

Combinational Logic Outlines

1. Introduction

- Scope *Reduce the scope, focus on fundamental*
- Boolean Algebra (Review)
 - Definition
 - Duality
 - AND/OR Gates
 - Expressions vs Circuits
- Handy Tools
 - DeMorgan's Theorem
 - Consensus Theorem
 - Shannon's Expansion*inversion*
bottom up
top down

2. Specification

3. Synthesis

K map

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CSE 140, Lecture 2

Combinational Logic

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HW: Team discussion encouraged

Work on the solution by oneself

Exam: Open book

No discussion

2 days 90% 80 minutes exam
10% open questions

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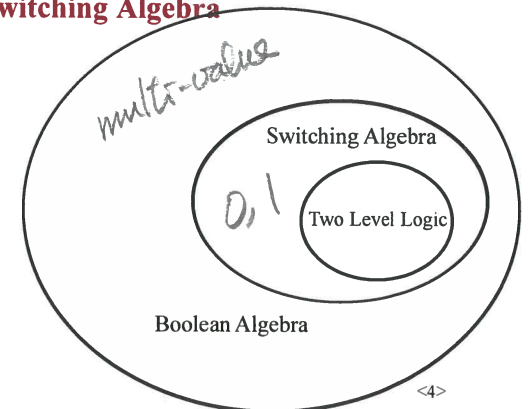
1.1 Combinational Logic: Scope

- Description
 - **Language:** e.g. C Programming, Verilog, VHDL
 - **Boolean algebra**
 - **Truth table:** Powerful engineering tool
- Design
 - Schematic Diagram
 - Inputs, Gates, Nets, Outputs *AND, OR, NOT*
- Goal
 - Validity: **correctness**, turnaround time
 - Performance: power, timing, **cost**
 - Testability: yield, diagnosis, robustness

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Scope: Boolean algebra, switching algebra, logic

- Boolean Algebra: multiple-valued logic, i.e. each variable have multiple values.
- Switching Algebra: binary logic, i.e. each variable can be either 1 or 0.
- **Boolean Algebra $\neq$ Switching Algebra**



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Scope: Switching Algebra (Binary Values)

- Typically consider only two discrete values:
 - 1's and 0's
 - 1, TRUE, HIGH
 - 0, FALSE, LOW
- 1 and 0 can be represented by specific voltage levels, rotating gears, fluid levels, etc.
- Digital circuits usually depend on specific voltage levels to represent 1 and 0
- Bit*: Binary digit

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1.2 Boolean Algebra (Review)

George Boole, 1815-1864

- Born to working class parents: Son of a shoemaker
- Taught himself mathematics and joined the faculty of Queen's College in Ireland.
- Wrote *An Investigation of the Laws of Thought* (1854): **systematize Aristotle's logic**
- Introduced binary variables
- Introduced the three fundamental logic operations: AND, OR, and NOT.



SAT/SMT

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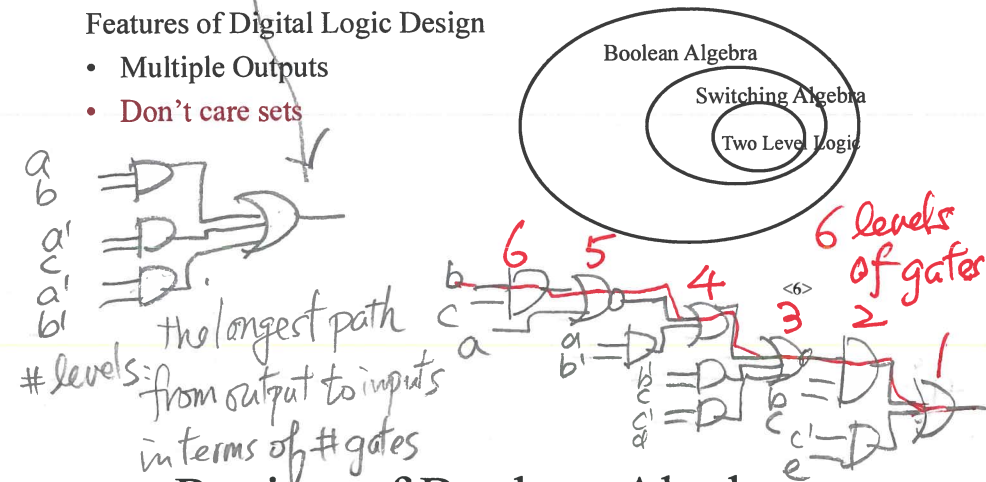
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Scope: Levels of Logic

- Multiple Level Logic: Many layers of two level logic with some inverters, e.g. $((((a+bc)'+ab') + b'c + c'd)'bc + c'e)$
(A network of two level logic)
- Two Level Logic: Sum of products, or product of sums, e.g. $ab + a'c + a'b', (a'+c)(a+b')(a+b+c')$

Features of Digital Logic Design

- Multiple Outputs
- Don't care sets



Review of Boolean Algebra

Let B be a nonempty set with two 2-input operations, a 1-input operation $'$ (complement), and two distinct elements 0 and 1. Then B is called a Boolean algebra if the following axioms hold.

- Associative laws: $(a+b)+c=a+(b+c)$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
- Commutative laws: $a+b=b+a$, $a \cdot b=b \cdot a$
- Distributive laws: $a+(b \cdot c)=(a+b) \cdot (a+c)$,
 $a \cdot (b+c)=a \cdot b + a \cdot c$
- Identity laws: $a+0=a$, $a \cdot 1=a \Rightarrow a \cdot 0=0$, $a+1=1$
- Complement laws: $a+a'=1$, $a \cdot a'=0$ **Annulment theorem**

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Review of Boolean Algebra: Duality

Associative laws	$(a+b)+c=a+(b+c)$	$(a \cdot b) \cdot c=a \cdot (b \cdot c)$
Commutative laws	$a+b=b+a$	$a \cdot b=b \cdot a$
Distributive laws	$a+(b \cdot c)=(a+b) \cdot (a+c)$	$a \cdot (b+c)=a \cdot b+a \cdot c$
Identity laws	$a+0=a$	$a \cdot 1=a$
Complement laws	$a+a'=1$	$a \cdot a'=0$

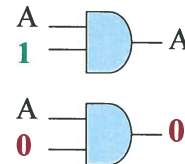
Duality: We swap all operators between (+,.) and interchange all elements between (0,1).

For a theorem if the statement can be proven with the laws of Boolean algebra, then the duality of the statement is also true.

1.3 Switching functions: Operators and Digital Logic Gates

id	A	B	Y
0	0	0	0
1	0	1	0
2	1	0	0
3	1	1	1

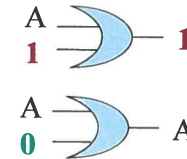
AND
 $Y=AB$



Input 0 dominates Y
0 blocks the output
1 passes signal A

A	B	Y
0	0	0
0	1	1
1	0	1
1	1	1

OR
 $Y=A+B$



Input 1 dominates Y
0 passes signal A
1 blocks the output

A	Y
0	1
1	0

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1.3 Switching functions: Operators and Digital Logic Gates

id	A	B	C	Y
0	0	0	0	0
1	0	0	1	0
2	0	1	0	0
3	0	1	1	0
4	1	0	0	0
5	1	0	1	0
6	1	1	0	0
7	1	1	1	1

AND
 $Y=ABC$

0 blocks the output
1 passes signal A
For AND, only one row is true (minterm)

a product term with all variables appear in the exp.

A	B	C	Y
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

OR
 $Y=A+B+C$

0 passes signal A
1 blocks the output
For OR, only one row is false (maxterm)

a sum term with all variables appear in the exp

1.3 Switching functions: Example on AND and OR

id	A	B	C	Y
0	0	0	0	0
1	0	0	1	1
2	0	1	0	0
3	0	1	1	0
4	1	0	0	0
5	1	0	1	0
6	1	1	0	0
7	1	1	1	0

AND
 $Y=A'B'C$

0 blocks the output
1 passes signal A
For AND, only one row is true (minterm)

a product term with all variables appear in the exp

A	B	C	Y
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	1

OR
 $Y=A'+B'+C$

0 passes signal A
1 blocks the output
For OR, only one row is false (maxterm)

a sum term with all variables appear in the exp

Canonical $f = m_1 + m_4 + m_5 = abc + abc' + abc$

$g = M_0 + M_1 + M_3 + M_4 + M_6 = \Pi M(0,1,3,4,6)$

Switching Expressions vs Logic Diagrams

Switching expression is related to logic implementation

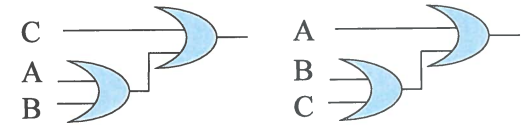
- Switching Expression: #literals, #operators
- Schematic Diagram: #gates, #nets, #pins

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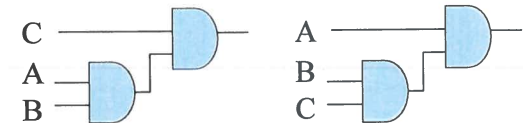
Laws and Logic Diagrams

Associativity Laws

$$(A+B) + C = A + (B+C)$$



$$(AB)C = A(BC)$$



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Laws and Logic Diagrams

Identity Laws

$$A \cdot 1 = A \quad A + 1 = 1$$

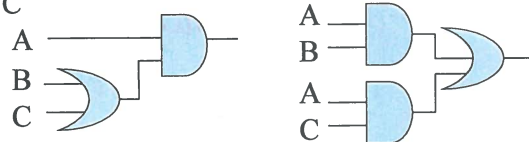
$$A \cdot 0 = 0 \quad A + 0 = A$$

Complement Laws

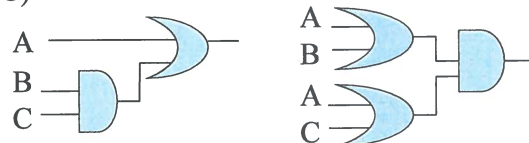
$$A + A' = 1 \quad A \cdot A' = 0$$

Distributive Laws

$$A \cdot (B+C) = A \cdot B + A \cdot C$$

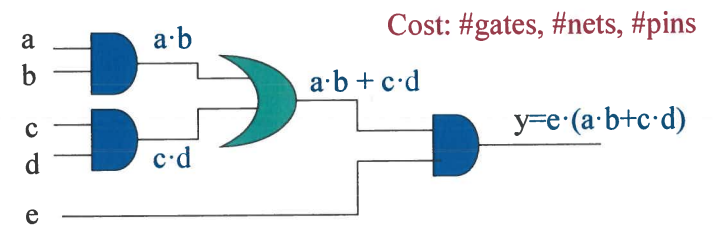


$$A+B \cdot C = (A+B) \cdot (A+C)$$



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Switching Expression and Logic



Cost: #gates, #nets, #pins

Schematic Diagram:

5 primary inputs

1 primary output

4 gates (3 ANDs, 1 OR)

9 signal nets

12 pins

Boolean Algebra:

5 variables

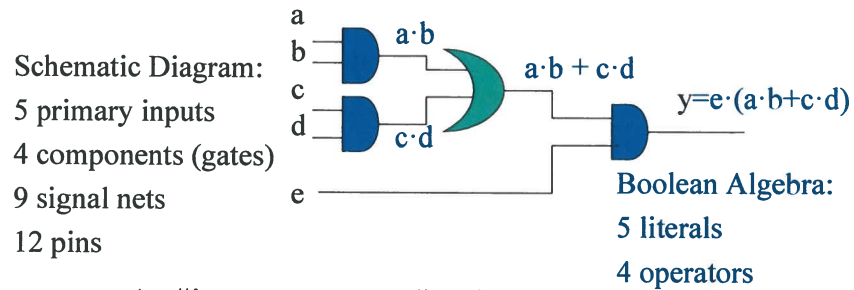
1 expression

4 operators (3 ANDs, 1 OR)

5 literals

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Switching Expression and Logic



- | | |
|------------|----------------------------------|
| A. #inputs | I. #variables |
| B. #gates | II. #operators |
| C. #nets | III. #variables + #operators |
| D. #pins | IV. #literals + 2 #operators - 1 |
| E. None | |

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Example: $f(a,b,c) = ab + a'b + a'b'$

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Example: $f(a,b,c) = (a' + c)(a + b')(a + b + c)$

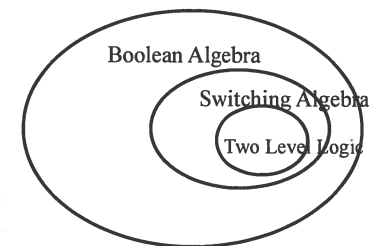
1.4 Handy Tools

Boolean Algebra

- DeMorgan's Law: Complements
- Consensus Theorem

Switching Logic

- Shannon's Expansion
- Truth Table
- Karnaugh Map (single output, two level logic)



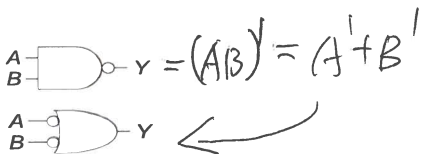
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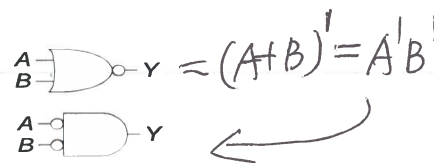
DeMorgan's Theorem and Digital Logic

T12. DeMorgan's Theorem $(A+B)' = A'B'$ $(AB)' = A' + B'$

- $Y = (AB)' = A' + B'$



- $Y = (A + B)' = A'B'$



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DeMorgan's Theorem: Bubble Pushing

- Pushing bubbles backward (from the output) or forward (from the inputs) changes the body of the gate from AND to OR or vice versa.
- Pushing a bubble from the output back to the inputs puts bubbles on all gate inputs.



- Pushing bubbles on *all* gate inputs forward toward the output puts a bubble on the output and changes the gate body.



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Consensus Theorem

- $AB + AC + B'C = AB + B'C$
- $(A+B)(A+C)(B'+C) = (A+B)(B'+C)$

The consensus of AB, B'C is: ?

if $AC=1$ then $AB+B'C=1$ logic

Exercise: to prove the reduction using

- (1) Venn Diagrams,
- (2) Boolean algebra,
- (3) Logic simulation and
- (4) Shannon's expansion

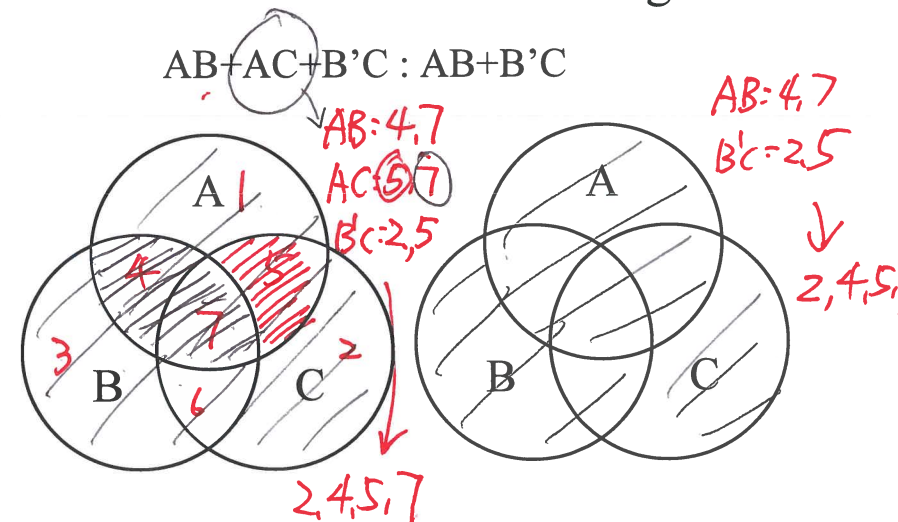
$$AB \cup B'C \supset AC \text{ set}$$

$$B: A \rightarrow AC$$

$$B': C$$

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Consensus Theorem: Venn Diagrams



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Consensus Theorem: Boolean Algebra

- $AB+AC+B'C$
 $=AB+B'C$
- $(A+B)(A+C)(B'+C)$
 $=(A+B)(B'+C)$

$LHS = AB+AC+B'C$
 $= AB+AC+1 \cdot B'C$ Identity
 $= AB+AC+(B+B')B'C$ Comp
 $= AB+ABC+AB'B'C+B'C$ Dist
 $= AB(1+C)+(A+1)B'C$ Dist
 $= AB+B'C$ Annulment
 $= RHS$

$LHS = (A+B)(A+C)(B'+C)$
 $= (A+B)(A+C+0)(B'+C)$
 $= (A+B)(A+C+BB')$ Comp
 $= (A+B)(A+C+BB')(B'+C)$
 $= (A+B)(A+C)(A+C+BB')(B'+C)$
 $= (A+B)(A+C)(A \cdot 0 + B'+C)$
 $= (A+B)(B'+C)$

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Consensus Theorem: Logic Simulation

$$f(A,B,C) = AB+AC+B'C$$

$$g(A,B,C) = AB+B'C$$



Index	A	B	C	AB	AC	B'C	f	g
0	0	0	0	0	0	0	0	0
1	0	0	1	0	0	1	1	1
2	0	1	0	0	0	0	0	0
3	0	1	1	0	0	0	0	0
4	1	0	0	0	0	0	0	0
5	1	0	1	0	1	1	1	1
6	1	1	0	1	0	0	1	1
7	1	1	1	1	1	0	1	1

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Consensus

Triangular inequality

$$d(A,B) + d(B,C) \geq d(A,C)$$

Quantum computing: entanglement.

Bell's inequality

$$(A, \bar{B}) \cup (B, \bar{C}) \supset A\bar{C}$$

$$\underline{AB' + BC'} \supset AC'$$

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Consensus Theorem (Examples)

Which one is not a consensus of the expressions on the right.

A. B

$$1. A+A'B = A \cdot 1 + A'B + 1 \cdot B$$

B. BC

$$2. A+A'BC = A + A'B + B = A+B$$

C. (B+C)D

$$3. A(B+C) + A'D = AB + AC + A'D$$

D. BCD

$$4. ABC + A'D + BCDE$$

E. BCDE

$$\begin{aligned}
 &AB + AB' \\
 &= AB + AB' + A \\
 &= A
 \end{aligned}$$

$$\begin{aligned}
 &BCD + BCDE \\
 &= BCD \cdot 1 + BCDE \\
 &= BCD(1+E) \\
 &= BCD
 \end{aligned}$$

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Shannon's Expansion

- Shannon's expansion assumes a switching algebra system
- Divide a switching function into smaller functions
- Pick a variable x , partition the switching function into two cases: $x=1$ and $x=0$

$$f(x,y,z,\dots) = xf(x=1,y,z,\dots) + x'f(x=0,y,z,\dots)$$

- For example

$$f(x) = xf(1) + x'f(0)$$

$$f(x,y) = xf(1,y) + x'f(0,y)$$

$f(x) = x'$: $f(1)=0$, $f(0)=1$
 $f(x) = xf(1) + x'f(0)$
 $= x \cdot 0 + x' \cdot 1 = x'$

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Shannon's Expansion: Example

$$f(x,y,z) = xf(x=1,y,z) + x'f(x=0,y,z)$$

$$f(x,y) = (x + f(x=0,y))(x' + f(x=1,y))$$

A. $?=0$

B. $?=1$

ABC f minterm
 001 1 $c \rightarrow db'c$
 101 0 $c \rightarrow a'b+c'$

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Shannon's Expansion (Example)

$$f(a,b,c) = a'b' + bc + ab'c$$

Shannon's expansion:

$$f(a,b,c) = af(1,b,c) + a'f(0,b,c)$$

$$f(1,b,c) = 1 \cdot b' + bc + 1 \cdot b'c = b' + bc + b'c = c$$

$$f(0,b,c) = 0 \cdot b' + bc + 0 \cdot b'c = bc + c = b' + c$$

$$f(a,b,c) = ac + a'(b' + c)$$

$$= ac + a'b' + a'c$$

$$= \underline{ac} + a'b' + \underline{a'c} + c$$

$$= a'b' + c$$

$b'c + c$
 $= bc + c \cdot 1$
 $= c(b+1)$
 $= c \cdot 1$
 $= c$

Shannon's Expansion

- Decompose the switching function into minterms

$$f(x,y) = xf(1,y) + x'f(0,y)$$

$$= x(yf(1,1) + y'f(1,0)) + x'(yf(0,1) + y'f(0,0))$$

$$= xyf(1,1) + xy'f(1,0) + x'yf(0,1) + x'y'f(0,0)$$

id	x	y	f(x,y)
0	0	0	f(0,0)
1	0	1	f(0,1)
2	1	0	f(1,0)
3	1	1	f(1,1)

If $f(0,0)=1 \rightarrow$ minterm $x'y'$
 If $f(0,1)=1 \rightarrow$ " $x'y$

Shannon's expansion can decompose a switching function into a truth table.

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