CSE 291 – Expanders – HW1

Question 1: maximal edge expansion of graphs

Let $G = (V, E)$ be a regular graph with $|V| = n$ nodes. Let $S \subseteq V$ be a random set, where each element is chosen independently with probability 50%.

(a) Prove that the expected size of the edge boundary of S is $|E|/2$.
(b) Use this to show that the edge expansion of G is at most $h(G) \leq \frac{1}{2} + o_n(1)$

Question 2: diameter of expander graphs

Let $G = (V, E)$ be a d-regular graph with $n = |V|$ nodes. Let $diam(G)$ denote the diameter of G, which is the maximal distance between a pair of nodes in the graph.

(a) Prove that $diam(G) \geq \Omega\left(\frac{\log n}{\log d}\right)$
(b) Assume that G is a λ-spectral expander with $\lambda \leq \frac{1}{3}$. Use the expander mixing lemma to prove that $diam(G) \leq O(\log n)$
(c) Use mixing of random walks to improve the bound for $\lambda \ll 1$: prove that $diam(G) \leq O\left(\frac{\log n}{\log\left(\frac{1}{\lambda}\right)}\right)$
(d) Use this to show that a d-regular λ-expander must have degree $d \geq \left(\frac{1}{\lambda}\right)^c$ for some constant $c > 0$. What is the best c that you can get?

Question 3: expander graphs are resilient to edge removal

In this question we show that expander graphs cannot be broken into small components by deleting few edges. We fix concrete values below for simplicity.

Prove that there is $\lambda_0 > 0$ such that the following holds. Let $G = (V, E)$ be a λ-spectral expander with $\lambda \leq \lambda_0$. If we delete 10% of the edges of G, there remains a connected component of size at least $|V|/2$.

Question 4: sharpened analysis of random walks avoiding a set

Prove theorem 2.7 in the notes: let $G = (V, E)$ be a λ-spectral expander, and let $B \subset V$ be a set of size $|B| = \beta|V|$. Recall that $P(B, t)$ denotes the probability that a t-step random walk, starting at a uniformly random vertex in V, stay entirely inside B. Prove that:

$$P(B, t) \leq (\beta(1 - \lambda) + \lambda)^t$$

In particular, for any $\beta, \lambda \in (0,1)$ the bound in nontrivial (decays exponentially with t)

Question 5: error reduction for two-sided algorithms

Prove theorem 2.9 in the notes: let A be a randomized algorithm for a decision problem, which answers correctly on each input with probability at least 2/3, and which uses k random bits. For some small enough constant λ, assume that there is a constant degree λ-expander on 2^k nodes. Show how to reduce the error probability to 2^{-t} using only $k + O(t)$ random bits.