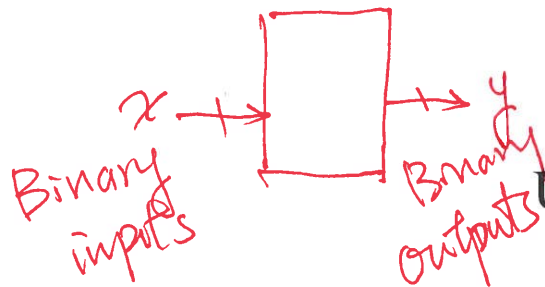
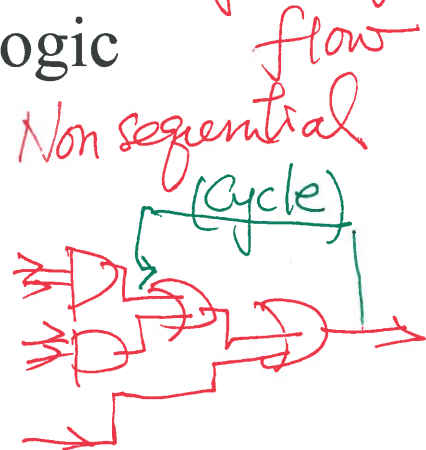


No memory
AND, OR, NOT Logic + 0, 1 constant
CSE 140, Lecture 2
Combinational Logic



CK Cheng
CSE Dept.
UC San Diego



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Combinational Logic Outlines

1. Introduction

- Scope
- Boolean Algebra (Review)
 - Definition
 - Duality
 - AND/OR Gates
 - Expressions vs Circuits
- Handy Tools
 - DeMorgan's Theorem
 - Consensus Theorem
 - Shannon's Expansion

Inversion

bottom up

top down

2. Specification

3. Synthesis

K maps

2

1.1 Combinational Logic: Scope

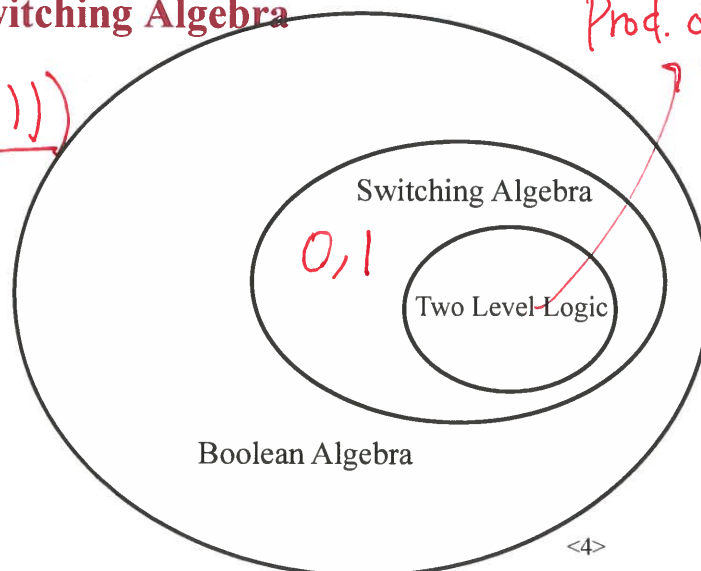
- Description
 - **Language:** e.g. C Programming, Verilog, VHDL
 - **Boolean algebra**
 - ✧ **Truth table: Powerful engineering tool**
- Design
 - Schematic Diagram
 - Inputs, Gates, Nets, Outputs
- Goal
 - Validity: **correctness**, turnaround time
 - Performance: power, timing, **cost**
 - Testability: yield, diagnosis, robustness

3

Scope: Boolean algebra, switching algebra, logic

- Boolean Algebra: multiple-valued logic, i.e. each variable have multiple values.
- Switching Algebra: binary logic, i.e. each variable can be either 1 or 0.
- **Boolean Algebra $\neq$ Switching Algebra**

$$f = a + b(c + d(e + f(q + h)))$$



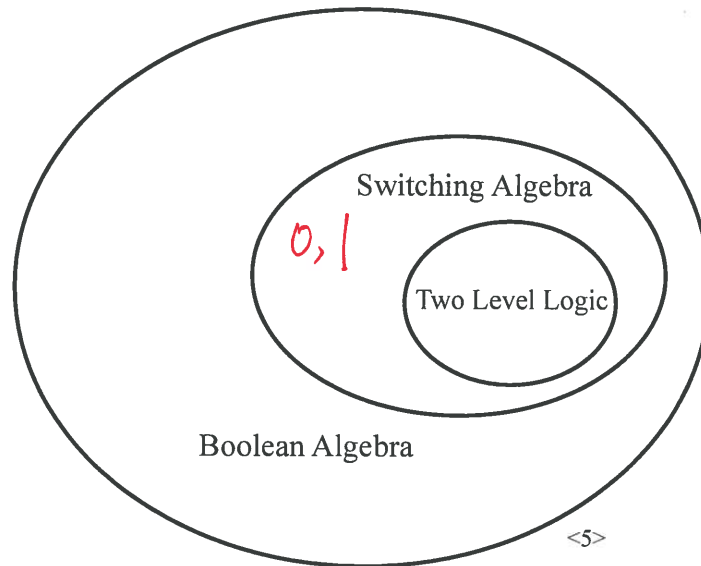
Scope: Boolean algebra, switching algebra, logic

- Boolean Algebra: multiple-valued logic, i.e. each variable have multiple values. **Boolean Algebra $\neq$ Switching Algebra**

Example: Set= $\{0, A, B, 1\}$, Operators $\&$, $+$ are defined in the following:

$\&$	0	A	B	1
0	0	0	0	0
A	0	A	0	A
B	0	0	B	B
1	0	A	B	1

$+$	0	A	B	1
0	0	A	B	1
A	A	A	1	1
B	B	1	B	1
1	1	1	1	1



Scope: Switching Algebra (Binary Values)

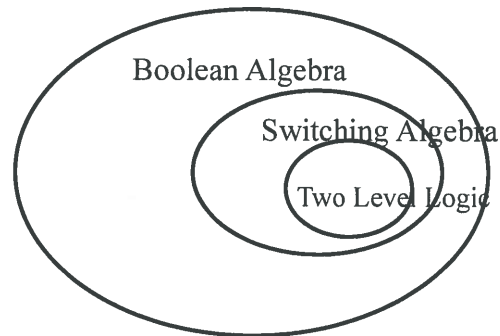
- Typically consider only two discrete values:
 - 1's and 0's
 - 1, TRUE, HIGH
 - 0, FALSE, LOW
- 1 and 0 can be represented by specific voltage levels, rotating gears, fluid levels, etc.
- Digital circuits usually depend on specific voltage levels to represent 1 and 0
- *Bit: Binary digit*

Scope: Levels of Logic

- Multiple Level Logic: Many layers of two level logic with some inverters, e.g. $((a+bc)'+ab')+b'c+c'd)bc+c'e$
(A network of two level logic)
- Two Level Logic: Sum of products, or product of sums, e.g. $ab + a'c + a'b'$, $(a'+c)(a+b')(a+b+c')$

Features of Digital Logic Design

- Multiple Outputs
- Don't care sets



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1.2 Boolean Algebra (Review)

George Boole, 1815-1864

- Born to working class parents: Son of a shoemaker
- Taught himself mathematics and joined the faculty of Queen's College in Ireland.
- Wrote *An Investigation of the Laws of Thought* (1854): **systematize Aristotle's logic**
- Introduced binary variables
- Introduced the three fundamental logic operations: AND, OR, and NOT.



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Review of Boolean Algebra

Let B be a nonempty set with two 2-input operations, a 1-input operation $'$ (complement), and two distinct elements 0 and 1. Then B is called a Boolean algebra if the following axioms hold.

- Associative laws: $(a+b)+c=a+(b+c)$, $(a \cdot b) \cdot c=a \cdot (b \cdot c)$
- Commutative laws: $a+b=b+a$, $a \cdot b=b \cdot a$
- Distributive laws: $a+(b \cdot c)=(a+b) \cdot (a+c)$,
 $a \cdot (b+c)=a \cdot b+a \cdot c$
- Identity laws: $a+0=a$, $a \cdot 1=a$
- Complement laws: $a+a'=1$, $a \cdot a'=0$

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Review of Boolean Algebra: Duality

Associative laws	$(a+b)+c=a+(b+c)$	$(a \cdot b) \cdot c=a \cdot (b \cdot c)$
Commutative laws	$a+b=b+a$	$a \cdot b=b \cdot a$
Distributive laws	$a+(b \cdot c)=(a+b) \cdot (a+c)$	$a \cdot (b+c)=a \cdot b+a \cdot c$
Identity laws	$a+0=a$	$a \cdot 1=a$
Complement laws	$a+a'=1$	$a \cdot a'=0$

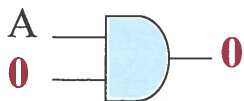
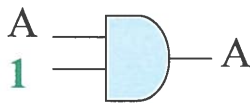
Duality: We swap all operators between $(+,\cdot)$ and interchange all elements between $(0,1)$.

For a theorem if the statement can be proven with the laws of Boolean algebra, then the duality of the statement is also true.

1.3 Switching functions: Operators and Digital Logic Gates

id	A	B	Y
0	0	0	0
1	0	1	0
2	1	0	0
3	1	1	1

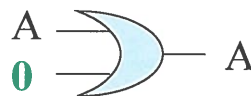
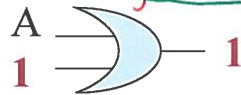
AND
 $Y=AB$



Input 0 dominates Y
0 blocks the output
1 passes signal A

A	B	Y
0	0	0
0	1	1
1	0	1
1	1	1

OR
 $Y=A+B$



Input 1 dominates Y
0 passes signal A
1 blocks the output

A	Y
0	1
1	0

NOT
 $Y=A'$

$f(a,b,c) = ab + a'b' + bc + bc' + ac + ac'$
find (a,b,c) , s.t. $f(a,b,c) = 1$
 s.t. $f(a,b,c) = 0$ ^{NP complete}

If $a=1 \Rightarrow b=0 \Rightarrow c=1 \Rightarrow f=1$
 If $a=0 \Rightarrow b=1 \Rightarrow c=0 \Rightarrow f=1$

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1.3 Switching functions: Operators and Digital Logic Gates

id	A	B	C	Y
0	0	0	0	0
1	0	0	1	0
2	0	1	0	0
3	0	1	1	0
4	1	0	0	0
5	1	0	1	0
6	1	1	0	0
7	1	1	1	1

AND
 $Y=ABC$

0 blocks the output
1 passes signal A
 For AND,
only one row is true (minterm)

$\rightarrow m_7 = 1$
 iff $a=b=c=1$

A	B	C	Y
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

OR
 $Y=A+B+C$

0 passes signal A
1 blocks the output
 For OR,
only one row is false (maxterm)

$M_0 = 0$ iff
 $A=B=C=0$

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1.3 Switching functions: Example on AND and OR

AND					OR				
id	A	B	C	Y		A	B	C	Y
0	0	0	0	0	$Y=A'B'C$	0	0	0	0
1	0	0	1	1	m_1	0	0	1	1
2	0	1	0	0		0	1	0	1
3	0	1	1	0		0	1	1	1
4	1	0	0	0		1	0	0	1
5	1	0	1	0		1	0	1	1
6	1	1	0	0		1	1	0	0
7	1	1	1	1		1	1	1	1

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Switching Expressions vs Logic Diagrams

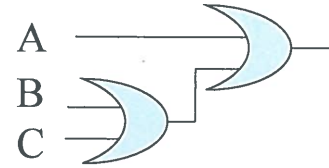
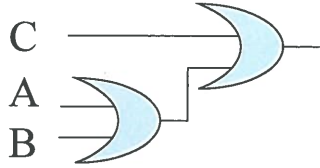
Switching expression is related to logic implementation

- Switching Expression: #literals, #operators
- Schematic Diagram: #gates, #nets, #pins

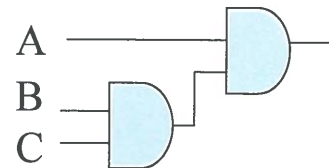
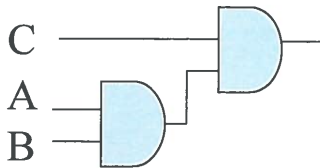
Laws and Logic Diagrams

Associativity Laws

$$(A+B) + C = A + (B+C)$$



$$(AB)C = A(BC)$$



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Laws and Logic Diagrams

Identity Laws

$$A \cdot 1 = A$$

$$A + 1 = 1$$

$$A \cdot 0 = 0$$

$$A + 0 = A$$

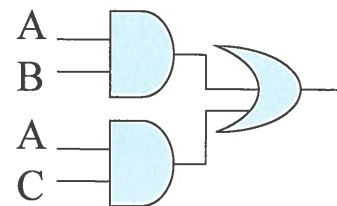
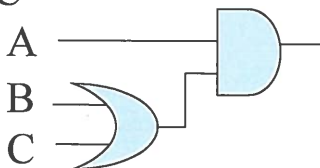
Complement Laws

$$A + A' = 1$$

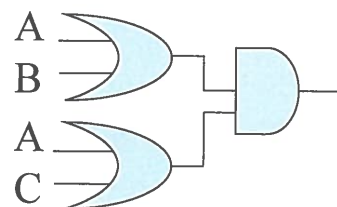
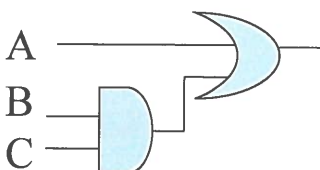
$$A \cdot A' = 0$$

Distributive Laws

$$A \cdot (B+C) = A \cdot B + A \cdot C$$

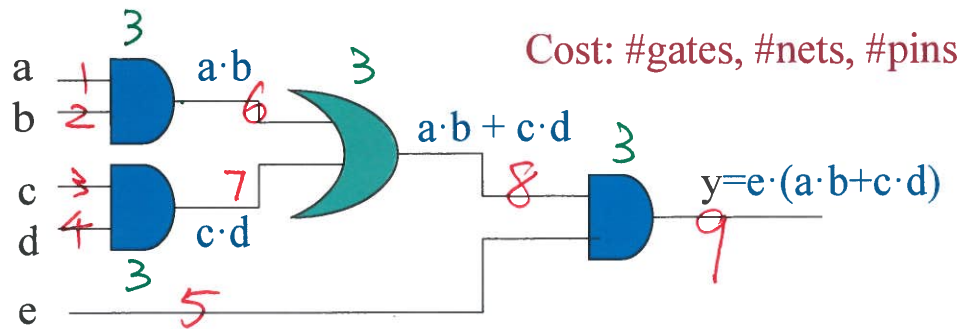


$$A+B \cdot C = (A+B) \cdot (A+C)$$



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Switching Expression and Logic



Schematic Diagram:

5 primary inputs

1 primary output

4 gates (3 ANDs, 1 OR)

9 signal nets

12 pins

Boolean Algebra:

5 variables

1 expression

4 operators (3 ANDs, 1 OR)

5 literals

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Switching Expression and Logic

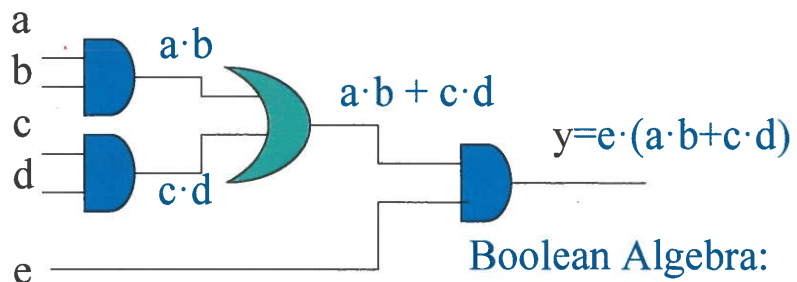
Schematic Diagram:

5 primary inputs

4 components (gates)

9 signal nets

12 pins



Boolean Algebra:

5 literals

4 operators

A. #inputs $\leftrightarrow$ I. #variables

B. #gates $\leftrightarrow$ II. #operators

C. #nets $\leftrightarrow$ III. #variables + #operators

D. #pins 12 IV. #literals + 2 #operators - 1

E. None

$$5 + 2 \times 4 - 1 = 12$$

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Example: $f(a,b,c) = ab + a'c + a'b'$

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Example: $f(a,b,c) = (a' + c)(a + b')(a + b + c)$

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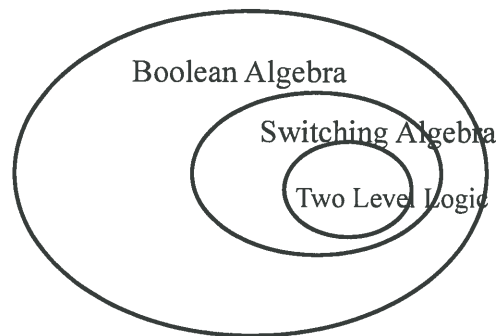
1.4 Handy Tools

Boolean Algebra

- DeMorgan's Law: Complements
- Consensus Theorem

Switching Logic

- Shannon's Expansion
- Truth Table
- Karnaugh Map (single output, two level logic)

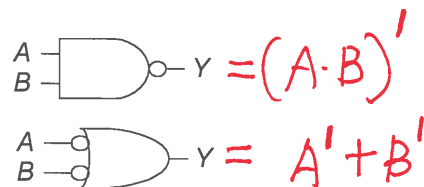


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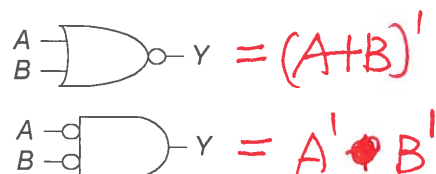
DeMorgan's Theorem and Digital Logic

T12. DeMorgan's Theorem $(A+B)' = A'B'$ $(AB)' = A' + B'$

- $Y = (AB)' = A' + B'$



- $Y = (A + B)' = A'B'$



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DeMorgan's Theorem: Bubble Pushing

- Pushing bubbles backward (from the output) or forward (from the inputs) changes the body of the gate from AND to OR or vice versa.
- Pushing a bubble from the output back to the inputs puts bubbles on all gate inputs.



- Pushing bubbles on *all* gate inputs forward toward the output puts a bubble on the output and changes the gate body.



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Consensus Theorem

- $AB + AC + B'C$
 $= AB + B'C$
- $(A+B)(A+C)(B'+C)$
 $= (A+B)(B'+C)$

The consensus of AB , $B'C$ is: ?

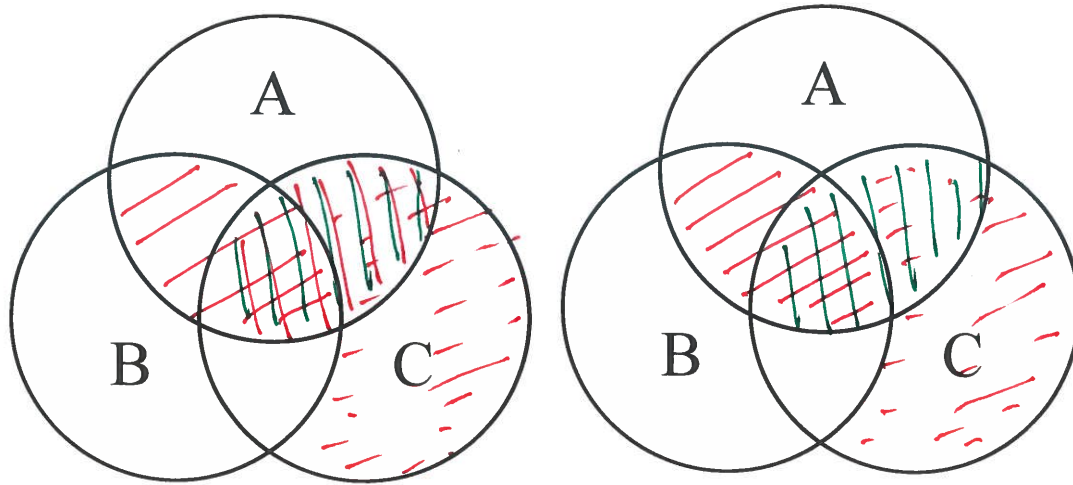
Exercise: to prove the reduction using

- (1) Venn Diagrams,
- (2) Boolean algebra,
- (3) Logic simulation and
- (4) Shannon's expansion

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Consensus Theorem: Venn Diagrams

$$AB+AC+B'C : AB+B'C$$



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Consensus Theorem: Boolean Algebra

$$\begin{aligned} & \bullet AB+AC+B'C \\ & =AB+B'C \end{aligned}$$

$$\begin{aligned} & \bullet (A+B)(A+C)(B'+C) \\ & =(A+B)(B'+C) \end{aligned}$$

$$\begin{aligned} AB+AC+B'C & \longleftrightarrow (A+B)(A+C)(B'+C) \\ & =AB+AC1+B'C \\ & =AB+AC(B+B')+B'C \\ & =AB+ABC+AB'C+B'C \\ & =AB(1+C)+(A+1)B'C \\ & =AB+B'C \end{aligned}$$

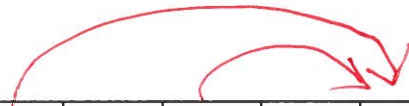
$$\begin{aligned} & (A+B)(A+C)(B'+C) \\ & =(A+B)(A+C+0)(B'+C) \\ & =(A+B)(A+C+BB')(B'+C) \\ & =(A+B)(A+B+c)(A+B'+c)(B'+C) \\ & =(A+B)0.c(A \cdot 0+B'+c) \\ & =(A+B)(B'+c) \end{aligned}$$

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Consensus Theorem: Logic Simulation

$$f(A,B,C) = AB + AC + B'C$$

$$g(A,B,C) = AB + B'C$$



Index	A	B	C	AB	AC	B'C	f	g
0	0	0	0	0	0	0	0	0
1	0	0	1	0	0	1	1	1
2	0	1	0	0	0	0	0	0
3	0	1	1	0	0	0	0	0
4	1	0	0	0	0	0	0	0
5	1	0	1	0	1	1	1	1
6	1	1	0	1	0	0	1	1
7	1	1	1	1	1	0	1	1

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$$ab + ab' + a = a$$

$$ab + a' + b = a' + b$$

$$\underline{ab'} + \underline{a'b} + \underline{b'b} = ab' + a'b$$

0

$$abe + a'cd \quad \underline{bcd}$$

consensus

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Consensus Theorem (Examples)

Which one is not a consensus of the expressions on the right.

A. B ——— 1. $A + A'B = A + B$

B. BC ——— 2. $A + A'BC = A + BC$

C. $(B+C)D$ ——— 3. $A(B+C) + A'D$

D. BCD ——— 4. $\underbrace{ABC + A'D}_{BCD} + BCDE = ABC + A'D$

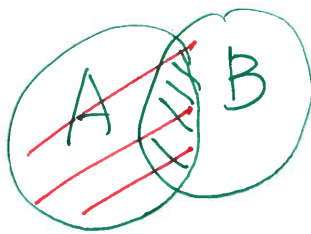
E. BCDE ——— 4. $\underbrace{ABC + A'D}_{BCD} + BCDE = ABC + A'D$

$$A(A+B) =$$

$$A \cdot A + AB$$

$$= A + AB$$

$$= A$$



$$A + \underline{AB} = A(1+B) = A \cdot 1 = A$$

Shannon's Expansion

- Shannon's expansion assumes a switching algebra system
- Divide a switching function into smaller functions
- Pick a variable x , partition the switching function into two cases: $x=1$ and $x=0$
 - $f(x,y,z,\dots) = xf(x=1,y,z,\dots) + x'f(x=0,y,z,\dots)$
- For example
 - $f(x) = xf(1) + x'f(0)$
 - $f(x,y) = xf(1,y) + x'f(0,y)$