Homework 2 covers the combinational logic specification, and implementation. For the first problem, we explore the structure of Boolean algebra. For the second two problems, we practice more on the specification and design using some popular functions. For the last two problems, we practice logic design using Karnaugh maps to derive the minimal sum of products and product of sums expressions.

1 Boolean Algebra

Given a mathematical system $M=\{0,a,b,c\},\#,$ and $\&$ where the two operators $\#$ and $\&$ are defined in the following two subtables.

(a)  
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<th>0</th>
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<td>a</td>
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(b)  
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Table 1: 2-input operators $\#$ and $\&$.

1. Verify whether the system is a Boolean algebra. Provide a sample of your derivation, i.e. no need to exhaust all combinations.
2. List the complements of elements 0, a, b, and c if the system is a Boolean algebra.
3. Does Shannon’s expansion remain correct and useful in this system? Explain why the Shannon’s expansion is not useful or how it can be useful?

2 Bit Counter

A bit counter reads four binary values $(x_3, x_2, x_1, x_0)$ and produces a three bit binary number $(y_2, y_1, y_0)$ to reflect the number of input bits which are true. For example, if $(x_3, x_2, x_1, x_0) = (1,0,1,1)$, then $(y_2, y_1, y_0) = (0,1,1)$, or if $(x_3, x_2, x_1, x_0) = (1,0,0,0)$, then $(y_2, y_1, y_0) = (0,0,1)$.

A. Write the truth table of the counter.
B. Express the functions in a minimal sum of products form.

3 Majority Function

A majority function produces output $f = 1$, when half of all, or more input bits are true, otherwise, the output $f = 0$. For a 3-input majority function example, we have $f(1,1,0) = 1$, and $f(0,1,0) = 0$.

A. Write the truth table of the 3-input majority function.
B. Express the 3-input majority function using a minimal sum of product expression. Illustrate your design with a logic diagram.
C. Construct a 2-input majority function using 3-input majority function modules as basic building blocks (no other gates except 0, 1 constants, and inverters).

D. Can we construct a 5-input majority function using 3-input majority function modules as basic building blocks (no other gates except 0, 1 constants, and inverters)? Use one sentence to explain your answer.

4 Minimal Sum of Products Expression

Implementation from truth tables to sum of products expressions.
1. Use Karnaugh map to simplify function
   \[ f(a, b, c) = \sum m(0, 3, 4, 6) + \sum d(1, 7). \]
   List all possible minimal two-level sum of products expressions. Show the switching functions. No need for the schematic diagram.
2. Use Karnaugh map to simplify function
   \[ f(a, b, c, d) = \sum m(0, 4, 6, 8, 9, 10, 13) + \sum d(1, 2, 7, 15). \]
   List all possible minimal two-level sum of products expressions. Show the switching functions. No need for the schematic diagram.
3. Use Karnaugh map to simplify function
   \[ f(a, b, c, d, e) = \sum m(1, 3, 4, 6, 9, 12, 14, 17, 19, 20, 25, 28, 29, 30) + \sum d(11, 23, 26). \]
   List all minimal two-level sum of products expressions. Show the switching functions. No need for the schematic diagram.

5 Minimal Product of Sums Expression

Implementation from truth tables to product of sums expressions.
1. Use Karnaugh map to simplify function
   \[ f(a, b, c) = \sum m(3, 6, 7) + \sum d(0, 5). \]
   List all possible minimal two-level product of sums expressions. Show the switching functions. No need for the schematic diagram.
2. Use Karnaugh map to simplify function
   \[ f(a, b, c, d) = \sum m(0, 4, 5, 8, 12, 15) + \sum d(2, 7, 10, 13, 14). \]
   List all possible minimal two-level product of sums expressions. Show the switching functions. No need for the schematic diagram.
3. Use Karnaugh map to simplify function
   \[ f(a, b, c, d, e) = \sum m(2, 10, 12, 13, 15, 18, 22, 23, 24, 28, 29, 30, 31) + \sum d(6, 8, 14, 26). \]
   List all minimal two-level product of sums expressions. Show the switching functions. No need for the schematic diagram.