Sensitivity Analysis by Adjoint Network

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Outline

• Introduction
• Tellegen’s Theorem
• Resistive Network
• Dynamic System
Introduction

Sensitivity Calculation

• Direct calculation: One simulation for each perturbation (Can work for multiple obj. functions).

• Adjoint network: One simulation for many possible perturbations (Work for one obj. function).
Tellegen’s Theorem

Tellegen’s Theorem: For a vector of branch voltages and branch currents, we have

\[ V_b^T I_b = \tilde{V}_b^T I_b = V_b^T \tilde{I}_b = \tilde{V}_b^T \tilde{I}_b = 0 \]

Note that the branch voltages \( V_b, \tilde{V}_b \) and branch currents \( I_b, \tilde{I}_b \) obey the KVL and KCL.
Tellegen’s Theorem

1. \( V_b^T I_b = 0 \)

\[ E^T V_n = V_b \]

\[
\begin{bmatrix}
1 & -1 & 0 & 0 \\
1 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
\nu_1 \\
\nu_2 \\
\nu_3 \\
\nu_4 \\
\end{bmatrix} =
\begin{bmatrix}
\nu_{12} \\
\nu_{13} \\
\nu_{20} \\
\nu_{30} \\
\nu_{34} \\
\nu_{40} \\
\end{bmatrix}
\]
Tellegen’s Theorem (con’t)

1. \( V_b^T I_b = 0 \)

\[
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 \\
\end{bmatrix}
\begin{bmatrix}
i_{12} \\
i_{13} \\
i_{20} \\
i_{30} \\
i_{34} \\
i_{40} \\
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}
\]

\[
V_b^T I_b = \left( E^T V_n \right)^T I_b = V_n^T E I_b = 0
\]
Tellegen’s Theorem (con’t)

II. \[ V^T_b \tilde{I}_b = 0 \]

Example: Two circuits with the same topology

\begin{align*}
&1 \text{ (2v)} & &1 \text{ (3v)} & &3 \text{ (0v)} & &2 \text{ (4v)} \\
&-1 & &1 & &0 & &-2 \\
&2 \text{ (4v)} & &-1 & &0 & &2 \\
& & & & & & & \\
&1 \text{ (3v)} & &4 \text{ (1v)} & &3 \text{ (1v)} & &4 \text{ (3v)} \\
& & & & & & & \\
& & & & & & & \\
&2 \text{ (2v)} & &2 & &-1 & &-1 \\
& & & & & & &
\end{align*}
Tellegen’s Theorem (con’t)

II. \( V_b^T \tilde{I}_b = 0 \)

Example case: Two circuits with the same topology

\[
\begin{bmatrix}
\tilde{v}_{12} \\
\tilde{v}_{13} \\
\tilde{v}_{20} \\
\tilde{v}_{30} \\
\tilde{v}_{34} \\
\tilde{v}_{40}
\end{bmatrix} =
\begin{bmatrix}
-2 \\
-1 \\
4 \\
3 \\
2 \\
1
\end{bmatrix}
= 
\begin{bmatrix}
\tilde{i}_{12} \\
\tilde{i}_{13} \\
\tilde{i}_{20} \\
\tilde{i}_{30} \\
\tilde{i}_{34} \\
\tilde{i}_{40}
\end{bmatrix} =
\begin{bmatrix}
-1 \\
1 \\
-1 \\
1 \\
0 \\
0
\end{bmatrix}
= 
\begin{bmatrix}
v_{12} \\
v_{13} \\
v_{20} \\
v_{30} \\
v_{34} \\
v_{40}
\end{bmatrix} =
\begin{bmatrix}
-2 \\
1 \\
2 \\
-1 \\
-4 \\
3
\end{bmatrix}
= 
\begin{bmatrix}
i_{12} \\
i_{13} \\
i_{20} \\
i_{30} \\
i_{34} \\
i_{40}
\end{bmatrix} =
\begin{bmatrix}
2 \\
-2 \\
2 \\
-1 \\
-1 \\
-1
\end{bmatrix}
\]

\[ V_b^T I_b = \tilde{V}_b^T \tilde{I}_b = \tilde{V}_b^T I_b = V_b^T \tilde{I}_b = 0 \]
Outline

• Tellegen’s Theorem

• Resistive Network

• Dynamic System
Resistive Network

• Objective Function

\[ y = \sum_{k \in A} f_k v_k + \sum_{k \in B} g_k i_k \]

– Set A: branch voltages \((i = 0)\)
– Set B: branch currents \((v = 0)\)

• Sensitivity Calculation

\[ \frac{\partial y}{\partial R_k} \quad \forall k \in D \]

– Set D: resistances
Example of Sensitivity Calculation

Given a circuit

\[
\begin{align*}
\text{The objective function} & \\
(e.g.) \ y &= v_1 + 2i_2 + 4i_3
\end{align*}
\]
Adjoint Network for Resistive Network

\[ y = \sum_{k \in A} f_k v_k + \sum_{k \in B} g_k i_k \]

original network

\[ \tilde{V}_b^T I_b - V_b^T \tilde{I}_b = \sum_{k \in A} (\tilde{v}_k i_k - v_k \tilde{i}_k) + \sum_{k \in B} (\tilde{v}_k i_k - v_k \tilde{i}_k) + \sum_{k \in D} (\tilde{v}_k i_k - v_k \tilde{i}_k) \]
For set A (branch voltage),

\[ v_k - f_k = \begin{cases} \text{original network} & \text{adjoint network} \\ v_i = -f_k \text{ where } \tilde{v}_k \text{ is unknown} \end{cases} \]

\[ \tilde{v}_k i_k - v_k \tilde{i}_k = 0 - v_k \times (-f_k) = f_k v_k \]
Adjoint Network for Resistive Network (con’t)

For set B (branch current),

\[ i_k \quad \text{where } v_k = 0 \]

original network

\[ \tilde{v}_k \tilde{i}_k - v_k i_k = g_k i_k - 0 \times \tilde{i}_k \]
\[ = g_k i_k \]

adjoint network

where \( \tilde{i}_k \) is unknown
Adjoint Network for Resistive Network (con’t)

For set D (resistance),

original network

\[ v_k = (R_k + \Delta R_k) \times i_k \]

adjoint network

\[ \tilde{v}_k = \tilde{i}_k R_k \]

\[ \tilde{v}_k i_k - v_k \tilde{i}_k = \tilde{i}_k R_k i_k - i_k (R_k + \Delta R_k) \tilde{i}_k = -i_k \Delta R_k \tilde{i}_k \]
Put it together...

\[
\begin{align*}
\begin{bmatrix} \tilde{v}_k^T \end{bmatrix} I_b - V_b^T \tilde{I}_b &= \sum_{k \in A} \left( \tilde{v}_k i_k - v_k \tilde{i}_k \right) + \sum_{k \in B} \left( \tilde{v}_k i_k - v_k i_k \right) + \sum_{k \in D} \left( \tilde{v}_k i_k - v_k \tilde{i}_k \right) = 0 \\
\sum_{k \in A} f_k V_k + \sum_{k \in B} g_k i_k &= \sum_{k \in D} i_k \Delta R_k \tilde{i}_k 
\end{align*}
\]

where \( \frac{\partial y}{\partial R_k} = \tilde{i}_k i_k \)
Outline

- Tellegen’s Theorem
- Resistive Network
- Dynamic System
Dynamic System

- **Objective Function**
  \[
  y = \sum_{k \in A} \int_{0}^{T} f_k(t)v_k(t)dt + \sum_{k \in B} \int_{0}^{T} g_k(t)i_k(t)dt
  \]
  - Set A: branch voltages
  - Set B: branch currents

- **Sensitivity Calculation**
  \[
  \begin{cases}
  \frac{\partial y}{\partial R_k} & \forall k \in D \\
  \frac{\partial y}{\partial C_k} & \forall k \in E
  \end{cases}
  \]
  - Set D: resistances
  - Set E: capacitances

(We omit L here which is similar to C)
Adjoint Network for Dynamic System

$$\sum_{0}^{T} \int \tilde{v}_b(T-t)i_b(t) - v_b(t)\tilde{i}_b(T-t)dt = \sum_{k \in A,B,D,E} \int_{0}^{T} \tilde{v}_k(T-t)i_k(t) - v_k(t)\tilde{i}_k(T-t)dt = 0$$
Adjoint Network for Dynamic System (con’t)

For set A (branch voltage),

original network

\[ + \quad v_k(t) \quad - \]

\[ v_k(t) \quad \text{where} \quad i_k(t) = 0 \]

adjoint network

\[ \tilde{i}_k(T - t) = -f_k(t) \]

where \( \tilde{v}_k(t) \) is unknown

\[
\int_{0}^{T} \tilde{v}_k(T - t)i_k(t) - v_k(t)i_k(T - t)dt = \int_{0}^{T} f_k(t)v_k(t)dt
\]
Adjoint Network for Dynamic System (con’t)

For set B (branch current),

original network

diagram with equations:

\[ i_k(t) \]

where \( v_k(t) = 0 \)

adjoint network

diagram with equations:

\[ \tilde{v}_k(T-t) = g_k(t) \]

where \( \tilde{i}_k(t) \) is unknown

\[
\int_0^T \tilde{v}_k(T-t)i_k(t) - v_k(t)\tilde{i}_k(T-t)dt = \int_0^T g_k(t)i_k(t)dt
\]
Adjoint Network for Dynamic System (con’t)

For set D (resistance),

\[ v_k(t) = (R_k + \Delta R_k) \times i_k(t) \]

\[ \tilde{v}_k(t) = \tilde{i}_k(t) R_k \]

\[
\int_0^T \tilde{v}_k(T-t)i_k(t) - v_k(t)\tilde{i}_k(T-t)dt = \int_0^T \tilde{i}_k(T-t)R_ki_k(t) - i_k(t)(R_k + \Delta R_k)\tilde{i}_k(T-t)dt
\]

\[
= -\int_0^T i_k(t) \Delta R_k \tilde{i}_k(T-t) dt
\]
Adjoint Network for Dynamic System (con’t)

For set E (capacitance),

original network

\[ v_k(t) \quad \text{where} \quad i_k(t) \, dt = (C_k + \Delta C_k) \, dv_k(t) \]

adjoint network

\[ \tilde{v}_k(t) \quad \text{where} \quad \tilde{i}_k(t) \, dt = C_k \, dv_k(t) \]
Adjoint Network for Dynamic System (con’t)

For set E (capacitance),

\[
\int_0^T \tilde{v}_k(T-t)i_k(t) - v_k(t)\tilde{i}_k(T-t)dt
\]

\[
\int_0^T \tilde{v}_k(T-t)i_k(t) = \int_0^T \tilde{v}_k(T-t)[C_k + \Delta C_k]dv_k(t)
\]

\[
= \int_0^T \tilde{v}_k(T-t)C_k dv_k(t) + \int_0^T \tilde{v}_k(T-t)\Delta C_k dv_k(t)
\]

\[
\int_0^T \tilde{v}_k(T-t)C_k dv_k(t) = \tilde{v}_k(T-t)C_k v_k(t)\bigg|_{t=0}^T - \int_0^T v_k(t)d(\tilde{v}_k(T-t)C_k)
\]

\[
= \tilde{v}_k(0)C_k v_k(T) - \tilde{v}_k(T)C_k v_k(0) - \int_0^T v_k(t)d(\tilde{v}_k(T-t)C_k)
\]

\[
\text{overall} \quad \int_0^T \tilde{v}_k(T-t)i_k(t) - v_k(t)\tilde{i}_k(T-t)dt = -\tilde{v}_k(T)C_k(0)v_k(0) + \int_0^T \tilde{v}_k(T-t)\Delta C_k \tilde{v}_k(t)dt
\]

\[
cancellation
\]

\[
\tilde{v}_k(0) = 0
\text{ by initial setting}
\]
Put it together...

\[
\sum_{b}^{T} \int_{0}^{T} \tilde{v}_b(T-t)i_b(t) - v_b(t)\tilde{i}_b(T-t)dt = \sum_{k\in A,B,D,E}^{T} \int_{0}^{T} \tilde{v}_k(T-t)i_k(t) - v_k(t)\tilde{i}_k(T-t)dt = 0
\]

\[
y = \sum_{k\in D}^{T} \int_{0}^{T} i_k(t)\tilde{i}_k(T-t)\Delta R_k dt + \sum_{k\in E}^{T} \tilde{v}_k(T)C_k v_k(0) - \sum_{k\in E}^{T} \int_{0}^{T} \tilde{v}_k(T-t)\Delta C_k(t)\dot{v}_k(t)dt
\]

Initial voltage, \(v_k(0)\), of capacitor \(k\) remains fixed when parameters \(R_i, C_i\) vary. Thus, this term is not relevant to the sensitivities.

\[
\left\{ \begin{array}{l}
\frac{\partial y}{\partial R_k} = \int_{0}^{T} i_k(t)\tilde{i}_k(T-t)dt \quad \forall k \in D \\
\frac{\partial y}{\partial C_k} = -\int_{0}^{T} \tilde{v}_k(T-t)\dot{v}_k(t)dt \quad \forall k \in E
\end{array} \right.
\]
Conclusion

• Adjoint network can derive sensitivities of all parameters for one objective function.
• The integration traces backward on the time domain for the dynamic adjoint network.
• The derivation uses Tellegen’s theorem which depends upon the circuit topology only.