Circuit Theory of Time Domain Adjoint Sensitivity

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Abstract—It was originally stated that convolution operations were required to implement adjoint sensitivity in the time domain. In this article, we revisit time-domain adjoint sensitivity with a circuit theoretic approach and an efficient solution is clearly stated in terms of device level. Key is the linearization of the energy storage elements (e.g., capacitance and inductance) and nonlinear memoryless elements (e.g., MOS, BJT DC characteristics) at each time step. Due to the finite precision of computation, numerical errors that accumulate across timesteps can arise in nonlinear elements. A methodology to suppress that error is introduced. Numerical results demonstrate that the proposed method achieves accuracy while significantly reducing computational runtime.

Index Terms—Adjoint circuit, adjoint methods, backward Euler (BE) integration approximation, perturbation, sensitivity analysis, Tellegen’s theorem, time domain, transient.

I. INTRODUCTION

Sensitivity analysis is crucial in both integrated circuit (IC) synthesis [1], [2], [3], [4] and verification [5], [6], [7], [8], [9] tasks. Generally, the goal of sensitivity analysis is to calculate the partial derivatives of circuit performances (e.g., node voltages, branch currents, and circuit specifications) with respect to circuit design parameters (e.g., transistor width/length) or process parameters (e.g., oxide thickness—\(T_{ox}\) and threshold voltage—\(V_{th}\)). These obtained derivatives can be used in a variety of tasks, including gradient-based circuit design and optimization [1], [2], [3], [4], noise analysis [5], fault detection [6], [7], or yield estimation [8], [9].

Many sensitivity analysis methods have been proposed in the literature. They can be broadly classified into two categories: 1) frequency-domain sensitivity analysis [1] and 2) time-domain sensitivity analysis [2], [12], [13], [14], [15]. In the late 1960s, adjoint methods [1], [2] were proposed to solve both cases. The essence of adjoint methods is to build an adjoint circuit, which possesses not only the same topology as the given circuit, but also appropriately designated circuit elements that relate systematically to their original counterparts. Then, all sensitivities can be expressed as the products of currents/voltages in the original and adjoint circuits. In the frequency domain, adjoint methods are succinct and elegant, having every linear(ized) circuit element and its corresponding adjoint representation tabulated very early on in the investigation of frequency domain adjoint methods [1]. In the pioneer work on this subject [5], such adjoint circuits have been employed for design optimization, and the computation of IC noise.

Alternatively, when time-domain adjoint sensitivity was originally introduced in [2], Director and Rohrer interpreted the necessary operation as the convolution of the backward-in-time adjoint responses with desired stored forward-in-time counterparts. This called for the original circuit to be simulated forward in time, with all of its Jacobian matrices stored on a timestep-by-timestep basis. Those stored Jacobians formed the basis for a related backward-in-time adjoint circuit analysis [2]. Such an approach worked well as evidenced by the results in [10]. However, the implementation entailed an excessive amount of memory since it required the storage of the Jacobian matrix at every timepoint. As such, this method did not catch on at large in circuit analysis.

Subsequently, time domain sensitivity took a different route after it was observed that sensitivities could be obtained via mere matrix manipulation, without invoking the convolution operation [11]. That set the stage for various contributions and implementations of time-domain transient sensitivity [12], [13], [14], [15], referred to as direct methods by some [13]. These direct methods start from writing down the general differential algebraic equation (DAE) form of a circuit, and then extract the sensitivity DAE with respect to the element of interest. Next, the sensitivity DAE could be simulated forward in time accompanying the original DAE with an identical time step.

In the ensuing paper, we return to a circuit theoretic interpretation of time-domain transient sensitivity to clearly show its electrical properties and compatibility with earlier interpretations. We provide insight and show efficient implementation and computation of time-domain sensitivity for both linear and nonlinear circuits. Specifically, we first linearize the energy storage elements (e.g., capacitance and inductance) and nonlinear memoryless elements (e.g., MOS, BJT DC characteristics) in the original circuit at a time point, yielding a circuit containing only resistances, conductances, and independent sources. Then we take advantage of the DC adjoint sensitivity method, and only need to simulate the corresponding adjoint circuit forward in time to the next time point. Again, a linearization is performed at this new time point, and we repeat this process iteratively until reaching the end of a specified time interval. Our numerical results demonstrate that the proposed method achieves accuracy while reducing runtime with respect to convolutional or incremental approaches.

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Our primary contribution is not to solve the time-domain sensitivity problem as many methods are available, but rather provide a circuit theoretic exposition of adjoint sensitivity in the time domain. In the meanwhile, the error analysis over sensitivity estimation and further corrections are made to attain acceptable accuracy.

The remainder of this article is organized as follows. In Section II, we briefly review the background of adjoint sensitivity analysis. In Section III-A, we propose our forward-in-time sensitivity analysis method and employ a linear circuit as an analytical example. In Section III-B, we extend sensitivity to handle coupled elements and controlled sources. In Section III-C, the modeling of nonlinear elements in the time domain is presented and a numerical simulation result is given for the test example. This method is generalized in Section IV, and the key implementation detail for accurate nonlinear circuits is presented. Section V demonstrates the effects of simulation parameters, and presents analysis on the overall complexity of the implemented adjoint method. In Section VI, the general nature of the method is demonstrated by way of several circuit examples. Finally, we conclude with Section VII.

II. BACKGROUND

A. DC Adjoint Sensitivity Analysis

As a simplified version of time-domain sensitivity, here, we briefly review the adjoint sensitivity analysis for a circuit containing only independent DC sources, resistances, and conductances. Given a circuit, we short-connect all independent voltage sources (if any), open all independent current sources (if any), and finally add a unit current source \( i_{ad} = 1 \text{ A} \) between the node of interest and ground, yielding the adjoint circuit [2]. This procedure is illustrated in Fig. 1.

Tellegen’s theorem [16] tells us

\[
\begin{align*}
\sum_B \psi_B(\tau) \cdot i_B(t) &= 0 \quad (1a) \\
\sum_B \phi_B(\tau) \cdot v_B(t) &= 0 \quad (1b)
\end{align*}
\]

where the branch voltage and current are denoted by \( v_B(t) \) and \( i_B(t) \), respectively, while in the adjoint circuit, they are denoted by \( \psi_B(\tau) \) and \( \phi_B(\tau) \). Here, \( \tau \) and \( \tau_B \) are the time variables of the original and adjoint circuit, respectively, and need not be identical [2], [16]. As we are now considering the DC case, \( \{v_B, i_B, \psi_B, \phi_B\} \) are all constant in time. Hence, we omit their dependences on time for brevity. Note that by convention, the reference directions of \( v_B \) (or \( i_B \)) and the corresponding \( \psi_B \) (or \( \phi_B \)) should be identical.

Consider a perturbation \( \delta \) in the original circuit that causes a shift of branch currents by \( \delta i_B \) and a shift of branch voltages by \( \delta v_B \). Reapplying Tellegen’s theorem gives us

\[
\begin{align*}
\sum_B \psi_B(\tau) \cdot (i_B + \delta i_B) &= 0 \\
\sum_B \phi_B(\tau) \cdot (v_B + \delta v_B) &= 0.
\end{align*}
\]

Subtracting (2) from (1), we obtain

\[
\begin{align*}
\sum_B \psi_B \cdot \delta i_B &= 0 \quad (3a) \\
\sum_B \phi_B \cdot \delta v_B &= 0 \quad (3b)
\end{align*}
\]

and, thus, most importantly

\[
\sum_B (\psi_B \cdot \delta i_B - \phi_B \cdot \delta v_B) = 0. \quad (4)
\]

The above equation can be further simplified based on the categories of branches

\[
\sum_{src} (\phi_{src} \delta v_{src} - \psi_{src} \delta i_{src}) = \sum_R (\psi_R \delta i_R - \phi_R \delta v_R) + \sum_G (\psi_G \delta i_G - \phi_G \delta v_G) \quad (5)
\]

where we have explicitly written out all branches, separating conductive branches (denoted \( G \)) and resistive branches (denoted \( R \)), and moved terms related to the independent sources (denoted src) to the other side of the equation as they are delivering power to the rest of the circuit.

In the adjoint circuit, all devices keep the same characteristic of the corresponding BCR yields

\[
\psi_R = R \cdot \phi_R. \quad (6a)
\]

A first-order perturbative expansion of the differential component of the corresponding BCR yields

\[
\begin{align*}
v_R &= R \cdot i_R \quad (6b) \\
v_R + \delta v_R &= (R + \delta R) \cdot (i_R + \delta i_R) \quad (6c) \\
(6c) - (6b) \Rightarrow \delta v_R &= \delta R \cdot i_R + R \cdot \delta i_R \quad (6d)
\end{align*}
\]

where we have neglected the second order \( \delta R \cdot \delta i_R \) term. Substituting (6) into (5) simplifies the summation over \( R \) on the right-hand side. After performing the same deduction on a conductive branch \( G \), we can rewrite (5) as follows:

\[
\sum_{src} (\phi_{src} \delta v_{src} - \psi_{src} \delta i_{src}) = -i_R \delta \phi_R + \sum_G v_G \psi_G \delta G. \quad (7)
\]

For illustrative purpose, let us apply (7) to the example shown in Fig. 1. It is obvious that the right-hand side of (7) reduces to \(- i_1 \phi_1 \delta R_1 - i_2 \phi_2 \delta R_2 \), while the left-hand side is made up of two terms related to \( v_{in} \) and \( \phi_{ad} \), respectively

\[
\begin{align*}
(\phi_{in} \delta v_{in} - \psi_{in} \delta i_{in}) + (\phi_{ad} \delta v_{ad} - \psi_{ad} \delta i_{ad}) &= (\phi_{in} \cdot 0 - 0 \cdot \delta i_{in}) + (1 \cdot \delta v_{ad} - \psi_{ad} \cdot 0) \\
&= \delta v_{ad} = \delta v_2. \quad (8)
\end{align*}
\]

Here, we emphasize that \( \delta v_{in} \) equals 0 because the DC voltage source is assumed to be ideal (i.e., no perturbation allowed) by convention. In summary, (7) reduces to

\[
\delta v_2 = -i_1 \phi_1 \delta R_1 - i_2 \phi_2 \delta R_2. \quad (9)
\]
After solving both the original and adjoint circuits, we substitute the values of \( \{i_1, i_2, \phi_1, \phi_2\} \) back into (9), yielding
\[
\delta v_2 = \frac{R_2 \cdot v_{in}}{(R_1 + R_2)^2} \cdot \delta R_1 - \frac{R_1 \cdot v_{in}}{(R_1 + R_2)^2} \cdot \delta R_2
\]
where the coefficients ahead \( \delta R_1 \) and \( \delta R_2 \) are the desired partial derivatives \( \partial v_2 / \partial R_1 \) and \( \partial v_2 / \partial R_2 \), respectively. These results coincide with what we would obtain by direct differentiation of \( v_2(t) \).

The deduction in (8) is true in general. Namely, when there are multiple independent sources in the original circuit, \( \delta v_{in} \) and \( \psi_{in} \) will be 0 for the voltage sources, while \( \delta i_{in} \) and \( \phi_{in} \) will be 0 for the current sources. That is, unless we are interested in sensitivities with respect to perturbations in those independent source values, the left-hand side of (7) will always reduce to \( \delta v_{ad} \). Thus, for a general linear circuit containing only independent sources, linear resistances, and conductances, the DC adjoint sensitivity method states
\[
\delta v_{ad} = \sum_R -i_R \delta R \psi + \sum_G \psi G \delta G.
\]

Several things are worth mentioning here. First, for the calculation of current sensitivity, we need to insert a voltage source \( \psi_{ad} = 1 \text{ V} \) in series with the branch of interest [2], as shown in Fig. 2. Second, one nodal voltage of interest corresponds to one adjoint circuit. Finally, as this is a DC case, we need not care about the evolution of time. However, for a general circuit to be analyzed in the time domain, we need to explicitly write the dependence of \( \{v_B, i_B\} \) and \( \{\psi_B, \phi_B\} \) on time. Tellegen’s theorem does not pose any constraints on the time variable \( t \) of \( v_B(t), i_B(t) \) or the time variable \( \tau \) of \( \psi_B(\tau), \phi_B(\tau) \). Nevertheless, to build the adjoint circuit when capacitances exist in the original circuit, Director and Rohrer [2] required that \( t \) and \( \tau \) summed up to a fixed constant. Put simply, the original circuit was simulated forward in time and the adjoint circuit backward in time, and the convolution of their resulting responses yielded the desired sensitivity.

**B. Frequency-Domain Adjoint Sensitivity Analysis**

It is well known that convolution in the time domain is equivalent to multiplication in the frequency domain. Thus, for a linear circuit containing capacitance, sensitivity analysis is convenient and elegant to perform in the frequency domain. Director and Rohrer [1] showed that the general expression for voltage sensitivity is just an extension to (11)
\[
\delta V_{ad} = \sum_R -i_R \Phi_R \delta R + \sum_G V_G \psi G \delta G + s \sum_C V_C \psi C \delta C.
\]

As an example, applying (12) to the RC circuit shown in Fig. 3 tells us
\[
\delta V_C = -i_R \Phi_R \delta R + s V_C \psi C \delta C.
\]
Note that now the adjoint excitation \( \psi_{ad}(s) = 1 \) corresponds to a unit impulse in the time domain. Solving the original and adjoint circuits gives
\[
I_R(s) = \frac{C}{1 + sRC}
\]
and
\[
V_C(s) = \frac{1}{1 + sRC}\psi_C(s) = \frac{R}{1 + sRC}.
\]
Substituting (14) and (15) to (13) gives
\[
\delta V_C = -\frac{C}{(1 + sRC)^2} \delta R - \frac{R}{(1 + sRC)^2} \delta C
\]
which coincides with direct differentiation of \( V_C(s) \) given in (14b). It is straightforward to analyze sensitivities of such an RC circuit in the frequency domain, while the same analysis is complicated to do so in the time domain since convolution of the calculated circuit responses is required. In the next section, we will show from a circuit theoretic perspective, an easy and intuitive time-domain sensitivity mechanism, which does not require convolution.

**III. FORWARD-IN-TIME ADJOINT ANALYSIS**

In this section, two circuit examples are used to demonstrate the key idea, and generalization follows in the next section. From now on, we will explicitly write the time dependence of equations for clarity.

**A. Linear Circuit Illustration—RC Circuit**

We begin the study of linear circuits with the notion of a circuit element’s companion model, demonstrated by the backward Euler (BE) integration approximation for a linear capacitance. To begin with, basic circuit theory states
\[
i_C(t) = C \cdot \frac{d}{dt} v_C(t).
\]
When a short time step \( \Delta t \) is involved, we can discretize the above equation in terms of the BE integration approximation [2]
\[
i_C(t + \Delta t) \approx C \cdot \frac{v_C(t + \Delta t) - v_C(t)}{\Delta t}
\]
Fig. 4. Illustration of the BE companion model for a capacitance.

Fig. 5. Illustration of (a) simplified circuit in terms of its BE companion model and (b) its adjoint. Here, the value of $R_C$ equals $\Delta t/C$.

and, thus

$$v_C(t + \Delta t) \approx \frac{\Delta t}{C} \cdot i(t + \Delta t) + v_C(t). \quad (19)$$

Namely, (19) implies a simplified Thevenin Equivalent model at time $t + \Delta t$ for a capacitance as shown in Fig. 4, allowing the bypassing of the need for derivative information when numerically solving a circuit. We emphasize that this BE simplified model is only valid for sufficiently small $\Delta t$. We could as easily use trapezoidal or any other implicit integration approximation to build such a model, but we elect to use the BE for ease of exposition.

Consider the RC circuit shown in Fig. 3(a) to demonstrate our proposed forward-in-time adjoint sensitivity approach at any time point $t$ within the interval $[0, T]$, we can simplify the original circuit by using the BE companion model. As shown in Fig. 5, it is easy to obtain the adjoint of the simplified circuit.

By analogy to (1)–(7), if we choose the same time variable for the simplified circuit and its adjoint (i.e., $t = \tau$) and assume the perturbations on circuit elements are independent of time, we can write

$$\sum_{\text{src}} [\phi_{\text{src}}(t + \Delta t) \delta v_{\text{src}}(t + \Delta t) - \psi_{\text{src}}(t + \Delta t) \delta i_{\text{src}}(t + \Delta t)] = -i_R(t + \Delta t) \phi_R(t + \Delta t) \delta R$$

$$- i_C(t + \Delta t) \phi_C(t + \Delta t) \delta R_C. \quad (20a)$$

Strictly, the subscripts of $i_C(t\Delta t)$ and $\phi_C(t\Delta t)$ should both be representing the current flowing through $R_C$ in the original and adjoint circuits, respectively, at time $t\Delta t$. Here, we have slightly abused the notation for concision. In the case of Fig. 5, there are one independent voltage source $v_{\text{in}}$, one BE companion voltage source $v_C(t)$ and one adjoint excitation source $\phi_{\text{ad}}$. For the terms related to $v_{\text{in}}$ and $\phi_{\text{ad}}$, (8) remains applicable. However, we need to be careful with the terms related to $v_C(t)$. The left-hand side of (20a) can be simplified as follows:

$$\delta v_{\text{ad}}(t + \Delta t) + \left( \phi_{\text{ad}} \delta v_{\text{ad}} - \psi_{\text{ad}} \delta i_{\text{ad}} \right)$$

related to $v_{\text{in}}$ and $\phi_{\text{ad}}$

$$\delta v_{\text{ad}}(t + \Delta t) + \left( \phi_C \delta v_C - 0 \cdot \delta i_C \right)$$

related to $v_C(t)$

$$= \delta v_{\text{ad}}(t + \Delta t) + \phi_C(t + \Delta t) \delta v_C(t) \quad (20b)$$

where $v_C$ and $i_C$ represent the companion voltage source model’s voltage and current. In the last step, we have taken advantage of the fact that the currents flowing through $R_C$ and $v_C(t)$ are identical in both the original and adjoint circuits. Thus, (20a) can be simplified as follows:

$$\delta v_C(t + \Delta t) + \phi_C(t + \Delta t) \delta v_C(t)$$

$$= -i_R(t + \Delta t) \phi_R(t + \Delta t) \delta R$$

$$- i_C(t + \Delta t) \phi_C(t + \Delta t) \delta R_C. \quad (20c)$$

Solving both the original and adjoint circuits yields

$$i_R(t + \Delta t) = i_C(t + \Delta t) = \frac{v_{\text{in}}(t + \Delta t) - v_C(t)}{R + \Delta t/C} \quad (21)$$

and

$$\phi_R(t + \Delta t) = \frac{\Delta t/C}{R + \Delta t/C}$$

$$\phi_C(t + \Delta t) = -\frac{R}{R + \Delta t/C}. \quad (22)$$

Substituting (21) and (22) back into (20c) and using the chain rule

$$\delta R_C = \frac{dR_C}{dC} \cdot \delta C = \frac{d}{dC} \left( \frac{\Delta t}{C} \right) \cdot \delta C = -\frac{\Delta t}{C^2} \cdot \delta C \quad (23)$$

we can obtain

$$\frac{1}{\Delta t} \left[ \delta v_C(t + \Delta t) - \frac{RC}{RC + \Delta t} \delta v_C(t) \right]$$

$$= -\frac{RC}{(RC + \Delta t)^2} \cdot [v_{\text{in}}(t + \Delta t) - v_C(t)] \cdot \left( \frac{\Delta R}{RC} + \frac{\Delta C}{C} \right). \quad (24)$$

Taking the limit $\Delta t \to 0$, we reformulate the above equation into the DAE

$$\delta v_C(t) + \frac{\delta v_C(t)}{RC} = -\frac{v_{\text{in}}(t) - v_C(t)}{RC} \cdot \left( \frac{\Delta R}{RC} + \frac{\Delta C}{C} \right). \quad (25)$$

Recall that we assume that $t$ is an arbitrary time point in $[0, T]$, and, thus, (25) holds over the entire interval $[0, T]$. The unit step response of the RC circuit shown in Fig. 3 is given by

$$v_C(t) = 1 - e^{-t/RC}. \quad (26)$$

We substitute (26), $v_{\text{in}}(t) = 1$, and the initial condition $\delta v_C(t) = 0$ into (25), and solving the DAE, yields

$$\delta v_C(t) = -\frac{t \cdot e^{-t/RC}}{RC} \cdot \left( \frac{\Delta R}{RC} + \frac{\Delta C}{C} \right) \quad (27)$$

which coincides with what we would obtain by direct differentiation of (26) with respect to $R$ and $C$. Moreover, the frequency-domain derivation for this RC example given in (16) is identical to (27) upon an inverse Laplace transform. These two approaches to the derivation both clearly indicate compatibility with convolution [2]. We emphasize that Branin [11], Cao et al. [12], Daldoss et al. [13], and Hoevear et al. [14] obtained the DAE shown in (25) from pure Mathematics, and, here, we have filled in the missing circuit theory.

It is beneficial to rethink (20c) for a general circuit with multiple resistive, conductive, capacitive, and inductive branches. Comparing (7) and (20), it is obvious that a conductance $G$ will add an additional term $\nu_G(t + \Delta t) \psi_G(t + \Delta t) \delta G$
on the right-hand side of (20c). An inductance \( L \) is the dual of a capacitance \( C \) in terms of the BE companion model. By analogy to (17)–(19), we have

\[
i_L(t + \Delta t) = \frac{\Delta t}{L} \cdot v_L(t + \Delta t) + i_L(t)
\]

and this BE Norton Equivalent companion model is shown in Fig. 6.

Similarly to (20), the BE companion model for an inductance will introduce an additional term on the left-hand side of (20c) due to \( i_L(t) \)

\[
\phi_i \delta v_{L} - \psi_i \delta i_L = 0 \cdot \delta v_{L} - \psi_i \delta i_L
\]

and an additional term \( v_L(t + \Delta t) \psi_i L(t + \Delta t) \delta G_L \) on the right-hand side of (20c) due to \( G_L \). Combining the impacts of resistances, conductances, capacitances, and inductances, we obtain the generalized form of (20c) for the linear cases as

\[
\delta v_n(t + \Delta t) = - \sum_R i_R(t + \Delta t) \phi_R(t + \Delta t) \delta R + \sum_G v_G(t + \Delta t) \psi_G(t + \Delta t) \delta G
\]

\[
- \sum_C [i_C(t + \Delta t) \phi_C(t + \Delta t) \delta R_C + \psi_C(t + \Delta t) \delta v_C(t)] + \sum_L [v_L(t + \Delta t) \psi_L(t + \Delta t) \delta G_L + \psi_L(t + \Delta t) \delta i_L(t)]
\]

where \( v_n \) represents an arbitrary branch voltage of interest in the original circuit, and we have defined

\[
\delta R_C = \frac{dR_C}{dC} \cdot \delta C = \frac{d}{dC} \left( \frac{\Delta t}{C} \right) \cdot \delta C = -\frac{\Delta t}{C^2} \cdot \delta C. \quad (31a)
\]

\[
\delta G_L = \frac{dG_L}{dl} \cdot \delta L = \frac{d}{dL} \left( \frac{\Delta t}{L} \right) \cdot \delta L = -\frac{\Delta t}{L^2} \cdot \delta L. \quad (31b)
\]

Note that all \( \{\psi, \phi\} \) in (30) should be evaluated in the adjoint circuit corresponding to the branch voltage of interest. Although (30) is an expression for voltage sensitivity, it also works for current sensitivity if we replace the symbol \( \delta v_n \) with \( -\delta i_n \) on its left-hand side, with the adjoint circuit built by inserting a unit voltage source as shown in Fig. 2.

As sensitivity approximation in (30) is applied to get the time domain transient sensitivity response. It is obvious that using a small time step \( \Delta t \) and small parameter deviation \( \delta p \) provides a good agreement between analytic and calculated sensitivities.

B. General Linear Devices—Coupled Elements

Coupled elements such as controlled sources are also straightforward to understand through this formulation. Such elements, with primary side one and secondary side two, can be described by any two of the following relations as outlined in [1], [2]:

\[
v_1(t) = f_1(x_1, x_2, p, t)
\]

\[
i_1(t) = g_1(x_1, x_2, p, t)
\]

\[
v_2(t) = f_2(x_1, x_2, p, t)
\]

\[
i_2(t) = g_2(x_1, x_2, p, t)
\]

where \( x_n \) are currents/voltages of each of the primary/secondary \((1/2)\) terminals of the element, \( p = [p_1, p_2, \ldots, p_d]^T \) is the vector of design parameters, and \( t \) is time. As an example, we consider a linear time invariant voltage controlled current source (VCCS) shown in Fig. 7(a). Such an element is described by the following expressions:

\[
i_1(t) = 0
\]

\[
i_2(t) = g_m v_1(t)
\]

where \( v_1 \) is the control voltage across the zero-valued current source input, \( g_m \) is the transconductance, and \( i_2 \) is the dependent source current. This element has associated adjoint relations of

\[
\phi_1(t) = g_m \psi_2(t)
\]

\[
\phi_2(t) = 0
\]

which describes a backward VCCS as outlined in [1] and [2] and illustrated in Fig. 7(b). Such relations for the other controlled source types are derived and tabulated in [1] and [2]. Using the terms in (33), the sensitivity expression for a VCCS is found to be

\[
\psi_2(t) \delta i_2(t) - \phi_2(t) \delta v_2(t) + \psi_1(t) \delta i_1(t) - \phi_1(t) \delta v_1(t) = \psi_2(t) \delta i_2(t) - \phi_1(t) \delta v_1(t)
\]

\[
\psi_2(t) \delta v_1(t) \delta g_m + g_m \delta v_1(t) - \psi_2(t) g_m \delta v_1(t) = \psi_2(t) \delta v_1(t) \delta g_m.
\]

This provides an additional term to (30) as shown in (35)

\[
\delta v_n(t + \Delta t) = - \sum_R i_R(t + \Delta t) \phi_R(t + \Delta t) \delta R
\]
Fig. 8. Hypothetical characteristic curve of nonlinear elements.

\[
+ \sum_i v_i(t + \Delta t) \psi_i(t + \Delta t) \delta G
- \sum_i [c_i(t + \Delta t) \phi_i(t + \Delta t) v_i(t + \Delta t) \delta v_i(t)]
+ \sum_i [v_i(t + \Delta t) \psi_i(t + \Delta t) \delta G_i + \psi_i(t + \Delta t) \delta I_i(t)]
+ \sum_i [\psi_1(t + \Delta t) \left( \frac{\partial v_1(t + \Delta t)}{\partial v} \right)^T - \phi_1(t + \Delta t) \left( \frac{\partial v_1(t + \Delta t)}{\partial v} \right)^T] \delta \mathbf{p}
+ \sum_i [\psi_2(t + \Delta t) \left( \frac{\partial v_2(t + \Delta t)}{\partial v} \right)^T - \phi_2(t + \Delta t) \left( \frac{\partial v_2(t + \Delta t)}{\partial v} \right)^T] \delta \mathbf{p}
\]
\[
(35)
\]

where \(CS\) represents any one of the typical coupling elements (VCCS, VCVS, CCVS, and CCCS) as described in [1] and [2], and \(\delta \mathbf{p}\) is defined as the column vector of parameter deviations.

C. Nonlinear Circuit Case Study—RCD Circuit

In modern ICs, devices, such as BJTs and MOSFETs are indispensable. However, these circuit elements introduce nonlinearity, further complicating analysis. Consider the \(i-v\) characteristic of a nonlinear circuit element
\[
i(t) = f(v(t), \mathbf{p})
\]
where \(\mathbf{p} = [p_1, p_2, \ldots, p_d]^T\) represents the device parameters. For instance, an ideal diode has \(\mathbf{p} = [p_1, p_2]^T = [I_0, V_T]^T\) and
\[
i(t) = I_0 \cdot \left[ \exp(v(t)/V_T) - 1 \right].
\]
At time point \(t + \Delta t\), the linearization of the device corresponds to the tangent to the \(i-v\) characteristic curve as shown in Fig. 8, and satisfies
\[
G_{eq} \cdot v = i - i_{eq}
\]
where
\[
G_{eq} = \frac{\partial i}{\partial v}.
\]

Building upon (1)–(7), we can infer that the existence of a nonlinear element will add extra terms on the right-hand side of (30). As shown in Fig. 9, (38) implies a Norton equivalent model for the nonlinear element linearized at \(t + \Delta t\). The sensitivity contribution may be formulated as follows:
\[
\delta v = \psi \left[ \delta \phi G_{eq} + \delta i_{eq} \right]
\]
where
\[
\delta G_{eq} = \frac{\partial}{\partial \mathbf{p}} \frac{\partial i}{\partial v} \delta \mathbf{p}
\]
\[
\delta i_{eq} = \frac{\partial i}{\partial \mathbf{p}} \delta \mathbf{p} - \frac{\partial}{\partial v} \frac{\partial i}{\partial v} \delta v.
\]
Substituting these terms into (39) presents a tempting opportunity for cancelation of the second-order partial terms, rendering the sensitivity for nonlinear elements of the same form as those of simple linear conductances. However, such an approximation only remains valid so long as the linearization holds. As a nonlinear device traverses the \(i-v\) curve, the linearized terms \(G_{eq}\) and \(i_{eq}\) will vary. This variation does not occur in linear devices, and means that the only scenario in which such cancelation can be applied is when \(\Delta t\) is infinitesimally small.

The use of an infinitesimal \(\Delta t\) is obviously not practical to implement, but the use of a finite \(\Delta t\) introduces a local truncation-derivative error over the integration interval. If one defines the truncation error (TE) of a device as the difference between the actual point and estimated point shown in Fig. 8, then such error becomes negligible as a proper numerical integration process is given to minimize such error. But for time domain sensitivity analysis, the difference of the actual point and estimated point’s derivative with respect to the device parameters is critical and should not be ignored and will be named as the truncation-derivative error.

In the nonlinear device’s companion model, TE is treated as an additional branch current in the linearized element as shown in Fig. 10
\[
i_{TE}(t + \Delta t) = i_{actual}(t + \Delta t) - i_{estimated}(t + \Delta t)
= \frac{di}{dv} \cdot [v(t + \Delta t) - v(t)].
\]
As with many transient simulation methods, the Newton–Raphson method is applied to minimize this TE [17]. For any other \(\Delta t\), the truncation-derivative terms that represent higher order terms in the nonlinear sensitivity contribution, will not be canceled. Oftentimes, the scale of the TE is set by a tolerance threshold assigned by a transient simulation user (abstol, reltol, and vntol in many SPICE environments [18]). While this error is generally acceptable in transient simulation, this additional current must be factored into the sensitivity analysis.
as it can easily compound to be on the scale of the sensitivity itself via the contributions of the energy storage elements.

Specifically, the additional term introduced by the nonlinear element is

\[
\begin{align*}
\psi_G_{eq} \delta i_{eq} - \phi_G_{eq} \delta v_{eq} + (\psi_{eq} \delta v_{eq} - \phi_{eq} \delta v_{eq}) \\
+ \left( \psi_{eq} \delta v_{TE} - \phi_{eq} \delta v_{TE} \right) \\
= \psi_{eq} \delta i_P - \phi_{eq} \delta v_P + \psi_G_{eq} \delta i_{eq} \\
= \psi_{eq} \left[ \frac{\partial i}{\partial p} \cdot \delta v_P + \left( \frac{\partial i}{\partial p} \right)^T \delta P \right] - \phi_{eq} \delta v_P + \psi_{eq} \delta i_{eq}
\end{align*}
\]

where all the terms above are evaluated at \( t + \Delta t \), and we have defined the following column vectors:

\[
\frac{\partial i}{\partial p} = \begin{bmatrix}
\frac{\partial i}{\partial p_1} \\
\frac{\partial i}{\partial p_2} \\
\vdots \\
\frac{\partial i}{\partial p_4}
\end{bmatrix}^T.
\]

As assumed before, the parameter variation \( \delta p \) is taken to be time independent. This term updates the general sensitivity expression (35) to

\[
\delta v_n(t + \Delta t) = -\sum_k i_R(t + \Delta t) \delta R
\]

\[
+ \sum \psi_G(t + \Delta t) \delta G
\]

\[
- \psi_{eq} \left[ \frac{\partial i}{\partial p} \cdot \delta v_P + \left( \frac{\partial i}{\partial p} \right)^T \delta P \right] - \phi_{eq} \delta v_P + \psi_{eq} \delta i_{eq}
\]

where, \( \delta i_{eq} \) can be represented as

\[
\delta i_{eq}(t + \Delta t) = \frac{di(v(t + \Delta t))}{dp} \cdot \delta p - \frac{di(v(t), p)}{dp} \cdot \delta p
\]

\[
- \frac{di}{dv} \left[ \frac{dv(t + \Delta t)}{dp} - \frac{dv(t)}{dp} \right] \cdot \delta p
\]

\[
- \frac{di}{dvdp} \cdot [v(t + \Delta t) - v(t)] \cdot \delta p.
\]

Based on (41) and (45), the correction term \( \delta i_{eq} \) will be zero if an analytical integration process is given. Thus, to demonstrate an example in terms of theory with analytical sensitivity analysis, we analyze an RC-diode (RCD) circuit shown in Fig. 11 with the developed theory. The linearized model of the RCD circuit and the adjoint circuit are shown in Fig. 12. To begin with, we apply (44) to this circuit, yielding

\[
\delta v(t + \Delta t) = -i_R(t + \Delta t) \phi_R(t + \Delta t) \delta R
\]

\[
- [i_C(t + \Delta t) \phi_C(t + \Delta t) \delta R_C + \phi_C(t + \Delta t) \delta v(t)]
\]

\[
\frac{d}{dt} v(t + \Delta t) = \frac{1}{CR} \left[ v(t + \Delta t) - v(t) \right]
\]

\[
= \frac{v(t) - v_0}{CR} + \frac{v(t)}{CR} \frac{d}{dt} C = \frac{1}{C} \left[ \frac{\partial v}{\partial i_0} \cdot \delta i_0 + \frac{\partial v}{\partial V_T} \cdot \delta V_T \right].
\]

The above equation and the circuit equation

\[
C \frac{d}{dt} v(t) + i_0 \left[ \exp(v(t)/V_T) - 1 \right] + \frac{v(t)}{R} = \frac{v_{in}(t)}{R}
\]

form a system of ordinary differential equations (ODEs). Solving this system with the given the initial conditions \( v(0) = 0 \) and \( v_{in}(t) = 1 \) with a continuous-time ODE solver provides the sensitivities in good agreement with the
expected incremental results shown in Fig. 13. To calculate the incremental sensitivities, the transient response \( v_{\text{out}}(t, I_0, V_T) \) of the RCD circuit shown in Fig. 11 is simulated when all parameters are set to their nominal values. Next, one more transient run is invoked to obtain \( v_{\text{out}}(t, I_0, V_T + \delta V_T) \) if parameter \( V_T \) is perturbed.

Combining these two transient responses, the incremental sensitivity of the time-domain transient with respect to parameter \( V_T \) is

\[
\frac{\partial v_{\text{out}}(t)}{\partial V_T} = \frac{v_{\text{out}}(t, I_0, V_T + \delta V_T) - v_{\text{out}}(t, I_0, V_T)}{\delta V_T}.
\]  

(51)

If the sensitivity curve is differentiable, then the incremental sensitivity will converge no matter what perturbation is performed (e.g., \( +\delta V_T \) or \( -\delta V_T \)). This converged estimation is the benchmark gold standard sensitivity. From the above, it is clear that multiple transient runs are required for determining the incremental (gold standard) sensitivity with respect to each parameter. This limitation is one of the primary attractive features of the adjoint approach, which can be used simultaneously to obtain the sensitivities with respect to all parameters.

In the case of the RCD circuit, a continuous time ODE was found but this is rarely the case in general. To avoid the truncation-derivative error as shown in (45), the analytical form of adjoint circuit computation in the device level derived from (4) can be represented as follows:

\[
\psi \delta i - \phi \delta v = \int_{t}^{t+\Delta t} \left( \psi \frac{di}{dt} - \phi \frac{dv}{dt} \right) dt.
\]

(52)

In the integration process as shown in (52), one can assume the adjoint branch voltage \( \psi \) keeps the same value and leave the whole integration to the time derivative of \( v \) as

\[
\psi \cdot \int_{t}^{t+\Delta t} \frac{di}{dt} dt - \phi \cdot \int_{t}^{t+\Delta t} \frac{dv}{dt} dt
\]

(53)

or the time derivative of \( v \) remains a constant ramp between \( v(t) \) and \( v(t + \Delta t) \) but \( \psi \) needs to be correspondingly refined to a proper value as follows:

\[
\frac{di}{dt} \cdot \int_{t}^{t+\Delta t} \psi dt - \frac{dv}{dt} \cdot \int_{t}^{t+\Delta t} \phi dt.
\]

(54)

Since (53) is practically impossible, finding the refined adjoint value becomes the key to correction and a reasonable guess of adjoint branch voltage \( \psi \) is between \( \psi(t) \) and \( \psi(t + \Delta t) \), but its exact value depends on the selected integration method. In other words, the adjoint circuit needs to be re-evaluated at the end of each time intervals to get rid of the truncation-derivative error.

The mechanics of handling the \( \delta i_{\text{TE}} \) term are covered in Section IV and more general notation must be introduced in matrix form.

IV. IMPLEMENTATION IN GENERAL

A. Matrix Form Generalization

To render the aforementioned expressions practical, we reconsider them in matrix form for general circuits in this section. To begin with, we will consider the circuit instead of as a single device, but in the nodal analysis formulation. In transient analysis, under the BE approximation, we have

\[
f(v_n(t + \Delta t)) + \frac{Q(v_n(t + \Delta t)) - Q(v_n(t))}{\Delta t} = i_{\text{TE}}(t + \Delta t)
\]

(55)

where \( f \) represents the collection of all currents flowing through memoryless devices, \( Q \) stands for the charge/flux of all memory elements, and \( i_{\text{TE}} \) is the summation of every nonlinear device’s TE. Here, we have explicitly written out the dependency [e.g., \( f = f(v_n(t + \Delta t)) \)] for later simplicity. Additionally, we consider all possible output nodes’ sensitivity in terms of a single-column vector. Based on the information of \( f, Q \) and \( i_{\text{TE}} \) from (55), the column vector sensitivity can be generalized to

\[
\delta v_n(t + \Delta t) = \Psi \cdot \left( \frac{\partial Q}{\partial v_n}(t + \Delta t) \cdot \frac{\partial v_n}{\partial p}(t) \cdot \delta p \right)
\]

\[
- \Psi \cdot \frac{\partial f}{\partial p}(t + \Delta t) \cdot \delta p
\]

\[
- \Psi \cdot \frac{\partial Q}{\partial v_n}(t + \Delta t) \cdot \frac{\partial Q}{\partial v_n}(t) \cdot \delta p
\]

\[
- \Psi \cdot \delta i_{\text{TE}}(t + \Delta t)
\]

(56)

where \( \Psi \) is the adjoint state matrix.

When choosing \( v_x \) as the output node of interest, \( x = \{1, 2, \ldots, N\} \), the corresponding adjoint state vector \( \Psi_x \) can be calculated via solving

\[
\left( \frac{\partial f}{\partial v_n}(t + \Delta t) + \frac{\partial Q}{\partial v_n}(t + \Delta t) \right)^T \Psi_x = e_x
\]

(57)

where the term in brackets is the circuit Jacobian denoted \( J \) and \( e_x \) is a column vector with its \( x \)th element equal to one and all others equal to zero. Each such vector represents a unit impulse excitation at the various nodes of the adjoint.
circuit. If we have $N$ (output) nodes in the original circuit, $N$ unit impulse excitations are required, rendering the right-hand side to be an identity matrix. This implies that the overall adjoint state matrix shown in (56) is the combination of adjoint state vectors
\[ \Psi = [\Psi_1 \ \Psi_2 \ \cdots \ \Psi_N]^T. \tag{58} \]

The proposed time domain adjoint sensitivity analysis shares the same Jacobian matrix with the transient analysis. As the same Jacobian matrix has already been LU factorized to advance the transient integration, runtime can be saved, as the transposed LU factors are all that are needed to calculate the adjoint quantities \cite{19}. If the Jacobian were a full matrix, the complexity of its LU factorization would be $O(N^3)$ where $N$ is the number of nodes. The complexity of the associated forward and backward substitution (FBS) would be $O(N^2)$. But there are $N$ of those, so, theoretically, the overall transient simulation with sensitivity should be roughly twice as computationally costly as without. For real circuits, the Jacobian is sparse, and the complexity of its LU-factorization is more like $O(N^{1.5})$.

\section*{B. Efficiently Handling Truncation-Derivative Error}

Based on (56), to calculate sensitivity with respect to a certain parameter, both sides of (56) are divided by $\Delta t$ for all parameters. The left-hand side approximates $\delta v_n(t)/\Delta t$ in the limit as $\Delta t$ tends to zero, while the truncation-derivative error term on the right-hand side correspondingly becomes $\delta i_{TE}(t+\Delta t)/\Delta t$. While $\delta i_{TE}$ should be reasonably small with an accurate transient response, the parameter deviation $\delta p$ is also chosen to be sufficiently small to get rid of the high order error caused by large parameter deviation. This makes the contribution of $\delta i_{TE}(t+\Delta t)/\Delta t$ a non-negligible term that will propagate through the entire simulation via the energy storage elements. The nominal $TE$ $i_{TE}(v_n(t+\Delta t), p)$ are available as
\[ i_{TE}(v_n(t+\Delta t), p) = f(v_n(t+\Delta t), p) + \frac{q(v_n(t+\Delta t), p) - q(v_n(t), p)}{\Delta t}. \tag{59a} \]

Based on (45), $\delta i_{TE}(t+\Delta t)/\Delta t$ can be represented in terms of matrix form as
\[ \frac{\partial}{\partial p} i_{TE}(t+\Delta t) = \frac{\partial f}{\partial p}(t+\Delta t) - \frac{\partial f}{\partial p}(t) - J \left[ \frac{\partial v_n}{\partial p}(t+\Delta t) - \frac{\partial v_n}{\partial p}(t) \right] - \frac{\partial}{\partial p} [v(t+\Delta t) - v(t)]. \tag{59b} \]

The calculation of $\delta i_{TE}(t+\Delta t)/\Delta t$ requires the sensitivity information at time point $(t+\Delta t)$ which is the sensitivity of the next state. To address that, an initial guess for the next state sensitivity is necessary and there are two ways to estimate it. If the sensitivity curve is assumed to be differentiable, then $\delta v_n(t+\Delta t)/\Delta t$ can be extrapolated using a backwards difference as
\[ \frac{\partial v_n}{\partial p}(t+\Delta t) \approx 2 \frac{\partial v_n}{\partial p}(t) - \frac{\partial v_n}{\partial p}(t-\Delta t). \tag{60} \]

With the extrapolated sensitivity being carried to (59b), a refined sensitivity is available through (56). Another way to attain the initial guess of $\delta v_n(t+\Delta t)/\Delta t$ is to calculate it from (56) with the truncation-derivative error being ignored. The sensitivity estimation is close to the gold standard result only if the guessed sensitivity result being carried to (59b) and the refined result from (56) are close.

The abovementioned truncation-derivative error cancelation mechanism is derived from the (45) which assumes the adjoint state matrix remains the same from $t$ to $(t+\Delta t)$. Another error cancelation method uses a refined adjoint state matrix to replace the original one at (56) and ignores the $\delta v_n(t+\Delta t)/\Delta t$ term. The rough guess of the refined adjoint state matrix would be between $\Psi(t+\Delta t)$ and $\Psi(t)$, but in reality, the refined adjoint matrix requires to be evaluated based on jacobian matrix $J(t+\Delta t, p + \delta p)$ to perform truncation-derivative error cancelation properly
\[ J^{-1}(t + \Delta t, p + \delta p) \]
\[ = \left[ \frac{\partial f}{\partial v_n}(t + \Delta t, p + \delta p) + \frac{\partial q}{\partial v_n}(t + \Delta t, p + \delta p) \right]^{-1} \tag{61} \]

Based on the information from the nominal transient response, $J^{-1}(t, p), J^{-1}(t + \Delta t, p),$ and $J(t, p)$ are known as shown in Section IV. An efficient way of calculating the cancelation matrix is using matrix inversion approximation as
\[ J^{-1}(t + \Delta t, p + \delta p) \approx J^{-1}(t + \Delta t, p) - J^{-1}(t + \Delta t, p) \cdot [J(t + \Delta t, p + \delta p) - J(t + \Delta t, p)] \]
\[ \cdot J^{-1}(t + \Delta t, p). \tag{62} \]

To summarize the above, there are two types of error corrections in this sensitivity analysis. The first method eliminates the truncation-derivative error by assuming the adjoint state matrix remains the same from $t$ to $t + \Delta t$ and adding correction vector as shown in (59b). The second type of correction directly uses the refined adjoint state matrix, this refined adjoint state matrix coordinate with both forward-in-time and backward-in-time which assumes the excision ramp remains the same slope from $t$ to $t + \Delta t$.

Based on the above, the sensitivities of all outputs with respect to all parameters are available and the related algorithmic flow is shown in Algorithm 1. The proposed sensitivity estimation method works for general circuit examples and becomes the key contribution of this article, even those including nonlinear capacitance/inductance. It is of note that in Algorithm 1, step 6 can oftentimes be bypassed for many linear parameters, and typically converges in one iteration. Additionally, this method can be applied for any kind of integration approximation (Euler, Trapezoidal, Gear, etc.) and with adaptively changed $\Delta t$, as only the evaluation of $\delta i_{TE}(t+\Delta t)/\Delta t$ must be updated accordingly. The effectiveness of the truncation-derivative error iteration is discussed in the next section.

\section*{V. PERFORMANCE EVALUATION}

In this section, the evaluation of the sensitivity estimation method is performed analytically and numerically for linear and nonlinear examples. We start with the linear RC circuit excited by a unit step function (26). The calculated sensitivities for each of the parameters $\{R$ and $C\}$ are analytically
Algorithm 1 Time Domain Adjoint Sensitivity Method

(This algorithm focuses on the time interval from \( t \) to \( t + \Delta t \))

1. Define
   i. The parameter vector of interest
   ii. The threshold tolerances of sensitivity \( tol \)
2. Perform the typical BE-based transient simulation evaluation at time, noting that:
   i. The Jacobian matrix \( J(P) \) has been LU factorized as \( J(P) = LU \)
   ii. Branch currents and charges are known information
3. Calculate every adjoint state vector as described in (57)
4. The initial guess of the sensitivity can be obtained either by (60) or by (56) with truncation-derivative error being ignored. Carry the initial guess of sensitivity to (59b) and get \( \delta_{TE}(t+\Delta t)/\delta p \)
5. Carry the calculated \( \delta_{TE}(t+\Delta t)/\delta p \) to (56) and get the refined sensitivity with respect to the parameter of interest.
6. Evaluate the difference between refined sensitivity and the initial guess
   i. If the difference is less than \( tol \), sensitivity simulation moves to \( t + 2\Delta t \)
   ii. If the difference is larger than \( tol \), those refined sensitivities are fed into (59b) and \( \delta_{TE}(t+\Delta t)/\delta p \) is reevaluated.
7. End

Fig. 14. Gold standard sensitivity versus sensitivity estimated by (30) over a linear RC circuit with \( R = 1 \) and \( C = 1 \).

represented as follows:

\[
\frac{dv_C(t)}{dR} = -\frac{t}{R^2C}e^{-t/RC} \quad (63a)
\]

\[
\frac{dv_C(t)}{dC} = -\frac{t}{CR^2}e^{-t/RC}. \quad (63b)
\]

Such analytical sensitivity evaluation is our gold standard, and our method estimates sensitivities based on (30). Both the gold standard and estimated sensitivities are plotted in Fig. 14, and show good agreement. Fig. 15(a), demonstrates the increasing accuracy of the method as the time step \( \Delta t \) in (30) is reduced.

Next, consider the previously discussed RCD circuit as shown in Fig. 11. Apply a numerical solution to it and incremental sensitivity is used to calculate the gold standard sensitivity. Fig. 15(b), demonstrates once again the impact of picking a smaller time step \( \Delta t \).

To view the significance of the sensitivity estimation with and without the truncation-derivative error correction, the operational amplifier (opamp) circuit shown in Fig. 16 is considered. Fig. 17, shows the capability of the truncation-derivative error minimization to allow accurate estimation of the parameter sensitivity. Based on our research, the nonlinear error grows faster (needs to be considered) when the circuit scale or the nonlinearity of its devices increases. As an example, the RCD circuit has a relatively small dependence on the nonlinear error, as the nonlinearity is only active for a small period of time, and, thus, can provide almost satisfactory results even when truncation-derivative error is not well handled.

Because the practical complexities involved in circuit simulation cannot be easily theorized, we instead opt to use a variety of example circuits (further shown in the next section) to verify the claim that the computational overhead is akin to the runtime of the original circuit simulation without sensitivity. We conclude that transient circuit adjoint sensitivity of all state variables with respect to all variable parameters can be obtained at roughly the cost of a single incremental parameter change sensitivity computation. To quantify the complexity, we define the complexity ratio as matrix runtime with sensitivity divided by matrix runtime without sensitivity. Note that the matrix runtime only includes the time of matrix manipulation as we know the device
model evaluation is efficiently executed in industrial simulators. The complexity ratios for selected circuit examples are tabulated in Table I. Even though the correction contributes computational overhead, the complexity ratio remains around two which means that the sensitivities of all state variables (practically, all node voltages) with respect to all parameters can be obtained for roughly the cost of an additional simulation.

When compared to the direct method, the proposed method solves for all sensitivities simultaneously, while the direct method must calculate the sensitivities on a parameter-by-parameter basis [12], [13], [14], [15].

**TABLE I**

<table>
<thead>
<tr>
<th>Figure</th>
<th>Circuit name</th>
<th>Complexity Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fig. 21</td>
<td>RC Chain</td>
<td>1.562</td>
</tr>
<tr>
<td>Fig. 25(a)</td>
<td>SRAM</td>
<td>1.814</td>
</tr>
<tr>
<td>Fig. 25(b)</td>
<td>OPAMP</td>
<td>1.973</td>
</tr>
<tr>
<td>Fig. 25(c)</td>
<td>BANDGAP</td>
<td>2.027</td>
</tr>
</tbody>
</table>

**Fig. 17.** Output voltage sensitivities without and with correction over the output transient of an operational amplifier excited with a step function input. +1% parameter changes are applied to find incremental sensitivity.

**Fig. 18.** Schematic for an RLC circuit.

**Fig. 19.** Deviation of the voltage on $R$ in the RLC circuit is plotted versus time at the left-hand side and their related errors are plotted in the right-hand side.

**Fig. 20.** Circuit schematic for an $RC$ chain.

**Fig. 21.** Deviation of the voltage on $C_4$ in the $RC$ chain circuit is plotted versus time.
TABLE II
CIRCUIT EXAMPLES

<table>
<thead>
<tr>
<th>Characteristics</th>
<th>Circuit name</th>
<th>Chosen Perturbed Components</th>
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</thead>
<tbody>
<tr>
<td>Linear</td>
<td>RLC, RC chain</td>
<td>$R, L, C$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$R_1 \sim R_4, C_1 \sim C_4$</td>
</tr>
<tr>
<td>Digital</td>
<td>Inverter, SRAM</td>
<td>$M_p, M_n$</td>
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<td></td>
<td>$M_1, M_2$</td>
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<tr>
<td>Analog</td>
<td>Opamp, Bandgap</td>
<td>$M_5, C_L$</td>
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<tr>
<td></td>
<td></td>
<td>$Q_1 \sim Q_5, C_L$</td>
</tr>
</tbody>
</table>

VI. NUMERICAL EXAMPLES

In this section, we verify the proposed sensitivity analysis in terms of several circuit examples as listed in Table II. For comparison purposes, we also implement the gold standard incremental method. Specifically, the gold standard method simulates twice, once for the nominal circuit and once with the parameter perturbed. Then, it calculates their difference. All circuits are tested on an open-source sensitivity simulator developed upon the XYCE platform [20].

As a linear circuit example, we select a series RLC circuit (Fig. 18). We chose the voltage across $R$ at time points $t = [0, 10 \text{ ms}, 20 \text{ ms}, \ldots, 16 \text{ s}]$ as the desired output of the circuit when stimulated with a unit step excitation. For simulation purposes, we chose $R = 1.2 \ \Omega$, $C = 1 \ \text{F}$, and $L = 1 \ \text{H}$. Relative +1% deviations are assumed for all circuit elements. As shown in the first column of Fig. 19, compared with the gold standard results, our proposed method can accurately predict the sensitivities. Fig. 20 demonstrates the effectiveness of the method on an RC chain circuit, where we chose the voltage on $C_4$ at time points $t = [0, 1 \text{ ms}, 2 \text{ ms}, \ldots, 10 \text{ s}]$ as the output of interest. For illustrative purposes, we set all resistances to 1 $\Omega$ and all capacitances to 1 $\text{F}$. Furthermore, relative +1% deviations are assumed for all circuit elements $R_1$–$R_4$, $C_1$–$C_4$. As shown in Fig. 21, compared with the gold standard results, our proposed method can accurately predict the sensitivities. In this example, performing adjoint sensitivity analysis once returns all sensitivities simultaneously which guarantees its potential to applications with all parameters be perturbed in time domain, and shows the efficiency in large-scale circuits with lots of parameters that makes adjoint methods attractive to begin with.

More circuit examples are given in Fig. 22(a)–(c). Both the specific observation node and the parameters of interest are marked in the red boxes. Relative +1% deviations are assumed for all of these parameters. As shown in Fig. 22(d)–(f) the adjoint sensitivities match well with the gold standard results.

Fig. 23 shows a CMOS inverter circuit, where we chose $v_{\text{out}}$ at time points $t = [0, 1 \text{ ns}, 2 \text{ ns}, \ldots, 5000 \text{ ns}]$ as the desired output. For illustrative purposes, we chose $\beta p = 4.0 \times 10^{-4} \ \text{A/V}^2$, $v_{\text{thp}} = -0.4 \ \text{V}$, and $C_L = 1 \text{pF}$. Relative +1% deviations are assumed for all these parameters. Fig. 24 shows that once again, our adjoint sensitivities match well with the gold standard results.

All the abovementioned examples are stable circuits which have converged DC points, and some may want to apply...
time-domain sensitivity analysis to circuit with oscillators. Thus, Fig. 25 shows an oscillator circuit and $v_{\text{out}}$ is chosen to be the observation point. The simplified original circuit of the oscillator and its adjoint circuit are shown in Fig. 26. Fig. 27 shows the time domain sensitivity response for the oscillator circuit, other than the result itself, the order of magnitude of the sensitivity with respect to different parameters essentially reveal which parameters are the most critical and which are not. With a proper preprocessing, some applications of time domain sensitivity can be greatly simplified with pruned result.

More details regarding the adjoint sensitivity approximation and the incremental sensitivity are shown in Table III, which lists the maximum difference between both over the whole runtime. Even though the difference typically is small, observing a relative error metric provides a more telling observation. The relative error is calculated as the ratio of the absolute error to the size of the sensitivity. Such a metric shows that all sensitivities are calculated with acceptable accuracy, and it demonstrate the expected trend of significant nonlinearities such as the FET switching in the inverter showing smaller accuracies compared to analog circuit perturbations.

VII. CONCLUSION

In this article, the circuit theoretic basis of the direct adjoint sensitivity approach is described. An efficient time domain adjoint sensitivity approach is proposed that employs a Backward-Euler companion model to linearize all the energy storage elements (e.g., capacitance and inductance) and nonlinear memoryless elements (e.g., MOS, BJT DC characteristics) at each time step. A comprehensive error analysis over the proposed sensitivity estimation is discussed and a related strategy to eliminate such error is deployed that is demonstrated to be effective for time domain studies of various analog and digital circuits. Overall, this approach obtains the sensitivities of all outputs (node voltages and specified branch currents) with respect to all parameters while entailing no
new LU-factorization. The set of Forward-and-Back substitutions to obtain the adjoint variables has less overhead than the incremental approach, often cited, and advocated earlier [14], and more importantly calculates all sensitivities at the cost of roughly a single additional simulation.

REFERENCES


