

Reducing Incompletely Specified Functions

Which of the following assignment would result in an implementation with the fewest gates?

Id	a	b	F(a,b)
0	0	0	X
1	0	1	0
2	1	0	1
3	1	1	0

- A. $F(0,0)=1$
 B. $F(0,0)=0$
 C. Neither A or B
 D. Both A and B

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Boolean Algebra
Consensus Theorem
Shannon's Exp

Implementation

Other Gates

Specification



AND, OR, NOT

Schematic Diagram

Net list,

Switching expression

Obj min cost
 (max performance)



Search in solution space

Cost: wires, gates



Literals, product terms,
 sum terms

For two level logic (sum of products or product of sums),
 we want to minimize # of terms, and # of literals

$$f(a,b,c) = \underbrace{ab}_{12} + \underbrace{a'b'}_{34} + \underbrace{a'c}_{56}$$

3 prod terms
 6 literals

W3A
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Implementation: Specification => Logic Diagram

Karnaugh Map: A 2-dimensional truth table

Flow 1: Boolean Algebra

1. Specification
2. Truth table
3. Sum of products (SOP) or product of sums (POS) canonical form
4. Reduced expression using Boolean algebra
5. Schematic diagram of two level logic

Flow 2: K Map

1. Specification
2. Truth Table
3. Karnaugh Map (truth table in two dimensional space)
4. Reduce using K'Maps
5. Reduced expression (SOP or POS)
6. Schematic diagram of two level logic

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Truth Table vs. Karnaugh Map

2-variable function, $f(A,B)$

ID	A	B	f(A,B)
0	0	0	f(0,0)
1	0	1	f(0,1)
2	1	0	f(1,0)
3	1	1	f(1,1)

② ①

	B=0	B=1
A=0	f(0,0) 0	f(0,1) 1
A=1	f(1,0) 2	f(1,1) 3

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Truth Table

An example of 2-variable function, $f(A,B)$

ID	A	B	$f(A,B)$	minterm
0	0	0	0	0
1	0	1	1	$A'B$
2	1	0	1	AB'
3	1	1	1	AB

$$f(A,B) = A'B + AB' + AB$$

Function can be represented by sum of minterms:

$$f(A,B) = A'B + AB' + AB$$

This is not optimal however!

We want to minimize the number of literals and terms.

$$f(A,B) = \underbrace{A'B + AB'}_{B} + \underbrace{AB}_{B} + \underbrace{B}_{B} + \underbrace{A}_{A}$$

Consensus Theorem

$$= B + A$$

2 terms
2 literals

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To minimize the number of literals and terms.

We factor out common terms –

$$f(A,B) = A'B + AB' + AB$$

$$= \underline{A'B} + \underline{AB'} + \underline{AB + AB} \quad \text{Idempotent theorem}$$

$$= (A' + A)B + A(B' + B) = B + A$$

$$A + A = A$$

$$AB + AB = AB$$

Hence, we have

$$f(A,B) = A + B$$

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How can we guarantee the most reduced expression was reached?

- Boolean expressions can be minimized by combining terms
- K-maps minimize equations graphically

ID	A	B	$f(A,B)$
0	0	0	$f(0,0)$
1	0	1	$f(0,1)$
2	1	0	$f(1,0)$
3	1	1	$f(1,1)$

	B=0	B=1
A=0	$A'B'$	$A'B$
A=1	AB'	AB

→ A

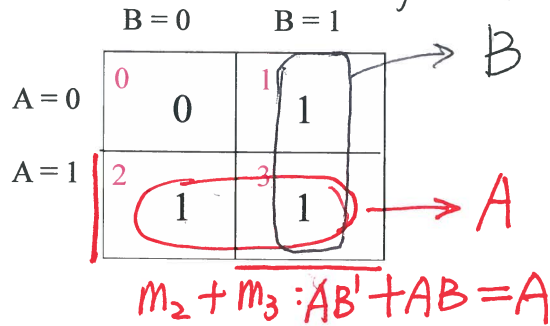
$$f(A,B) = A + B$$

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K-Map: Truth Table in 2 Dimensions

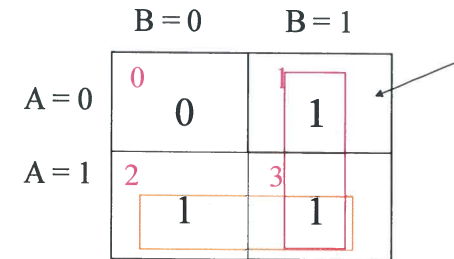
I D	A	B	f(A,B)
0	0	0	0
1	0	1	1
2	1	0	1
3	1	1	1

$A+B$



K-Map: Truth Table in 2 Dimensions

I D	A	B	f(A,B)
0	0	0	0
1	0	1	1
2	1	0	1
3	1	1	1



$f(A,B) = A + B$

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$f(A,B,C) = \overline{AB} + A'B'$

$(1,1) \Leftrightarrow AB$
 $(0,0) \Leftrightarrow A'B'$

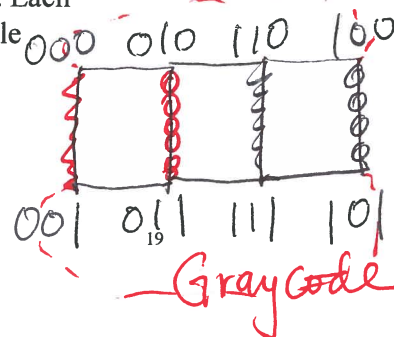
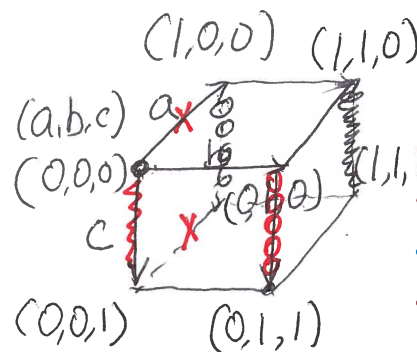
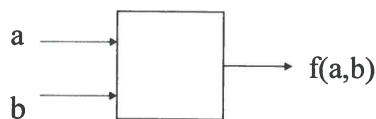
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Two-Variable K-maps

id	a	b	f(a,b)
0	0	0	f(0,0)
1	0	1	f(0,1)
2	1	0	f(1,0)
3	1	1	f(1,1)

possible 2-variable functions:

For 2 variables as inputs, we have $4=2^2$ entries. Each entry can be 0 or 1. Thus we have $16=2^4$ possible functions.



Representation of k-Variable Func.

A cube of 4 variables: (A,B,C,D)

Boolean Expression

Truth Table

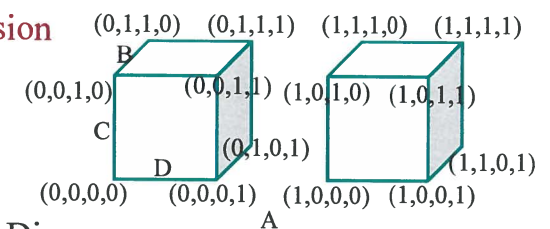
Cube

K Map

Binary Decision Diagram

And-Inverter Graph

– Ref: R. Brayton and A. Mishchenko, “ABC: an academic industrial-strength verification tool,” ACM Int. Conf. on Computer Aided Verification, pp. 24-40, 2010.



Truth table

Id	a	b	c	f(a,b,c)
0	0	0	0	1 abc'
1	0	0	1	0
2	0	1	0	1 abc'
3	0	1	1	0
4	1	0	0	1 abc'
5	1	0	1	0
6	1	1	0	1 abc'
7	1	1	1	0

Corresponding three variable K-map

Gray code (a,b)

	(0,0)	(0,1)	(1,1)	(1,0)
c = 0	0 1	2 1	6 1	4 1
c = 1	1 0	3 0	7 0	5 0

$$f(a,b,c) = c'$$

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$$\begin{aligned}
 m_0 + m_2 &: a'b'c' \\
 m_4 + m_6 &: ac' \\
 \Sigma m(0,2,4,6) &= c' \\
 f(a,b,c) &= c'
 \end{aligned}$$

Karnaugh Maps (K-Maps)

- K-maps minimize equations graphically
- Note that the label decides the order in the map

id

A	B	C	Y
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	0
1	1	1	0

	AB	00	01	11	10
C	0	1	0	0	0
1	1	0	0	0	0

$$a'b' \Leftrightarrow a'b'$$

	AB	00	01	11	10
C	0	$\bar{A}\bar{B}\bar{C}$	$\bar{A}B\bar{C}$	$AB\bar{C}$	$A\bar{B}\bar{C}$
1	1	$\bar{A}BC$	$\bar{A}BC$	ABC	$A\bar{B}C$

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K-map

- Circle 1's in adjacent squares
- Find rectangles which correspond to product terms in Boolean expression

A	B	C	Y
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	0
1	1	1	0

	AB	00	01	11	10
C	0	1	0	0	0
1	1	0	0	0	0

$$y(A,B) = A'B'C' + A'B'C = A'B'(C' + C) = A'B'$$

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Another 3-Input truth table

Id	a	b	c	f(a,b,c)
0	0	0	0	0
1	0	0	1	0
2	0	1	0	1
3	0	1	1	0
4	1	0	0	1
5	1	0	1	1
6	1	1	0	X
7	1	1	1	1

Corresponding K-map

$$f(a,b,c) = \sum m(4,5,6,7) + \sum m(2,6) = a + bc'$$

	(0,0)	(0,1)	(1,1)	(1,0)
c = 0	0	1	X	1
c = 1	0	0	1	1

$$\sum m(4,5,6,7) : a$$

$$\textcircled{1} \sum m(2) : a'bc'$$

$$\textcircled{2} \sum m(2,6) : a'bc' + abc' = bc'$$

Another 3-Input truth table

Id	a	b	c	f(a,b,c)
0	0	0	0	0
1	0	0	1	0
2	0	1	0	1
3	0	1	1	0
4	1	0	0	1
5	1	0	1	1
6	1	1	0	X
7	1	1	1	1

Corresponding K-map

	(0,0)	(0,1)	(1,1)	(1,0)
c = 0	0	1	X	1
c = 1	0	0	1	1

$$f(a,b,c) = a + bc'$$

3-input truth table

Id	a	b	c	f(a,b,c,d)
0	0	0	0	1
1	0	0	1	1
2	0	1	0	X
3	0	1	1	0
4	1	0	0	1
5	1	0	1	1
6	1	1	0	0
7	1	1	1	0

Corresponding K-map

	(0,0)	(0,1)	(1,1)	(1,0)
c = 0	1	X	0	1
c = 1	1	0	0	1

3-Input Truth table

Id	a	b	c	f(a,b,c,d)
0	0	0	0	1
1	0	0	1	1
2	0	1	0	X
3	0	1	1	0
4	1	0	0	1
5	1	0	1	1
6	1	1	0	0
7	1	1	1	0

Corresponding K-map

	(0,0)	(0,1)	(1,1)	(1,0)
c = 0	1	X	0	1
c = 1	1	0	0	1

Q: In the above case the minimal Boolean expression is obtained when the don't care term is:

- A. Included i.e. Circled in with one or more minterms that evaluate to 1
- B. Excluded when grouping the minterms that evaluate to one
- C. Either approach gives the minimal expression

Q: In the above case the minimal Boolean expression is obtained when the don't care term is:

- A. Included i.e. Circled in with one or more minterms that evaluate to 1
- B. Excluded when grouping the minterms that evaluate to one
- C. Either approach gives the minimal expression

3-input truth table with Don't cares

Id	a	b	c	f(a,b,c,d)
0	0	0	0	1
1	0	0	1	1
2	0	1	0	X
3	0	1	1	0
4	1	0	0	1
5	1	0	1	1
6	1	1	0	0
7	1	1	1	0

Corresponding K-map

	$b = 1$			
	(0,0)	(0,1)	(1,1)	(1,0)
$c = 0$	0 1	2 X	6 0	4 1
$c = 1$	1 1	3 0	7 0	5 1
	$a = 1$			

$$f(a,b,c) = b'$$

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Consensus Theorem in View of K Maps

$$\text{Consensus Theorem: } A'B + AC + BC = A'B + AC$$

A	B	C	Y
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

Y	AB	00	01	11	10
C	0	0	1	0	0
1	0	0	1	1	1

Y	AB	00	01	11	10
C	0	$\bar{A}\bar{B}\bar{C}$	$\bar{A}B\bar{C}$	$AB\bar{C}$	$A\bar{B}\bar{C}$
1	0	$\bar{A}\bar{B}C$	$\bar{A}BC$	ABC	$A\bar{B}C$

BC

$$B' + C'$$

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