CSE 291g: HW 3

Due 11:59pm, 05/24/21

Problem 1: Flag Complex

The goal of this problem is to prove that the Flag Complex of $\mathbb{F}_q^d$ is a one-sided $O\left(\sqrt{\frac{1}{q}}\right)$-spectral expander so long as $d \ll \sqrt{q}$. Recall from the notes:

Definition (Flag Complex on $\mathbb{F}_q^d$)**.** *Let q be a prime power. A* ***complete flag*** *of $\mathbb{F}_q^d$ is a strict containment sequence of $d-1$ subspaces $\{0\} \subset V_1 \subset \ldots \subset V_{d-1} \subset \mathbb{F}_q^d$. Let $Gr(d,q)$ denote the set of subspaces of $\mathbb{F}_q^d$ of any dimension. The Flag Complex on $\mathbb{F}_q^d$ is the $(d-1)$-dimensional simplicial complex on vertex set $Gr(d,q)$ whose top level faces are given by the complete flags.*

Part (a): Bipartite Spectral Expanders

The Flag Complex can be viewed as a multipartite complex. It will therefore be useful to understand some properties of bipartite spectral expanders to start. Given a bipartite graph $G = (L, R, E)$, let

$$M = \begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix}$$

be the (square) adjacency matrix indexed by $L \cup R$. In the next part, we will see that it is easier to analyze the spectrum of the *two-step random walks* $A^T A$ and AA^T rather than M directly. In this part, you will show the two are closely related. In particular, show that:

1. If λ is an eigenvalue of M, then λ^2 is an eigenvalue of $A^T A$ and AA^T
2. If $\lambda \neq 0$ is an eigenvalue of $A^T A$ or AA^T, then $\pm\sqrt{\lambda}$ are eigenvalues of M.

Part (b): Flag Complex in Two Dimensions

Next, we analyze the Flag Complex of $\mathbb{F}_q^3$, which is a bipartite graph $G = (L, R, E)$ where L is given by the set of lines in $\mathbb{F}_q^3$ (1-dimensional subspaces, generated by a nonzero element) and R is the set of planes (2-dimensional subspaces, generated by a pair of nonzero elements which are not a multiple of each other). We will analyze G through the two-step random walk from lines to planes to lines.

1. Prove that there are q^2+q+1 distinct lines and q^2+q+1 distinct planes in $\mathbb{F}_q^3$.
2. Prove that every plane contains $q+1$ distinct lines, and every line is adjacent to $q+1$ distinct planes.
3. Using these facts, prove that the two-step walk $A^T A$ has the following form:

$$A^T A = \frac{q}{(q+1)^2} I + \frac{1}{(q+1)^2} J$$

   where I is the identity matrix and J is the all ones matrix.

4. Combine this with Part (a) to prove that G is a one-sided $\frac{1}{\sqrt{q}}$-spectral expander.

Part (c): Flag Complex in d dimensions

We are ready to prove the main result. By Oppenheim's theorem, it is enough to prove that all links are connected and that $(d-3)$-links are good one-sided expanders. Let's start by proving every link of dimension less than $d-3$ is connected (OPTIONAL[1]). We break the proof into two parts:

1. Prove that $X_\emptyset$ is connected.
2. Given $j > 2$ and $\sigma \in X(d-j-1)$, let $i_1 \leq \ldots \leq i_j$ denote the dimensions missing from σ.[2]
   (a) Prove that if there exists a gap, i.e. ℓ such that $i_\ell + 1 < i_\ell$, X_σ is connected.
   (b) Otherwise, prove the link is connected by reducing to 1.

We now turn our attention to $(d-3)$-links (REQUIRED). Every $\sigma \in X(d-3)$ has the following structure:

$$\sigma = \{V_1 \subset V_2 \subset \ldots \subset V_{i-1} \subset V_{i+1} \subset \ldots \subset V_{j-1} \subset V_{j+1} \subset \ldots \subset \mathbb{F}_q^d\},$$

i.e. a complete flag missing two arbitrary subspaces of dimensions i and j. There are two cases of interest:

1. Prove that when $j = i+1$, X_σ is isomorphic to the 2-dimensional Flag Complex (hint: first consider $i = 1, j = 2$, then try to reduce general i, j to this case)
2. Prove that when $j > i+1$, X_σ is a complete bipartite graph with $q+1$ vertices on each side.

Finally, assuming $d \ll \sqrt{q}$, combine these facts with Part (b) to prove that the flag complex on $\mathbb{F}_q^d$ is a one-sided $O\left(\sqrt{\frac{1}{q}}\right)$-spectral expander

Problem 2: Oppenheim's Theorem

Prove Oppenheim's Trickling-down Theorem for **two-sided** local spectral expanders. That is:

Theorem (Oppenheim's Trickling Down Theorem)**.** *Let (X, Π) be a d-dimensional weighted simplicial complex satisfying the following two properties:*

1. *Every link of dimension $d-2$ is a **two-sided** γ-spectral expander*
2. *Every link of dimension $\leq d-2$ is connected.*

*Then (X, Π) is a **two-sided** $\frac{\gamma}{1-(d-2)\gamma}$-local spectral expander.*

In fact, it's possible to prove a stronger result: negative eigenvalues actually *improve* through trickle-down!

Bonus Problem (no points): Let X be a 3-dimensional complex. Prove that if the smallest negative eigenvalue across 1-links is η, then the smallest negative eigenvalue of the graph underlying X is at least $\frac{\eta}{1-\eta} > \eta$.

Notice that this lets us turn one-sided local-spectral expanders into two-sided local-spectral expanders just by truncating the complex at some dimension $k \ll d$!

[1] You may choose skip this part and assume the links are connected without proof

[2] By definition, σ is a partial flag (a complete flag with some subspaces removed). A dimension is "missing" if its corresponding subspace has been removed.