CSE245: Computer-Aided Circuit Simulation and Verification

Lecture Note 5.1

Numerical Integration

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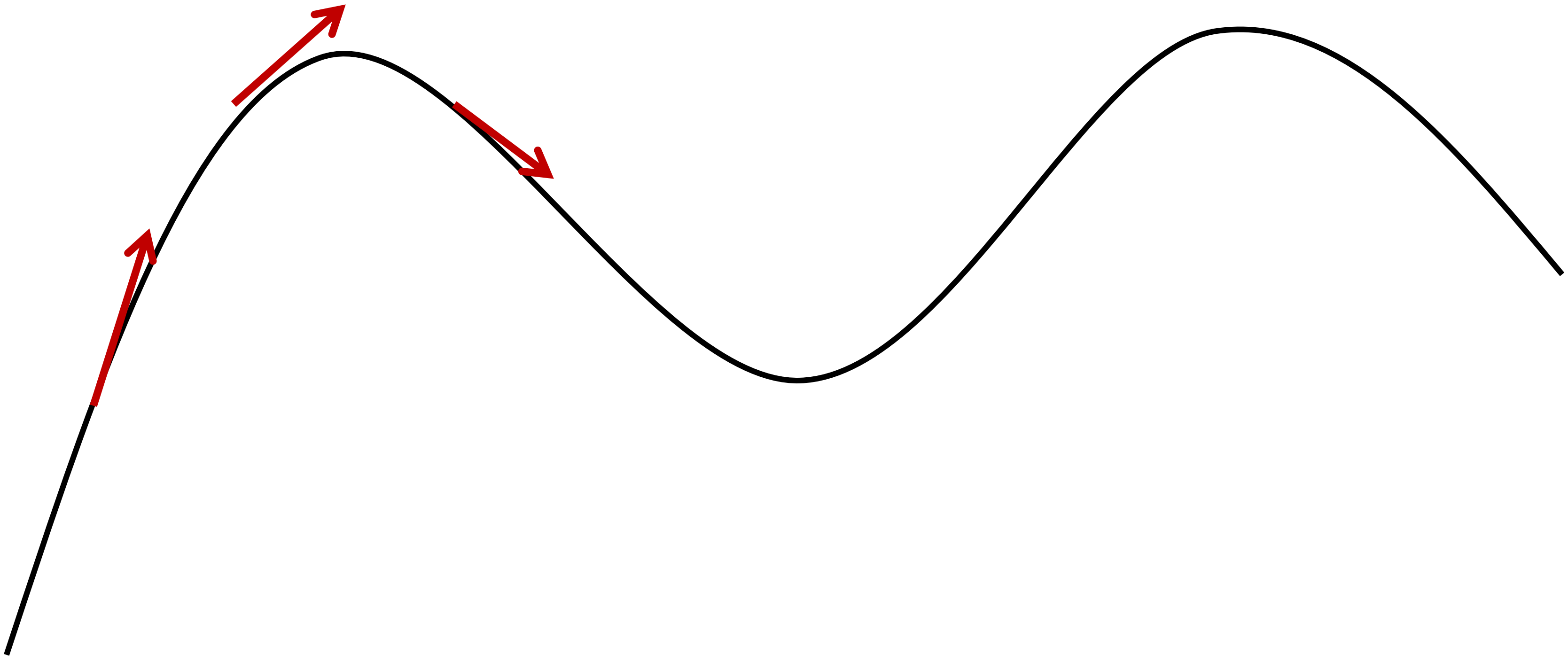
Integration Methods



Numerical Integration: Outline

- Multistep Multivalue Methods
  - Multistep Methods
  - Multivalue Methods
  - Predictor-Corrector Methods
- Rosenbrock Methods (Runge Kutta)
  - Several Euler-style steps
- Richardson extrapolations (Bulirsch-Stoer)
- Matrix Exponential Integration
- Keywords: Taylor expansion

Multiple Step Integration: Stability

For a system $x'=\lambda x$, let $q= \lambda h$.

The integration formula is $\sum_{i=0,k} a_i x_{n-i}+hb_i x'_{n-i}=0$.

We set $(a_0+qb_0)z^k+ (a_1+qb_1)z^{k-1} \ldots+(a_k+qb_k)=0$

There are k roots, r_i, of the polynomial eq.

The generic solution is $x_n=c_1 r_1^n+c_2 r_2^n+\ldots+c_k r_k^n$

but for multiplicity root, we have

$$x_n=\ldots+(c_{i0}+c_{i1}n+\ldots+c_{im}n^{m-1})r_i^n+\ldots$$

If $|r_i|< 1$ for all i, the system is stable

Else for $|r_i|= 1$ but not multiplicity root, the system remains stable

Multiple Step Integration: Stability

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We set $(a_0+qb_0)z^k + (a_1+qb_1)z^{k-1} \ldots + (a_k+qb_k)=0$

Examples using FE, BE, TZ methods

Method	a_0	a_1	b_0	b_1	root
FE	1	-1	0	-1	$1+q$
BE	1	-1	-1	0	$1/(1-q)$
TR	1	-1	-1/2	-1/2	$(1+q/2)/(1-q/2)$

Predictor-Corrector Methods: Adams-Bashforth

ODE: $dx/dt=f(x)$

Predictor

$x_{n+1}=x_n+h/12(23f_{n-1}-16f_{n-1}+5f_{n-2})+O(h^4)$

Corrector

$x_{n+1}=x_n+h/12(5f_{n+1}+8f_n-f_{n-1})+O(h^4)$

Rosenbrock and Runge-Kutta Methods

Rosenbrock Method:

ODE: $dx/dt=f(x)$

Stepsize: h

Process:

$x(t_0+h)=x_0+\sum_{i=1,s} b_i k_i$

$(1-rhf')k_i=hf(x_0+\sum_{j=1,i-1} a_{ij}k_j)+hf'\sum_{j=1,i-1} r_{ij}k_j$, $i=1,\ldots,s$

Runge-Kutta: $r=r_{ij}=0$ for all ij.

Rung-Kutta Method (4^{th} order)

For 4^{th} order RK method, we evaluate the derivatives four times: once at the initial points, twice at trial midpoints, and once at a trial endpoint. The final solution is calculated from the 4 derivatives.

$k_1= hf(t_n, x_n)$

$k_2=hf(t_n+0.5h, x_n+0.5k_1)$

$k_3= hf(t_n+0.5h, x_n+0.5k_2)$

$k_4= hf(t_n+h, x_n+k_3)$

$x_{n+1}=x_n+1/6k_1+1/3k_2+1/3k_3+1/6k_4+O(h^5)$

Bulirsch-Stoer Method

Caveats:

- Nonsmooth function: RK
- Contain singular points: RK
- Very smooth and right-hand sides expensive to compute: Predictor-corrector

Bulirsch-Stoer Method

Approach:

- Modified midpoint method
- Extrapolation
- Stepsize control

Bulirsch-Stoer Method: midpoint method

Given $dx/dt=f(t,x)$, H and n, set $h=H/n$

$z_0=x(t)$

$z_1=z_0+hf(t,z_0)$

$z_{m+1}=z_{m-1}+2hf(t+mh,z_m)$ for $m=1,2, .., n-1$

$x(t+H)$: $x_n=1/2[z_n+z_{n-1}+hf(t+H,z_n)]$

Error: $x_n-x(t+H)=\sum_{i=1}a_i h^{2i}$ [1,2,3]

Example

Sequence: (Deuflhard) k:n=

0:2, 1:4, 2:6, 3:8, 4:10, 5:12, 6:14,…

Bulirsch-Stoer Method: Extrapolation

$$
\begin{array}{llll}
T_{00} & & & \\
T_{10} & T_{11} & & \\
T_{20} & T_{21} & T_{22} & \\
\ldots & \ldots & \ldots & \ldots
\end{array}
$$

$T_{k0}=x_k$

$T_{k,j+1}=T_{kj}+(T_{kj}-T_{k-1,j})/[(n_k/n_{k-j})^2-1]$, $j=0,1,\ldots,k-1$

Solution: T_{kk}

Error: $|T_{kk}-T_{k,k-1}|$

Err_k: H^{2k+1}

Bulirsch-Stoer Method: Stepsize Control

Stepsize $H_k=H\, s_1(s_2/err_k)^{1/(2k+1)}$

where constants s_1, s_2<1 are safety factors

Complexity

$a_0=n_0+1$

$a_{k+1}=a_k+n_{k+1}$

Work per unit step $W_k=a_k/H_k$

Strategy: minimize W_k ([4].17.3.3)

Example:

For $y(x+H)\approx y_n+(y_n-y_{n/2})/3$, we use 1.5 derivative evaluations per step h.

For Runge-Kutta, it takes 4 evaluations.

References

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