^T \end{bmatrix} \quad \text{for real vectors} \qquad N \times N$$

Matrix-based transforms using orthonormal basis vectors

- Forward and inverse transformations

$$\mathbf{t} = \mathtt{A}\mathbf{f}$$
$$\mathtt{A}^{-1}\mathbf{t} = \mathbf{f}$$
$$\mathtt{A}^{H}\mathbf{t} = \mathbf{f} \text{ for complex } \mathtt{A}$$
$$\mathtt{A}^{T}\mathbf{t} = \mathbf{f} \text{ for real } \mathtt{A}$$

Note forward transformation kernel $r(x,u)$ is not needed

Note $\mathtt{A}^H$ for complex $\mathtt{A}$ and $\mathtt{A}^T$ for real $\mathtt{A}$ is

$$\begin{matrix} & u \\ x & \end{matrix}\begin{bmatrix} \mathbf{r}_0^\top \\ \mathbf{r}_1^\top \\ \vdots \\ \mathbf{r}_{N-1}^\top \end{bmatrix}$$

where

$$\mathbf{r}_x = \begin{matrix} u \\ \downarrow \end{matrix}\begin{bmatrix} r(x,0) \\ r(x,1) \\ \vdots \\ r(x,N-1) \end{bmatrix}$$

- Properties

Unitary matrix $\mathtt{A}$

$$\mathtt{A}^H\mathtt{A} = \mathtt{A}\mathtt{A}^H = \mathtt{A}^*\mathtt{A}^T = \mathtt{A}^T\mathtt{A}^* = \mathtt{I}$$
$$\mathtt{A}^{-1} = \mathtt{A}^H$$

Orthogonal matrix $\mathtt{A}$

$$\mathtt{A}^T\mathtt{A} = \mathtt{A}\mathtt{A}^T = \mathtt{I}$$
$$\mathtt{A}^{-1} = \mathtt{A}^T$$

Matrix-based transforms using biorthonormal basis vectors

- In matrix form

$$\underset{u}{\begin{bmatrix} T(0) \\ T(1) \\ \vdots \\ T(N-1) \end{bmatrix}} = \underset{u}{\begin{bmatrix} \tilde{\mathbf{s}}_0^H \\ \tilde{\mathbf{s}}_1^H \\ \vdots \\ \tilde{\mathbf{s}}_{N-1}^H \end{bmatrix}} \underset{x}{\begin{bmatrix} f(0) \\ f(1) \\ \vdots \\ f(N-1) \end{bmatrix}} \quad \text{for complex vectors}$$

$$\begin{bmatrix} T(0) \\ T(1) \\ \vdots \\ T(N-1) \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{s}}_0^T \\ \tilde{\mathbf{s}}_1^T \\ \vdots \\ \tilde{\mathbf{s}}_{N-1}^T \end{bmatrix} \begin{bmatrix} f(0) \\ f(1) \\ \vdots \\ f(N-1) \end{bmatrix} \quad \text{for real vectors}$$

$$\mathbf{t} = \tilde{\mathbf{A}}\mathbf{f}$$

$$\mathbf{f} = \mathbf{A}^H\mathbf{t} \text{ for complex } \mathbf{A}$$

$$\mathbf{f} = \mathbf{A}^T\mathbf{t} \text{ for real } \mathbf{A}$$

Use dual vectors for forward transformation only

Matrix-based transform using basis vectors

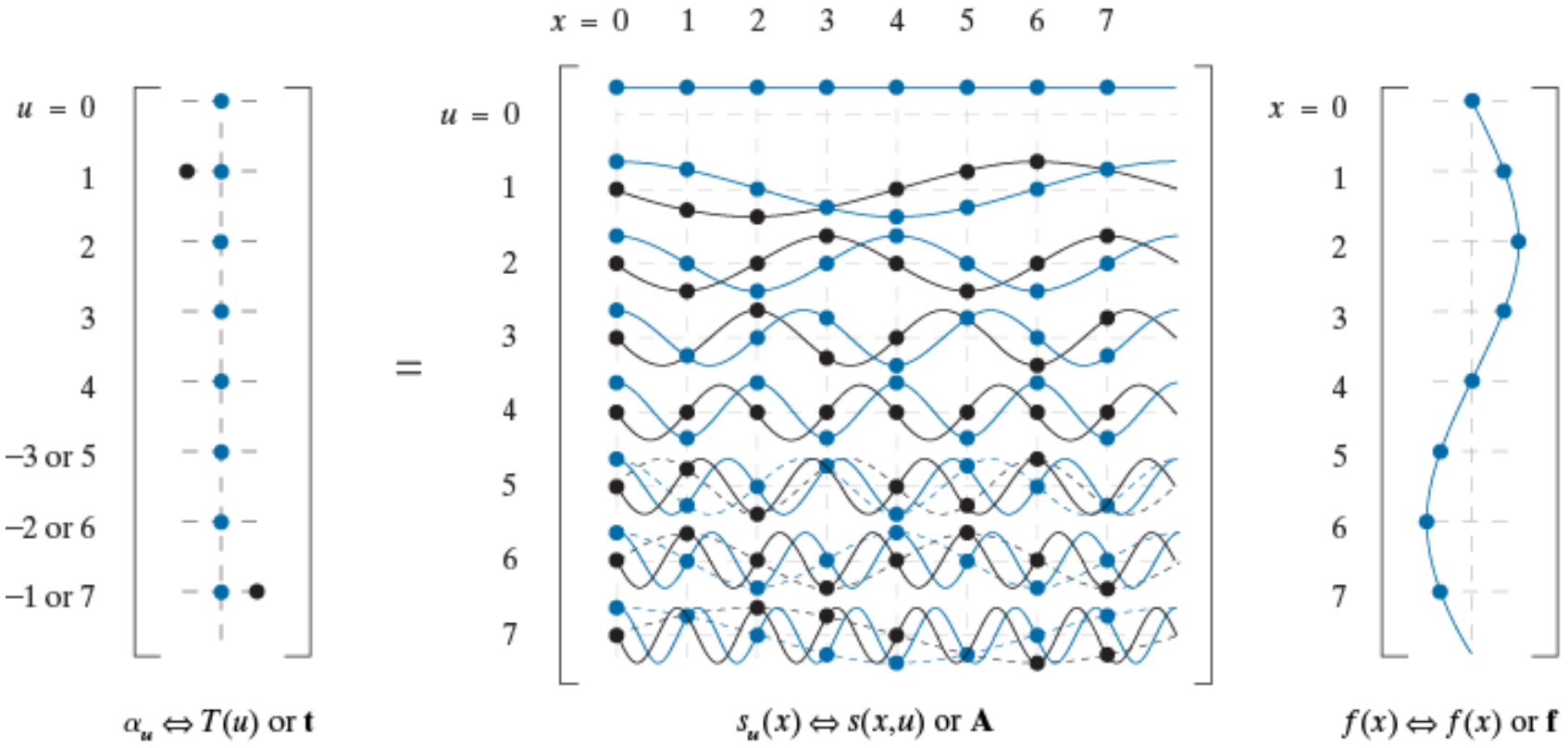
- Example: 1D discrete Fourier transform (DFT)

$$T(u) = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} f(x) e^{-j2\pi ux/N} \text{ for } u = 0, 1, 2, \ldots, N-1$$

$$\mathbf{t} = \mathbf{Af}, \text{ where } \mathbf{A} = \begin{bmatrix} \mathbf{s}_0^H \\ \mathbf{s}_1^H \\ \vdots \\ \mathbf{s}_{N-1}^H \end{bmatrix}, \text{ where } s_u(x) = \frac{1}{\sqrt{N}} e^{j2\pi ux/N}$$

Matrix-based transform using basis vectors

- Example: $N = 8$, DFT of $f(x) = \sin(2\pi x/N)$



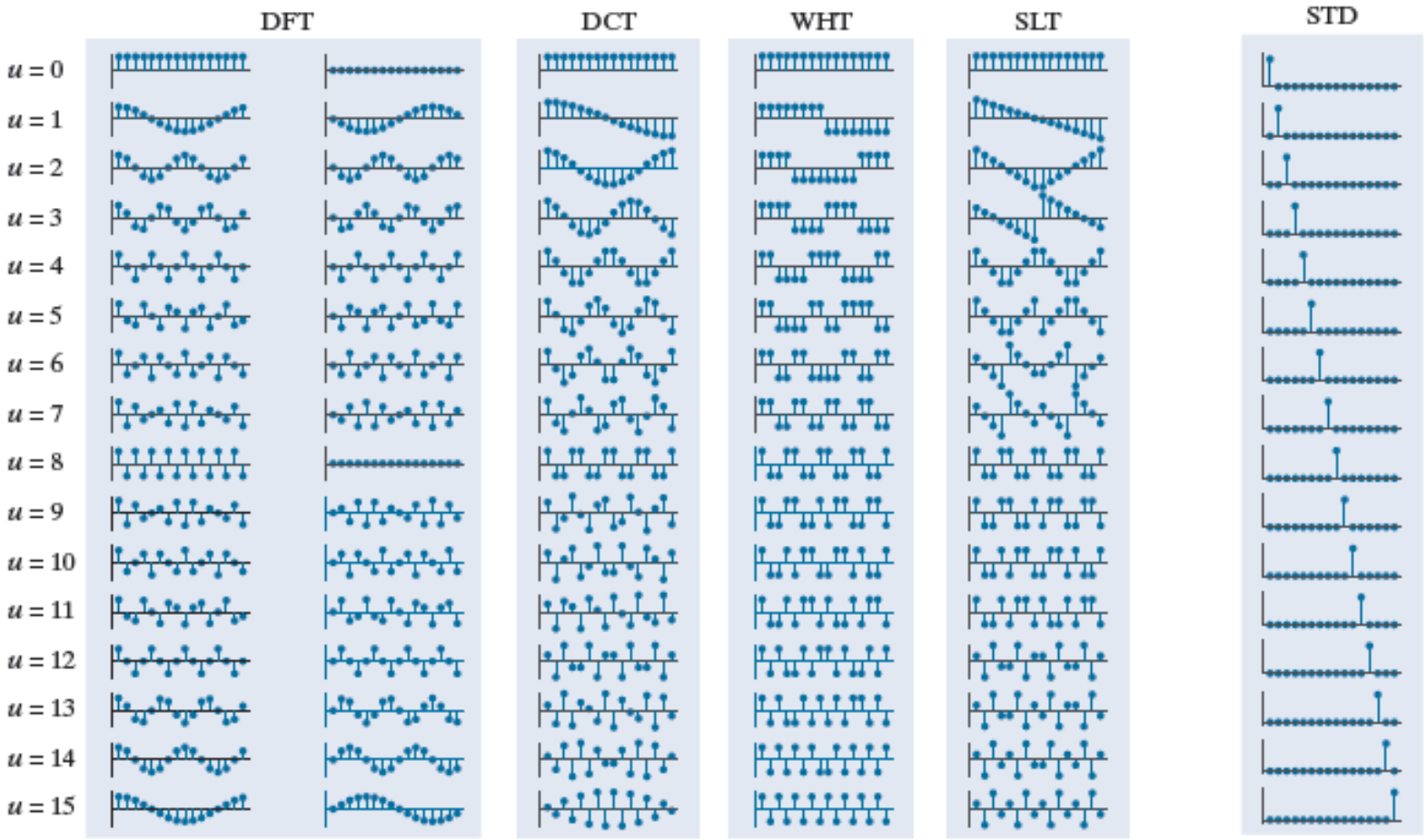
real part + imaginary part

Matrix-based transforms using basis vectors

- Discrete Fourier transform (DFT)
- Discrete Hartley transform (DHT)
- Discrete cosine transform (DCT)
- Discrete sine transform (DST)
- Walsh-Hadamard (WHT)
- Slant (SLT)
- Haar (HAAR)
- Daubechies (DB4)
- Biorthogonal B-spline (BIOR3.1)

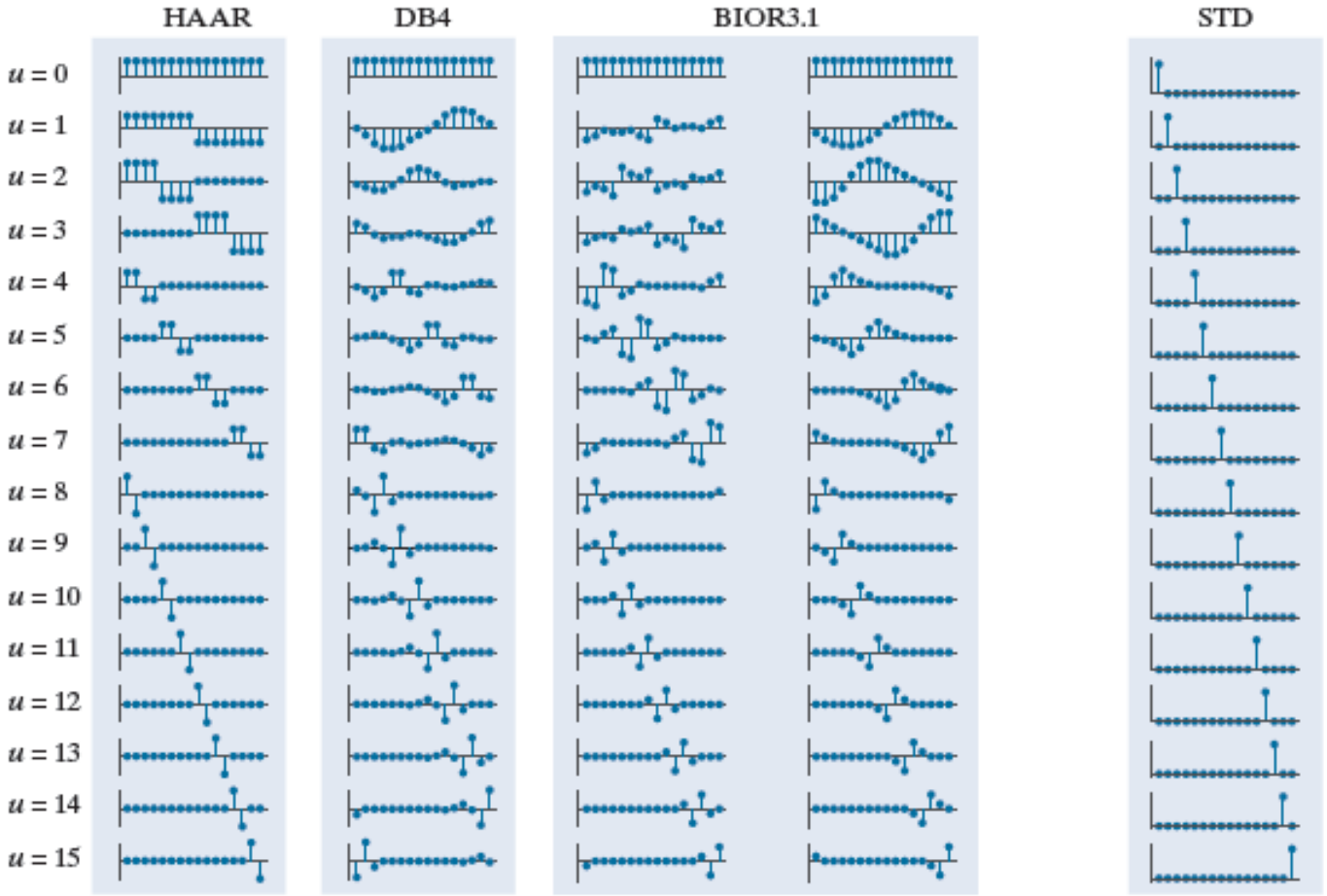
Basis vectors of matrix-based 1D transforms

$N = 16$



Basis vectors of matrix-based 1D transforms

$N = 16$



basis dual

Standard basis (for reference)

Matrix-based transforms in two dimensions

$$T(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) r(x,y,u,v)$$ Forward transform

$$f(x,y) = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} T(u,v) s(x,y,u,v)$$ Inverse transform

where

x, y are spatial variables

u, v are transform variables

$T(u,v)$ is the transform of $f(x,y)$

$f(x,y)$ is the inverse transform of $T(u,v)$

$r(x,y,u,v)$ is a forward transformation kernel

$s(x,y,u,v)$ is an inverse transformation kernel

Matrix-based transforms in two dimensions

If $s(x, y, u, v) = s_1(x, u)s_2(y, v)$, then s is called seperable.
If $s(x, y, u, v) = s_1(x, u)s_1(y, v)$, then s is called symmetric.
If r and s are separable and symmetric, then

$$\mathtt{T} = \mathtt{A}_M \mathtt{F} \mathtt{A}_N^T$$

$$\mathtt{F} = \mathtt{A}_M^{*T} \mathtt{T} \mathtt{A}_N^* \text{ for complex } \mathtt{A}_M \text{ and } \mathtt{A}_N$$

$$\mathtt{F} = \mathtt{A}_M^T \mathtt{T} \mathtt{A}_N \text{ for real } \mathtt{A}_M \text{ and } \mathtt{A}_N$$

Note forward transformation kernel is not needed

where

$$\mathtt{T} = \begin{array}{c} \\ u \\ \\ \\ \end{array} \overset{v}{\begin{bmatrix} T(0,0) & T(0,1) & \cdots & T(0,N-1) \\ T(1,0) & T(1,1) & \cdots & T(1,N-1) \\ \vdots & \vdots & \ddots & \vdots \\ T(M-1,0) & T(M-1,1) & \cdots & T(M-1,N-1) \end{bmatrix}}, \qquad M \text{ x } N$$

$$\mathtt{A}_M = \begin{array}{c} \\ u \\ \\ \\ \end{array} \overset{x}{\begin{bmatrix} s_1(0,0) & s_1(1,0) & \cdots & s_1(M-1,0) \\ s_1(0,1) & s_1(1,1) & \cdots & s_1(M-1,1) \\ \vdots & \vdots & \ddots & \vdots \\ s_1(0,M-1) & s_1(1,M-1) & \cdots & s_1(M-1,M-1) \end{bmatrix}}, \text{ and } \quad M \text{ x } M$$

Transformation matrices

$$\mathtt{A}_N = \begin{array}{c} \\ v \\ \\ \\ \end{array} \overset{y}{\begin{bmatrix} s_1(0,0) & s_1(1,0) & \cdots & s_1(N-1,0) \\ s_1(0,1) & s_1(1,1) & \cdots & s_1(N-1,1) \\ \vdots & \vdots & \ddots & \vdots \\ s_1(0,N-1) & s_1(1,N-1) & \cdots & s_1(N-1,N-1) \end{bmatrix}} \qquad N \text{ x } N$$

Matrix-based transforms in two dimensions

- If r and s are separable and symmetric, and $M = N$, then
  - For orthonormal basis vectors

$$\mathtt{T} = \mathtt{A}\mathtt{F}\mathtt{A}^T$$

$$\mathtt{F} = \mathtt{A}^{*T}\mathtt{T}\mathtt{A}^* \text{ for complex } \mathtt{A}$$

$$\mathtt{F} = \mathtt{A}^T\mathtt{T}\mathtt{A} \text{ for real } \mathtt{A}$$

Note $\mathtt{A}^H$ for complex $\mathtt{A}$ and $\mathtt{A}^T$ for real $\mathtt{A}$ is

$$\begin{matrix} & u \\ x & \begin{bmatrix} \mathbf{r}_0^\top \\ \mathbf{r}_1^\top \\ \vdots \\ \mathbf{r}_{N-1}^\top \end{bmatrix} \end{matrix}$$

where

$$\mathbf{r}_x = \begin{matrix} u \\ \downarrow \\ \end{matrix} \begin{bmatrix} r(x,0) \\ r(x,1) \\ \vdots \\ r(x,N-1) \end{bmatrix}$$

  - For biorthonormal basis vectors

$$\mathtt{T} = \tilde{\mathtt{A}}\mathtt{F}\tilde{\mathtt{A}}^T$$

$$\mathtt{F} = \mathtt{A}^{*T}\mathtt{T}\mathtt{A}^* \text{ for complex } \mathtt{A}$$

$$\mathtt{F} = \mathtt{A}^T\mathtt{T}\mathtt{A} \text{ for real } \mathtt{A}$$

Use dual vectors for forward transformation only

Matrix-based transforms in two dimensions using basis images

- Forward transform

$$T(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)r(x,y,u,v)$$ Using transformation kernel

$$\mathtt{T} = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\mathtt{R}_{x,y}$$ Using basis images

where

$$\mathtt{T} = \begin{matrix} u \\ \\ \\ \\ \end{matrix}\overset{v}{\begin{bmatrix} T(0,0) & T(0,1) & \cdots & T(0,N-1) \\ T(1,0) & T(1,1) & \cdots & T(1,N-1) \\ \vdots & \vdots & \ddots & \vdots \\ T(N-1,0) & T(N-1,1) & \cdots & T(N-1,N-1) \end{bmatrix}}$$

Each $\mathrm{R}_{x,y}$ is a basis image

$$\mathtt{R}_{x,y} = \begin{matrix} u \\ \\ \\ \\ \end{matrix}\overset{v}{\begin{bmatrix} r(x,y,0,0) & r(x,y,0,1) & \cdots & r(x,y,0,N-1) \\ r(x,y,1,0) & r(x,y,1,1) & \cdots & r(x,y,1,N-1) \\ \vdots & \vdots & \ddots & \vdots \\ r(x,y,N-1,0) & r(x,y,N-1,1) & \cdots & r(x,y,N-1,N-1) \end{bmatrix}}$$

Matrix-based transforms in two dimensions using basis images

- Forward transformation

Each $\mathbf{R}_{x,y}$ is a basis image

$$\mathbf{R}_{x,y} = \begin{matrix} u \downarrow \\ \end{matrix} \begin{bmatrix} r(x,y,0,0) & r(x,y,0,1) & \cdots & r(x,y,0,N-1) \\ r(x,y,1,0) & r(x,y,1,1) & \cdots & r(x,y,1,N-1) \\ \vdots & \vdots & \ddots & \vdots \\ r(x,y,N-1,0) & r(x,y,N-1,1) & \cdots & r(x,y,N-1,N-1) \end{bmatrix} \begin{matrix} \\ \\ \\ \\ \end{matrix}$$

$v \rightarrow$

If $r(x,y,u,v)$ is separable and symmetric, then

$$\mathbf{R}_{x,y} = \mathbf{r}_x \mathbf{r}_y^T$$

where

$$\mathbf{r}_x = \begin{matrix} u \downarrow \\ \end{matrix} \begin{bmatrix} r(0,y) \\ r(1,y) \\ \vdots \\ r(N-1,y) \end{bmatrix} \quad \text{and} \quad \mathbf{r}_y = \begin{matrix} v \downarrow \\ \end{matrix} \begin{bmatrix} r(0,y) \\ r(1,y) \\ \vdots \\ r(N-1,y) \end{bmatrix}$$

Matrix-based transforms in two dimensions using basis images

- Inverse transform

$$f(x,y) = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} T(u,v)s(x,y,u,v)$$ Using transformation kernel

$$\mathbf{F} = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} T(u,v)\mathbf{S}_{u,v}$$ Using basis images

where

$$\mathbf{F} = \begin{matrix} x \\ \\ \\ \\ \end{matrix}\overset{y}{\begin{bmatrix} f(0,0) & f(0,1) & \cdots & f(0,N-1) \\ f(1,0) & f(1,1) & \cdots & f(1,N-1) \\ \vdots & \vdots & \ddots & \vdots \\ f(N-1,0) & f(N-1,1) & \cdots & f(N-1,N-1) \end{bmatrix}}$$

Each $\mathbf{S}_{u,v}$ is a basis image

$$\mathbf{S}_{u,v} = \begin{matrix} x \\ \\ \\ \\ \end{matrix}\overset{y}{\begin{bmatrix} s(0,0,u,v) & s(0,1,u,v) & \cdots & s(0,N-1,u,v) \\ s(1,0,u,v) & s(1,1,u,v) & \cdots & s(1,N-1,u,v) \\ \vdots & \vdots & \ddots & \vdots \\ s(N-1,0,u,v) & s(N-1,1,u,v) & \cdots & s(N-1,N-1,u,v) \end{bmatrix}}$$

Matrix-based transforms in two dimensions using basis images

- Inverse transform

Each $S_{u,v}$ is a basis image

$$\mathbf{S}_{u,v} = \begin{matrix} x \downarrow \\ \\ \end{matrix} \overset{y \rightarrow}{\begin{bmatrix} s(0,0,u,v) & s(0,1,u,v) & \cdots & s(0,N-1,u,v) \\ s(1,0,u,v) & s(1,1,u,v) & \cdots & s(1,N-1,u,v) \\ \vdots & \vdots & \ddots & \vdots \\ s(N-1,0,u,v) & s(N-1,1,u,v) & \cdots & s(N-1,N-1,u,v) \end{bmatrix}}$$

If $s(x,y,u,v)$ is separable and symmetric, then

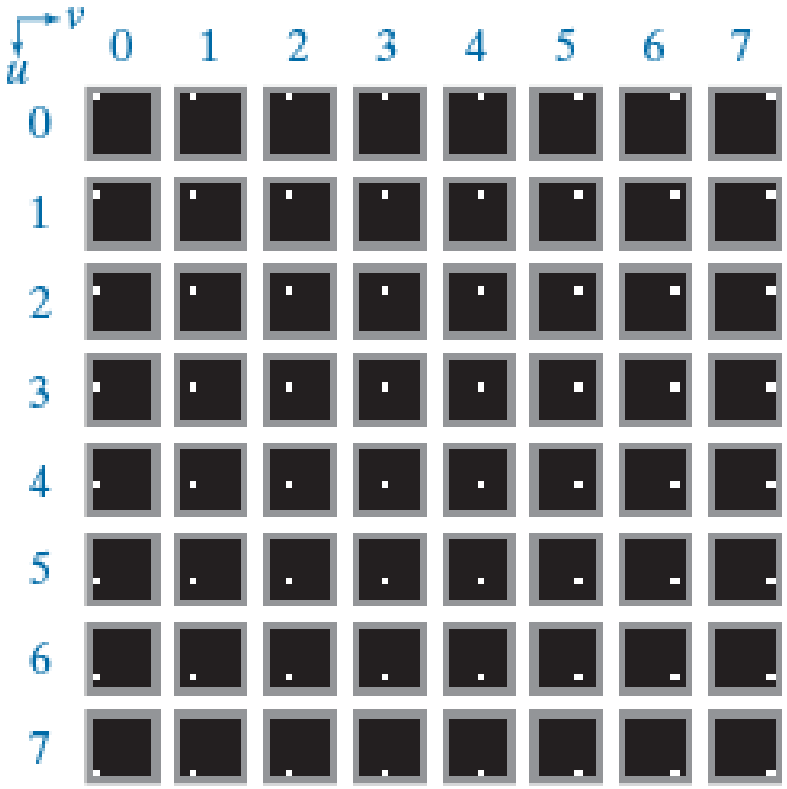
$$\mathbf{S}_{u,v} = \mathbf{s}_u \mathbf{s}_v^T$$

where

$$\mathbf{s}_u = \begin{matrix} x \downarrow \\ \\ \end{matrix} \begin{bmatrix} s(0,u) \\ s(1,u) \\ \vdots \\ s(N-1,u) \end{bmatrix} \quad \text{and} \quad \mathbf{s}_v = \begin{matrix} y \downarrow \\ \\ \end{matrix} \begin{bmatrix} s(0,v) \\ s(1,v) \\ \vdots \\ s(N-1,v) \end{bmatrix}$$

Basis images of matrix-based 2D transforms

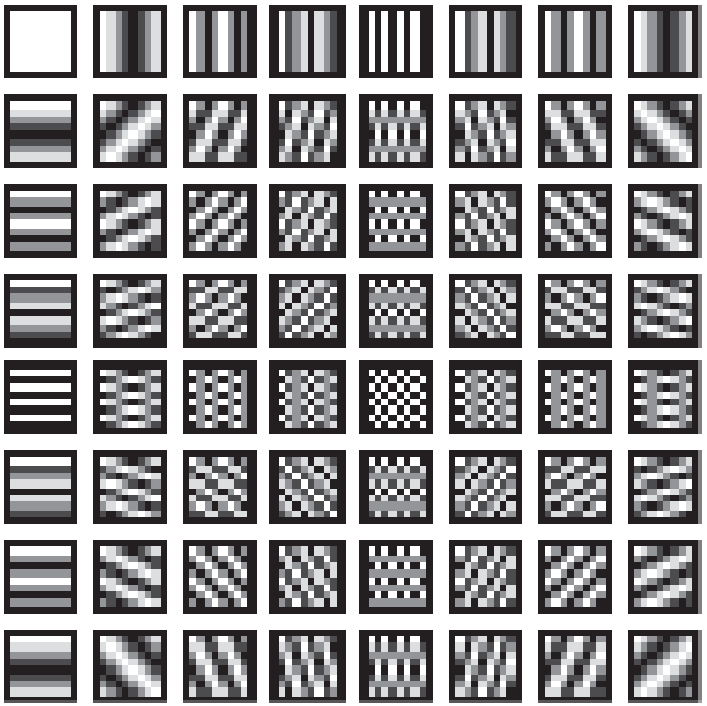
Standard basis images (for reference)



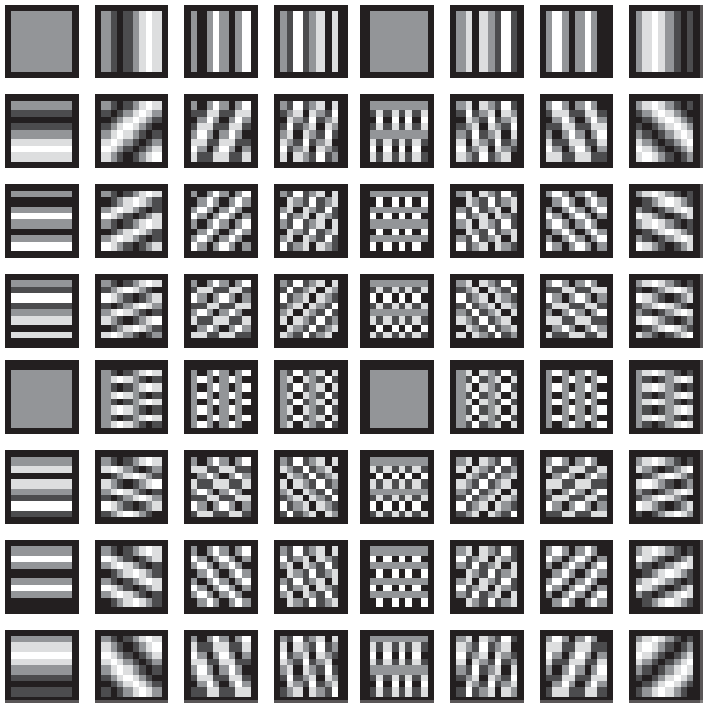
8-by-8 array of 8-by-8 basis images

Basis images of matrix-based 2D transforms

Discrete Fourier transform (DFT) basis images



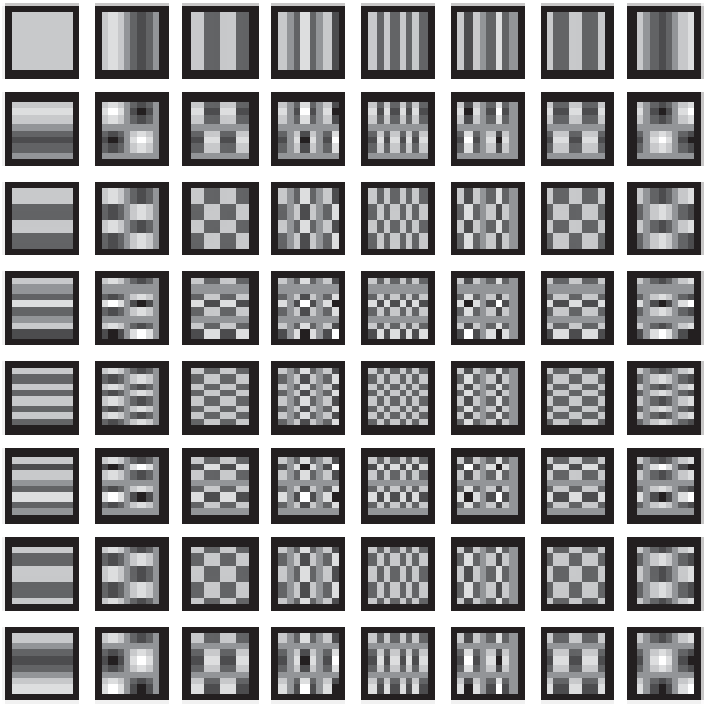
real part



imaginary part

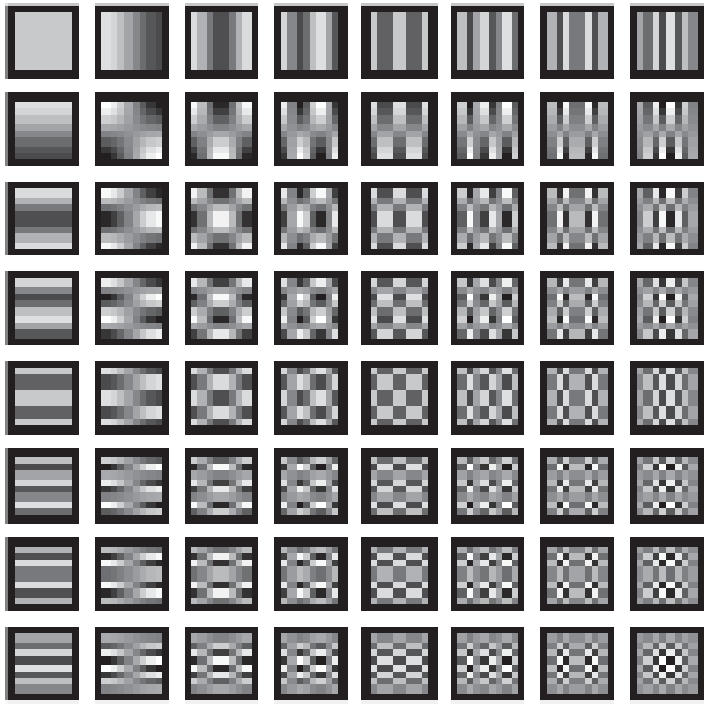
Basis images of matrix-based 2D transforms

Discrete Hartley transform (DHT) basis images



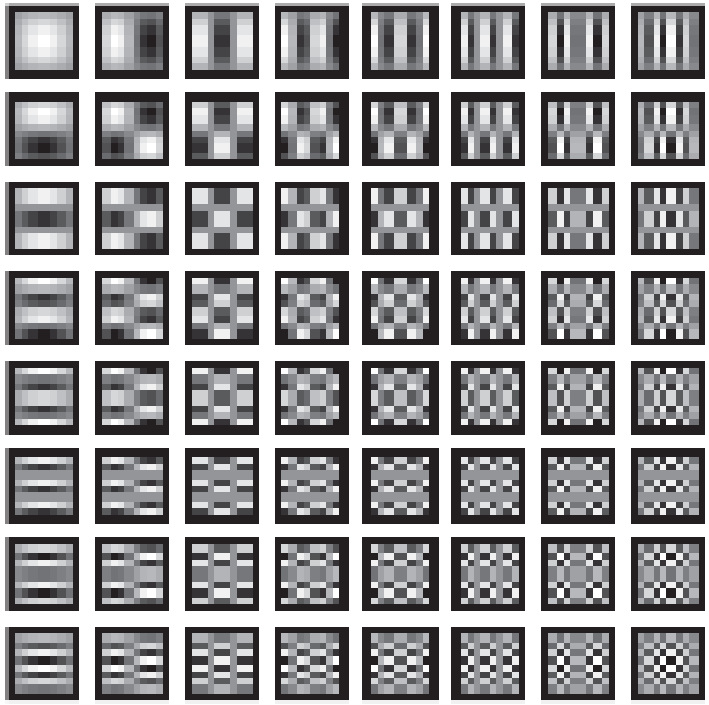
Basis images of matrix-based 2D transforms

Discrete cosine transform (DCT) basis images



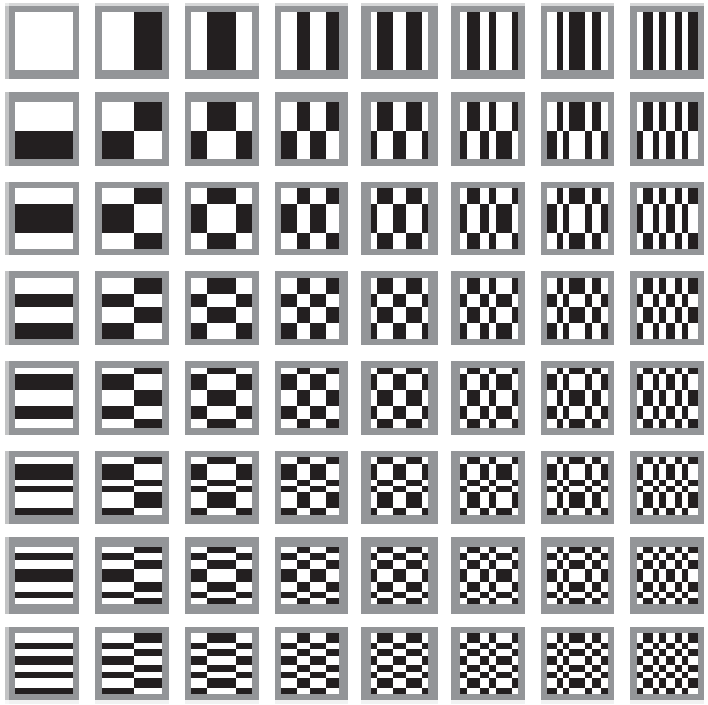
Basis images of matrix-based 2D transforms

Discrete sine transform (DST) basis images



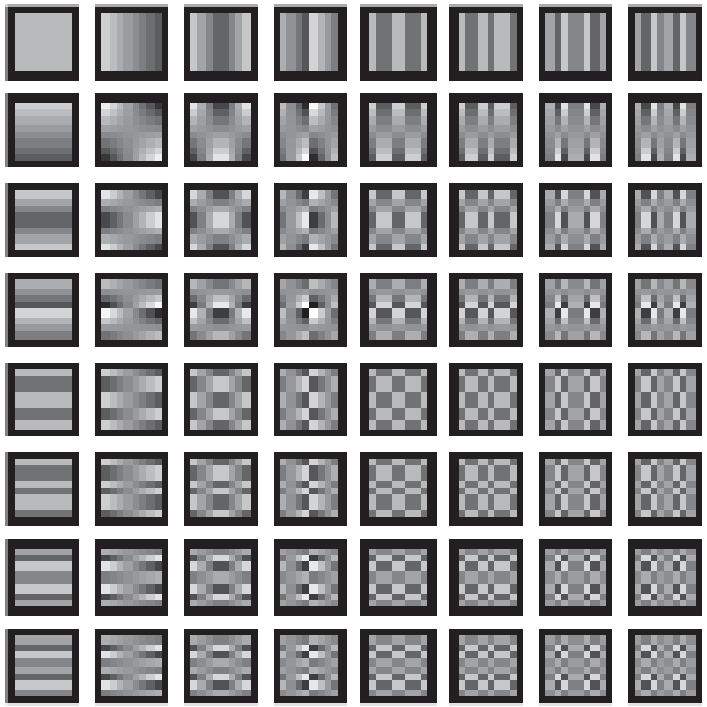
Basis images of matrix-based 2D transforms

Walsh-Hadamard transform (WHT) basis images



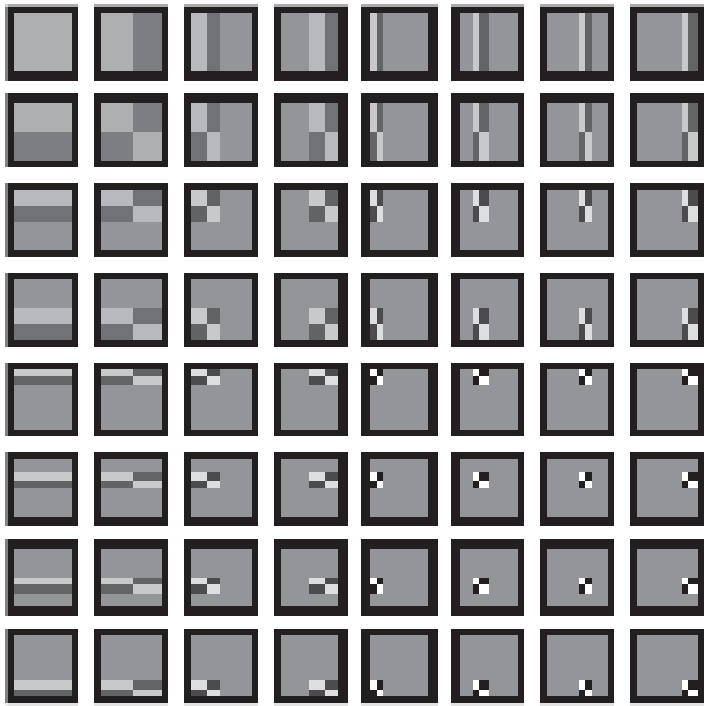
Basis images of matrix-based 2D transforms

Slant transform (SLT) basis images



Basis images of matrix-based 2D transforms

Haar transform (HAAR) basis images



Next Lecture

- Wavelet transforms
- Reading
  - Chapter 6: Wavelet and Other Image Transforms
    - Sections 6.10