LECTURE 21

LECTURE OUTLINE

- We enter a series of lectures on advanced topics
  - Gradient projection
  - Variants of gradient projection
  - Variants of proximal and combinations
  - Incremental subgradient and proximal methods
  - Coordinate descent methods
  - Interior point methods, etc
- Today's lecture on gradient projection
- Application to differentiable problems
- Iteration complexity issues

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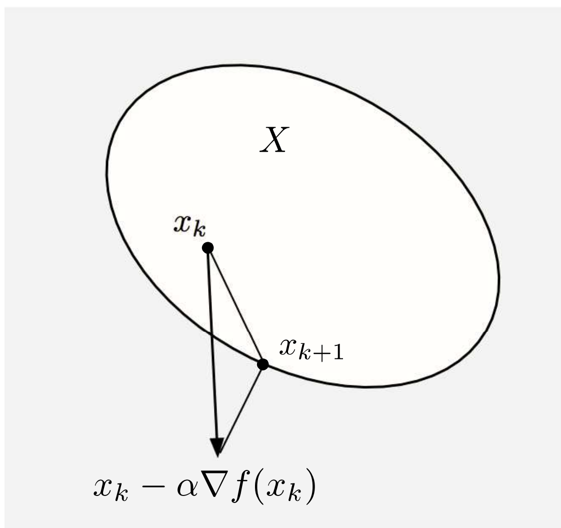
- Reference: The on-line chapter of the textbook

GRADIENT PROJECTION METHOD

- Let f be continuously differentiable, and X be closed convex.

- Gradient projection method:

$$x_{k+1} = P_X\big(x_k - \alpha_k \nabla f(x_k)\big)$$



- A specialization of subgradient method, but cost function descent comes into play

- $x_{k+1} - x_k$ is a feasible descent direction (by the projection theorem)

- $f(x_{k+1}) < f(x_k)$ if α_k: sufficiently small (unless x_k is optimal)

- α_k may be constant or chosen by cost descent-based stepsize rules

CONNECTION TO THE PROXIMAL ALGORITHM

- Linear approximation of f based on $\nabla f(x)$:

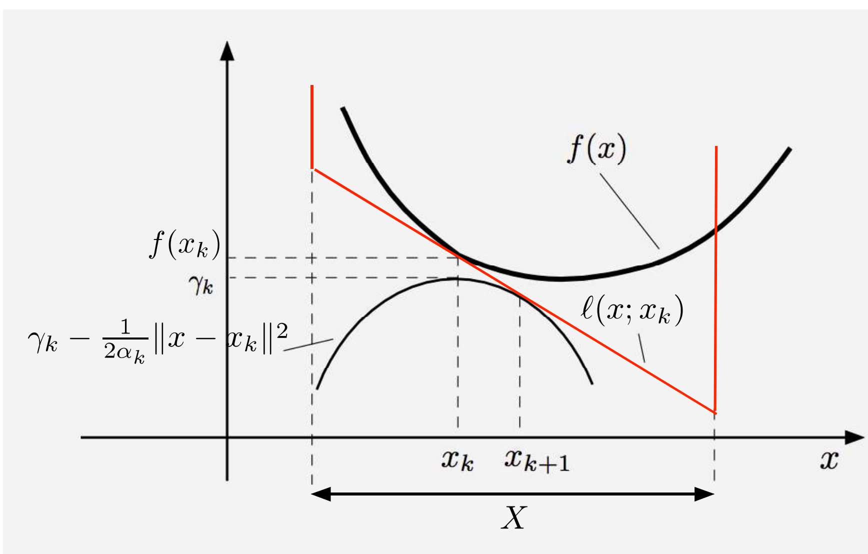
$$\ell(y;x) = f(x) + \nabla f(x)'(y-x), \qquad \forall\ x, y \in \Re^n$$

- For all $x \in X$ and $\alpha > 0$, we have

$$\frac{1}{2\alpha}\left\|y-\big(x-\alpha\nabla f(x)\big)\right\|^2 = \ell(y;x)+\frac{1}{2\alpha}\|y-x\|^2+\text{constant}$$

so

$$P_X\big(x-\alpha\nabla f(x)\big) \in \arg\min_{y\in X}\left\{\ell(y;x)+\frac{1}{2\alpha}\|y-x\|^2\right\}$$



- Three-term inequality holds: For all $y \in \Re^n$,

$$\|x_{k+1}-y\|^2 \leq \|x_k-y\|^2-2\alpha_k\big(\ell(x_{k+1};x_k)-\ell(y;x_k)\big)-\|x_k-x_{k+1}\|^2$$

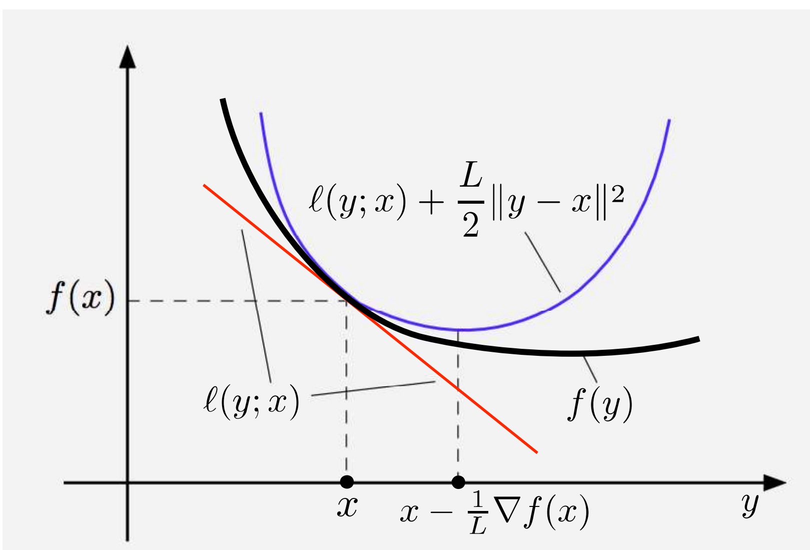
CONSTANT STEPSIZE - DESCENT LEMMA

- Consider constant α_k: $x_{k+1} = P_X\big(x_k - \alpha \nabla f(x_k)\big)$
- We need the gradient Lipschitz assumption

$$\big\|\nabla f(x) - \nabla f(y)\big\| \leq L\,\|x - y\|, \qquad \forall\ x, y \in X$$

- Descent Lemma: For all $x, y \in X$,

$$f(y) \leq \ell(y; x) + \frac{L}{2}\|y - x\|^2$$



- Proof idea: The Lipschitz constant L serves as an upper bound to the "curvature" of f along directions, so $\frac{L}{2}\|y-x\|^2$ is an upper bound to $f(y) - \ell(y; x)$.

CONSTANT STEPSIZE - CONVERGENCE RESULT

- Assume the gradient Lipschitz condition, and $\alpha \in (0, 2/L)$ (no convexity of f). Then $f(x_k) \downarrow f^*$ and every limit point of $\{x_k\}$ is optimal.

Proof: From the projection theorem, we have

$$\big(x_k - \alpha \nabla f(x_k) - x_{k+1}\big)'(x - x_{k+1}) \leq 0, \qquad \forall\, x \in X,$$

so by setting $x = x_k$,

$$\nabla f(x_k)'(x_{k+1} - x_k) \leq -\frac{1}{\alpha} \left\| x_{k+1} - x_k \right\|^2$$

- Using this relation and the descent lemma,

$$\begin{aligned} f(x_{k+1}) &\leq \ell(x_{k+1}; x_k) + \frac{L}{2} \|x_{k+1} - x_k\|^2 \\ &= f(x_k) + \nabla f(x_k)'(x_{k+1} - x_k) + \frac{L}{2} \|x_{k+1} - x_k\|^2 \\ &\leq f(x_k) - \left(\frac{1}{\alpha} - \frac{L}{2}\right) \|x_{k+1} - x_k\|^2 \end{aligned}$$

so $\alpha \in (0, 2/L)$ reduces the cost function value.

- If $\alpha \in \big(0, 2/L\big)$ and $\overline{x}$ is the limit of a subsequence $\{x_k\}_{\mathcal{K}}$, then $f(x_k) \downarrow f(\overline{x})$, so $\|x_{k+1} - x_k\| \to 0$. This implies $P_X\big(\overline{x} - \alpha \nabla f(\overline{x})\big) = \overline{x}$. **Q.E.D.**

STEPSIZE RULES

- Eventually constant stepsize. Deals with the case of an unknown Lipschitz constant L. Start with some $\alpha > 0$, and keep using α as long as

$$f(x_{k+1}) \leq \ell(x_{k+1}; x_k) + \frac{1}{2\alpha}\|x_{k+1} - x_k\|^2$$

is satisfied (this guarantees cost descent). When this condition is violated at some iteration, we reduce α by a certain factor, and repeat. (Satisfied once $\alpha \leq 1/L$, by the descent lemma.)

- A diminishing stepsize α_k, satisfying

$$\lim_{k\to\infty} \alpha_k = 0, \qquad \sum_{k=0}^{\infty} \alpha_k = \infty, \qquad \sum_{k=0}^{\infty} \alpha_k^2 < \infty.$$

Does not require Lipschitz condition or differentiability of f, only convexity of f.

- Stepsize reduction and line search rules - Armijo rules. These rules are based on cost function descent, and ensure that through some form of line search, we find α_k such that $f(x_{k+1}) < f(x_k)$, unless x_k is optimal. Do not require Lipschitz condition, only differentiability of f.

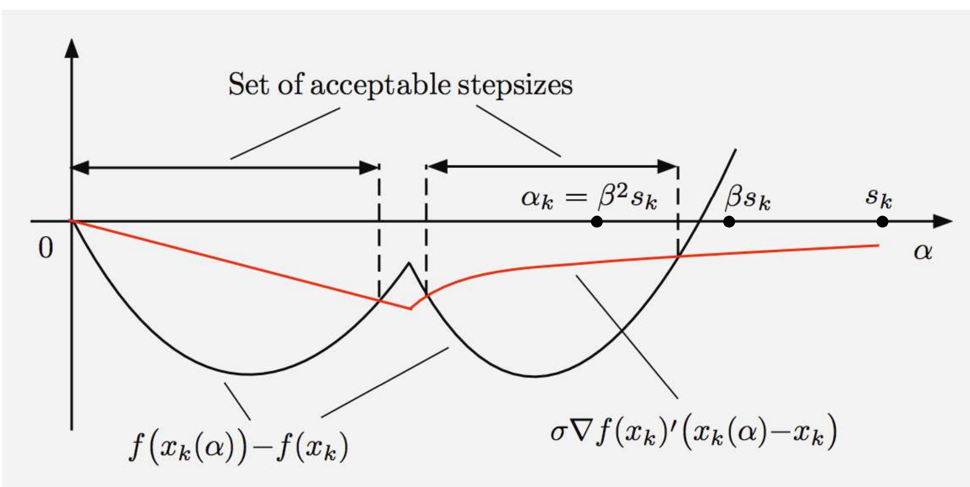
ARMIJO STEPSIZE RULES

- Search along the projection arc: $\alpha_k = \beta^{m_k} s$, where $s > 0$ and $\beta \in (0,1)$ are fixed scalars, and m_k is the first integer m such that

$$f(x_k) - f\big(x_k(\beta^m s)\big) \geq \sigma \nabla f(x_k)'\big(x_k - x_k(\beta^m s)\big),$$

with $\sigma \in (0,1)$ being some small constant, and

$$x_k(\alpha) = P_X\big(x_k - \alpha \nabla f(x_k)\big)$$



- Similar rule searches along the feasible direction

CONVERGENCE RATE - $\alpha_K \equiv 1/L$

- Assume f: convex, the Lipschitz condition, $X^* \neq \emptyset$, and the eventually constant stepsize rule. Denote $d(x_k) = \min_{x^* \in X^*} \|x_k - x^*\|$. Then

$$\lim_{k \to \infty} d(x_k) = 0, \qquad f(x_k) - f^* \leq \frac{Ld(x_0)^2}{2k}$$

Proof: Let $x^* \in X^*$ be such that $\|x_0 - x^*\| = d(x_0)$. Using the descent lemma and the three-term inequality,

$$\begin{aligned} f(x_{k+1}) &\leq \ell(x_{k+1}; x_k) + \frac{L}{2}\|x_{k+1} - x_k\|^2 \\ &\leq \ell(x^*; x_k) + \frac{L}{2}\|x^* - x_k\|^2 - \frac{L}{2}\|x^* - x_{k+1}\|^2 \\ &\leq f(x^*) + \frac{L}{2}\|x^* - x_k\|^2 - \frac{L}{2}\|x^* - x_{k+1}\|^2 \end{aligned}$$

Let $e_k = f(x_k) - f(x^*)$ and note that $e_k \downarrow$. Then

$$\frac{L}{2}\|x^* - x_{k+1}\|^2 \leq \frac{L}{2}\|x^* - x_k\|^2 - e_{k+1}$$

Use this relation with $k = k-1, k-2, \ldots$, and add

$$0 \leq \frac{L}{2}\|x^* - x_{k+1}\|^2 \leq \frac{L}{2}d(x_0)^2 - (k+1)e_{k+1}$$

GENERALIZATION - EVENTUALLY CONST. α_K

- Assume f: convex, the Lipschitz condition, $X^* \neq \emptyset$, and any stepsize rule such that

$$\alpha_k \downarrow \overline{\alpha},$$

for some $\overline{\alpha} > 0$, and for all k,

$$f(x_{k+1}) \leq \ell(x_{k+1}; x_k) + \frac{1}{2\alpha_k} \|x_{k+1} - x_k\|^2.$$

Denote $d(x_k) = \min_{x^* \in X^*} \|x_k - x^*\|$. Then

$$\lim_{k \to \infty} d(x_k) = 0, \qquad f(x_k) - f^* \leq \left(\frac{d(x_0)^2}{2\overline{\alpha}\, k} \right)$$

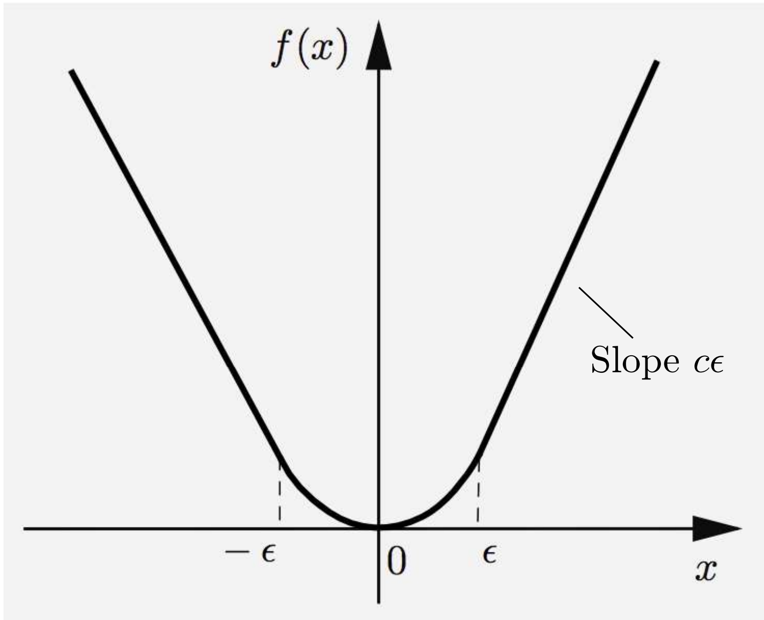
Proof: Show that

$$f(x_{k+1}) \leq f(x_k) - \frac{1}{2\alpha_k} \|x_{k+1} - x_k\|^2,$$

and generalize the preceding proof. **Q.E.D.**

- Applies to eventually constant stepsize rule.
- Error complexity $O(1/k)$, (k iterations produce $O(1/k)$ cost error), i.e., $\min_{\ell \leq k} f(x_\ell) \leq f^* + \frac{\text{const}}{k}$
- Iteration complexity $O(1/\epsilon)$, ($O(1/\epsilon)$ iterations produce ϵ cost error), i.e., $\min_{k \leq \frac{\text{const}}{\epsilon}} f(x_k) \leq f^* + \epsilon$

SHARPNESS OF COMPLEXITY ESTIMATE



- Unconstrained minimization of

$$f(x) = \begin{cases} \frac{c}{2}|x|^2 & \text{if } |x| \le \epsilon, \\ c\epsilon|x| - \frac{c\epsilon^2}{2} & \text{if } |x| > \epsilon \end{cases}$$

- With stepsize $\alpha = 1/L = 1/c$ and any $x_k > \epsilon$,

$$x_{k+1} = x_k - \frac{1}{L}\nabla f(x_k) = x_k - \frac{1}{c}c\,\epsilon = x_k - \epsilon$$

- The number of iterations to get within an ϵ-neighborhood of $x^* = 0$ is $|x_0|/\epsilon$.

- The number of iterations to get to within ϵ of $f^* = 0$ is proportional to $1/\epsilon$ for large x_0.