CSE 291 – Expanders – HW2

Question 1: equivalence of balanced codes, small biased sets, and Cayley expanders over F_2^n

We define three related objects:

Balanced codes: An ϵ-biased linear (n,k) code is a subspace $C \subset F_2^n$ of dimension k, such that for any nonzero codeword $x \in C$, its Hamming weight is very close to n/2; concretely: $\left(\frac{1-\epsilon}{2}\right) n \leq |x| \leq \left(\frac{1+\epsilon}{2}\right) n$

Small biased sets: A set $S \subset F_2^k$ is called ϵ-biased if for any nonzero $y \in F_2^k$, the inner products $\langle s, y \rangle$ for $s \in S$ obtain the values 0 or 1 about the same number of times; concretely, if

$$|[\langle s, y \rangle = 0] - [\langle s, y \rangle = 1]| \leq \epsilon$$

Cayley expanders over F_2^k: Let $S \subset F_2^k$. The Cayley graph $Cay(F_2^k, S)$ is an ϵ-spectral expander if its normalized eigenvalues satisfy $(|\lambda_2|, |\lambda_n|) \leq \epsilon$.

(a) Let $k < n$. Let M be a $k \times n$ matrix over F_2 of rank k. Prove that the following 3 properties are equivalent:
   (1) Let $R \subset F_2^n$ be the rows of M. They span an ϵ-biased linear (n,k) code
   (2) Let $S \subset F_2^k$ be the columns of M. They are an ϵ-biased set
   (3) Let $S \subset F_2^k$ be the columns of M. The Cayley graph $Cay\left(F_2^k, S\right)$ is an ϵ-spectral expander

(b) Let $S \subset F_2^k$ be a random set of size $n = O\left(\frac{k}{\epsilon^2}\right)$. Prove that with high probability, it is an ϵ-biased set. Conclude that there exist Cayley expander graphs over $G = F_2^k$ of logarithmic degree.

(c) Prove that this is essentially tight: any Cayley graph $Cay(F_2^k, S)$ with $|S| < k$ is not connected; in particular, it cannot be an expander.

Note: there are explicit constructions of small-biased sets getting closer to the bound obtained by the random construction. For example, the paper "Simple Constructions of Almost k-wise Independent Random Variables" give several constructions with $n = poly\left(\frac{k}{\epsilon}\right)$.

Question 2: existence of expanders with close to maximal vertex expansion

Prove claim 3.12 in the notes: There is a large enough constant $d = O(1)$ such that the following holds. For any large enough n, there exist $\left(n, \frac{3}{4}n; d\right)$ bipartite graphs which are (α, β) expanders for some constant $\alpha > 0$ and $\beta = \frac{3}{4}d$.

Question 3: Tree number

In this question you will prove the lower bound for the number of closed walks in a d-regular graph.

(a) Let T_d be the d-regular infinite tree. Let $t_{d,2k}$ denote the number of closed walks of length 2k starting at the root. We showed in class that $t_{d,2k} \geq (d-1)^k C_k$, where C_k is the Catalan number, counting the number of sequences in $\{-1,1\}^{2k}$ that sum to zero and that all their prefixes have non-negative sums. Prove using induction the formula $C_k = \frac{\binom{2k}{k}}{k+1}$.

(b) Prove that if G is a finite d-regular graph, then for any start node v, the number of closed walks of length 2k starting at v is at least $t_{d,2k}$.

Hint for (b): Prove T_d is a universal cover for d-regular graphs (see Lemma 4.5 in the notes)

Question 4: mixing time of lazy random walks

We saw that in order to prove that a random walk on a graph mixes, we need to bound $\lambda = (|\lambda_2|, |\lambda_n|)$. In some applications we will see later in the course, we only have a bound on λ_2. That is, we have a one-sided expander. In this question we will see that this suffices to bound the mixing time of a *lazy* random walk.

Concretely, let $G = (V, E)$ be a graph with normalized adjacency matrix M, which to recall corresponds to the standard (non-lazy) random walk on G. Consider a lazy random walk, where at each step, if we are at node $v \in V$, then with probability 50% we walk to a random neighbor u of v; and with probability 50% we stay at the node v. Let M' denote the transition matrix of the lazy random walk.

(a) Prove that $M' = \frac{M+I}{2}$ where I is the identity matrix. Use this to compute the eigenvalues of M'.

(b) Show that if $\lambda_2(M) \leq 1 - g$ then $\lambda(M') = max(|\lambda_2(M')|, |\lambda_n(M')|) \leq 1 - g/2$.

(c) Prove that the lazy random walk mixes fast on one-sided expanders. Concretely, if $\lambda_2(M) \leq 1 - g$ for any constant $g > 0$ then the mixing time of the lazy random walk is $T(\mathrm{G}, \varepsilon) = c \log(n/\epsilon)$ for some $c = c(g)$.