Rigid Body Motion

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Cross Product

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} a_y b_z - a_z b_y & a_z b_x - a_x b_z & a_x b_y - a_y b_x \end{bmatrix}$$

Properties of the Cross Product

- Non-commutative:

$$\mathbf{a} \times \mathbf{b} \neq \mathbf{b} \times \mathbf{a}$$

- Non-associative:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$$

Cross Product

$$\mathbf{a}\times\mathbf{b}=\begin{bmatrix}a_y b_z - a_z b_y & a_z b_x - a_x b_z & a_x b_y - a_y b_x\end{bmatrix}$$

$$\mathbf{c}=\mathbf{a}\times\mathbf{b}$$

$$c_x = 0\cdot b_x - a_z b_y + a_y b_z$$

$$c_y = a_z b_x + 0\cdot b_y - a_x b_z$$

$$c_z = -a_y b_x + a_x b_y + 0\cdot b_z$$

Cross Product

$$c_x = 0 \cdot b_x - a_z b_y + a_y b_z$$

$$c_y = a_z b_x + 0 \cdot b_y - a_x b_z$$

$$c_z = -a_z b_x + a_x b_y + 0 \cdot b_z$$

$$\begin{bmatrix} c_x \\ c_y \\ c_z \end{bmatrix} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \cdot \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

Cross Product

$$\begin{bmatrix} c_x \\ c_y \\ c_z \end{bmatrix} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \cdot \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

$$\mathbf{a} \times \mathbf{b} = \hat{\mathbf{a}} \cdot \mathbf{b}$$

$$\hat{\mathbf{a}} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

Hat Operator

- We've introduced the 'hat' operator which converts a vector into a *skew-symmetric* matrix ($\hat{\mathbf{a}}^T = -\hat{\mathbf{a}}$)
- This allows us to turn a cross product of two vectors into a dot product of a matrix and a vector
- This is mainly for algebraic convenience, as the dot product is associative (although still not commutative)

$$\hat{\mathbf{a}} \cdot \mathbf{b} = \mathbf{a} \times \mathbf{b}$$

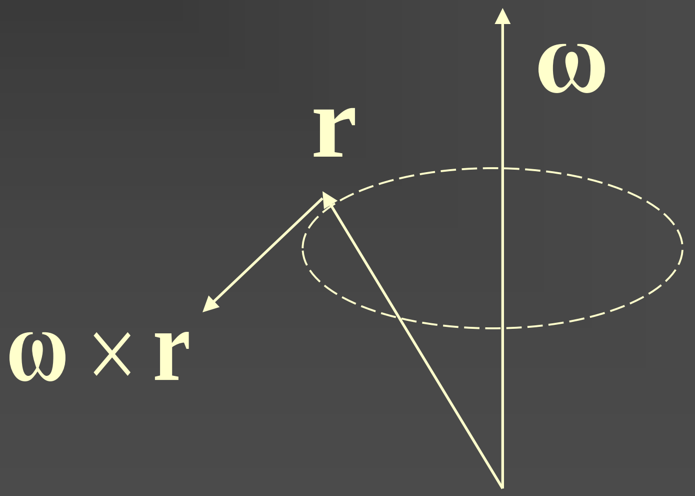
$$\hat{\mathbf{a}} \cdot \mathbf{b} \neq \mathbf{b} \cdot \hat{\mathbf{a}} \quad \text{(non commutative)}$$

$$\hat{\mathbf{a}} \cdot \left(\hat{\mathbf{b}} \cdot \mathbf{c}\right) = \left(\hat{\mathbf{a}} \cdot \hat{\mathbf{b}}\right) \cdot \mathbf{c} \qquad \text{(associative)}$$

Derivative of a Rotating Vector

- Let's say that vector **r** is rotating around the origin, maintaining a fixed distance
- At any instant, it has an angular velocity of **ω**

$$\frac{d\mathbf{r}}{dt} = \boldsymbol{\omega} \times \mathbf{r}$$



Derivative of Rotating Matrix

- If matrix **A** is a rigid 3x3 matrix rotating with angular velocity **ω**
- This implies that the **a**, **b**, and **c** axes must be rotating around **ω**
- The derivatives of each axis are **ω**x**a**, **ω**x**b**, and **ω**x**c**, and so the derivative of the entire matrix is:

$$\frac{d\mathbf{A}}{dt} = \boldsymbol{\omega} \times \mathbf{A} \qquad = \hat{\boldsymbol{\omega}} \cdot \mathbf{A}$$

Product Rule

- The product rule defines the derivative of products

$$\frac{d(ab)}{dt} = \frac{da}{dt}b + a\frac{db}{dt}$$

$$\frac{d(abc)}{dt} = \frac{da}{dt}bc + a\frac{db}{dt}c + ab\frac{dc}{dt}$$

Product Rule

- It can be extended to vector and matrix products as well

$$\frac{d(\mathbf{a}\cdot\mathbf{b})}{dt}=\frac{d\mathbf{a}}{dt}\cdot\mathbf{b}+\mathbf{a}\cdot\frac{d\mathbf{b}}{dt}$$

$$\frac{d(\mathbf{a}\times\mathbf{b})}{dt}=\frac{d\mathbf{a}}{dt}\times\mathbf{b}+\mathbf{a}\times\frac{d\mathbf{b}}{dt}$$

$$\frac{d(\mathbf{A}\cdot\mathbf{B})}{dt}=\frac{d\mathbf{A}}{dt}\cdot\mathbf{B}+\mathbf{A}\cdot\frac{d\mathbf{B}}{dt}$$

Eigenvalue Equation

- Lets say we have a known matrix **M** and we want to know if there is any vector **x** and scalar *s* such that

$$\mathbf{M}\mathbf{x} = s\mathbf{x}$$

- This is known as an eigenvalue equation, and for a NxN matrix, there should be up to N eigenvectors $\mathbf{x}_i$ and N eigenvalues s_i that satisfy the equation
- If **M** is a symmetric matrix (i.e., $\mathbf{M}^T = \mathbf{M}$) then all of the eigenvalues will be real numbers and the eigenvectors will be real, orthonormal vectors (otherwise, some of the eigenvalues/eigenvectors will be complex)

Symmetric Matrix

- If we have a symmetric matrix $\mathbf{M}$, we can *diagonalize* it:

$$\mathbf{M}_0 = \mathbf{A}^T \cdot \mathbf{M} \cdot \mathbf{A}$$

- Where $\mathbf{M}_0$ is a diagonal matrix and $\mathbf{A}$ is an orthonormal (pure rotation) matrix
- The columns of $\mathbf{A}$ are the eigenvectors of $\mathbf{M}$ and the diagonal elements in $\mathbf{M}_0$ are the corresponding eigenvalues
- The *symmetric Jacobi algorithm* is a simple and effective matrix algorithm for computing this diagonalization

Symmetric Matrix Diagonalization

- $$\mathbf{M} = \begin{bmatrix} M_{xx} & M_{xy} & M_{xz} \\ M_{xy} & M_{yy} & M_{yz} \\ M_{xz} & M_{yz} & M_{zz} \end{bmatrix}$$

$$\mathbf{M}_0 = \mathbf{A}^T \cdot \mathbf{M} \cdot \mathbf{A} \quad where \quad \mathbf{M}_0 = \begin{bmatrix} M_x & 0 & 0 \\ 0 & M_y & 0 \\ 0 & 0 & M_z \end{bmatrix}$$

Dynamics of Particles

Kinematics of a Particle

$\mathbf{x}$ position

$\mathbf{v} = \frac{d\mathbf{x}}{dt}$ velocity

$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{x}}{dt^2}$ acceleration

Mass, Momentum, and Force

m	mass
$\mathbf{p} = m\mathbf{v}$	momentum
$\mathbf{f} = \dfrac{d\mathbf{p}}{dt} = m\mathbf{a}$	force

Moment of Momentum

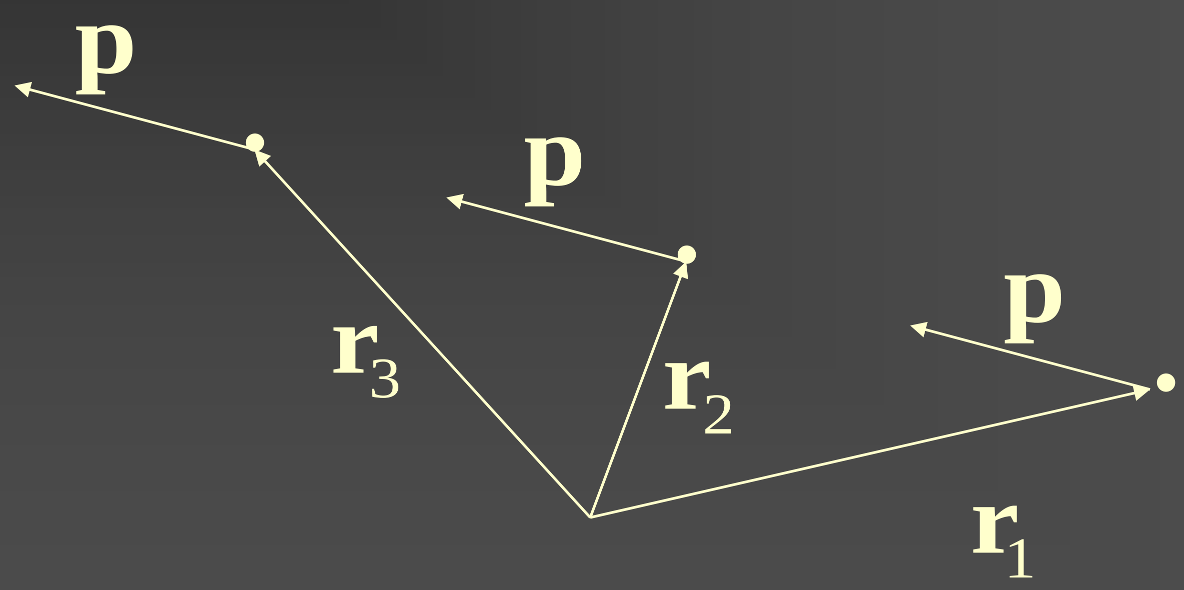
- The moment of momentum is a vector

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

- Also known as angular momentum (the two terms mean basically the same thing, but are used in slightly different situations)
- Angular momentum has parallel properties with linear momentum
- In particular, like the linear momentum, angular momentum is conserved in a mechanical system
- It is typically represented with a capital **L**, which is unfortunately inconsistent with our standard of using lowercase for vectors…

Moment of Momentum

- **L** is the same for all three of these particles

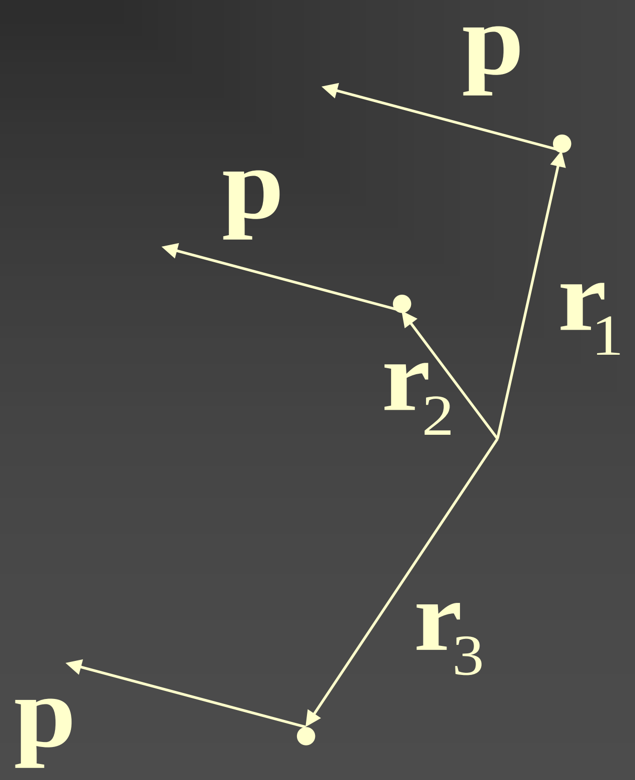


$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

Moment of Momentum

- **L** is different for all of these particles

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$



Moment of Force (Torque)

- The *moment of force* (or *torque*) about a point is the rate of change of the moment of momentum about that point

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}$$

Moment of Force (Torque)

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt} = \frac{d\mathbf{r}}{dt} \times \mathbf{p} + \mathbf{r} \times \frac{d\mathbf{p}}{dt}$$

$$\boldsymbol{\tau} = \mathbf{v} \times \mathbf{p} + \mathbf{r} \times \mathbf{f}$$

$$\boldsymbol{\tau} = \mathbf{v} \times \left(m\mathbf{v}\right) + \mathbf{r} \times \mathbf{f}$$

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{f}$$

Rotational Inertia

- **L**=**r**x**p** is a general expression for the moment of momentum of a particle
- In a case where we have a particle rotating around the origin while keeping a fixed distance, we can re-express the moment of momentum in terms of it’s angular velocity **ω**

Rotational Inertia

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

$$\mathbf{L} = \mathbf{r} \times (m\mathbf{v}) = m\mathbf{r} \times \mathbf{v}$$

$$\mathbf{L} = m\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r}) = -m\mathbf{r} \times (\mathbf{r} \times \boldsymbol{\omega})$$

$$\mathbf{L} = -m\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} \cdot \boldsymbol{\omega}$$

$$\mathbf{L} = \mathbf{I} \cdot \boldsymbol{\omega}$$

$$\mathbf{I} = -m\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}$$

Rotational Inertia

$$\mathbf{I} = -m\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}$$

$$\mathbf{I} = -m\begin{bmatrix} 0 & -r_z & r_y \\ r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{bmatrix}\cdot\begin{bmatrix} 0 & -r_z & r_y \\ r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{bmatrix}$$

$$\mathbf{I} = -m\begin{bmatrix} -r_y^2 - r_z^2 & r_x r_y & r_x r_z \\ r_x r_y & -r_x^2 - r_z^2 & r_y r_z \\ r_x r_z & r_y r_z & -r_x^2 - r_y^2 \end{bmatrix}$$

Rotational Inertia

$$\mathbf{I} = \begin{bmatrix} m\left(r_y^2 + r_z^2\right) & -mr_xr_y & -mr_xr_z \\ -mr_xr_y & m\left(r_x^2 + r_z^2\right) & -mr_yr_z \\ -mr_xr_z & -mr_yr_z & m\left(r_x^2 + r_y^2\right) \end{bmatrix}$$

$$\mathbf{L} = \mathbf{I} \cdot \boldsymbol{\omega}$$

Rotational Inertia

- The rotational inertia matrix I is a 3x3 matrix that is essentially the rotational equivalent of mass
- It relates the angular momentum of a system to its angular velocity by the equation

$$\mathbf{L} = \mathbf{I} \cdot \boldsymbol{\omega}$$

- This is similar to how mass relates linear momentum to linear velocity, but rotation adds additional complexity

$$\mathbf{p} = m\mathbf{v}$$

Systems of Particles

Systems of Particles

$$m_{total} = \sum_{i=1}^{n} m_i$$ total mass of all particles

$$\mathbf{x}_{cm} = \frac{\sum m_i \mathbf{x}_i}{\sum m_i}$$ position of center of mass

$$\mathbf{p}_{cm} = \sum \mathbf{p}_i = \sum m_i \mathbf{v}_i$$ total momentum

Velocity of Center of Mass

$$\mathbf{v}_{cm} = \frac{d\mathbf{x}_{cm}}{dt} = \frac{d}{dt}\frac{\sum m_i \mathbf{x}_i}{\sum m_i}$$

$$\mathbf{v}_{cm} = \frac{\sum m_i \frac{d\mathbf{x}_i}{dt}}{\sum m_i} = \frac{\sum m_i \mathbf{v}_i}{\sum m_i}$$

$$\mathbf{v}_{cm} = \frac{\mathbf{p}_{cm}}{m_{total}}$$

$$\mathbf{p}_{cm} = m_{total}\mathbf{v}_{cm}$$

Force on a Particle

- The change in momentum of the center of mass is equal to the sum of all of the forces on the individual particles
- This means that the resulting change in the total momentum is independent of the location of the applied force

$$\mathbf{p}_{cm} = \sum \mathbf{p}_i$$

$$\frac{d\mathbf{p}_{cm}}{dt} = \frac{d\sum \mathbf{p}_i}{dt} = \sum \frac{d\mathbf{p}_i}{dt} = \sum \mathbf{f}_i$$

Systems of Particles

- The total moment of momentum around the center of mass is:

$$\mathbf{L}_{cm} = \sum \mathbf{r}_i \times \mathbf{p}_i$$

$$\mathbf{L}_{cm} = \sum \left(\mathbf{x}_i - \mathbf{x}_{cm}\right) \times \mathbf{p}_i$$

Torque in a System of Particles

$$\mathbf{L}_{cm} = \sum \mathbf{r}_i \times \mathbf{p}_i$$

$$\boldsymbol{\tau}_{cm} = \frac{d\mathbf{L}_{cm}}{dt} = \frac{d\sum \mathbf{r}_i \times \mathbf{p}_i}{dt}$$

$$\boldsymbol{\tau}_{cm} = \frac{\sum d\left(\mathbf{r}_i \times \mathbf{p}_i\right)}{dt}$$

$$\boldsymbol{\tau}_{cm} = \sum \left(\mathbf{r}_i \times \mathbf{f}_i\right)$$

Systems of Particles

- We can see that a system of particles behaves a lot like a particle itself
- It has a mass, position (center of mass), momentum, velocity, acceleration, and it responds to forces

$$\mathbf{f}_{cm} = \sum \mathbf{f}_i$$

- We can also define it's angular momentum and relate a change in system angular momentum to a force applied to an individual particle

$$\boldsymbol{\tau}_{cm} = \sum \left(\mathbf{r}_i \times \mathbf{f}_i \right)$$

Internal Forces

- If forces are generated within the particle system (say from gravity, or springs connecting particles) they must obey Newton's Third Law (every action has an equal and opposite reaction)
- This means that internal forces will balance out and have no net effect on the total momentum of the system
- As those opposite forces act along the same line of action, the torques on the center of mass cancel out as well

Dynamics of Rigid Bodies

Kinematics of a Rigid Body

- For the position of the center of mass of the rigid body:

$$\mathbf{x}_{cm}$$

$$\mathbf{v}_{cm} = \frac{d\mathbf{x}_{cm}}{dt}$$

$$\mathbf{a}_{cm} = \frac{d\mathbf{v}_{cm}}{dt} = \frac{d^2\mathbf{x}_{cm}}{dt^2}$$

Kinematics of a Rigid Body

- For the orientation of the rigid body:

$\mathbf{A}$	3x3 orientation matrix
$\boldsymbol{\omega}$	angular velocity
$\bar{\boldsymbol{\omega}} = \frac{d\boldsymbol{\omega}}{dt}$	angular acceleration

Rigid Bodies

- We treat a rigid body as a system of particles, where the distance between any two particles is fixed
- We will assume that internal forces are generated to hold the relative positions fixed. These internal forces are all balanced out with Newton’s third law, so that they all cancel out and have no effect on the total momentum or angular momentum
- The rigid body can actually have an infinite number of particles, spread out over a finite volume
- Instead of mass being concentrated at discrete points, we will consider the density as being variable over the volume

Rigid Body Mass

- With a system of particles, we defined the total mass as:

$$m = \sum_{i=1}^{n} m_i$$

- For a rigid body, we will define it as the integral of the density ρ over some volumetric domain Ω

$$m = \int_{\Omega} \rho \, d\Omega$$

Rigid Body Center of Mass

- The center of mass is:

$$\mathbf{x}_{cm} = \frac{\int \rho \mathbf{x} d\Omega}{\int \rho d\Omega}$$

Rotational Inertia of a Particle

- Recall that the rotational inertia for a single particle of mass *m* as position **r** is:

$$\mathbf{I} = \begin{bmatrix} m\left({r_y}^2 + {r_z}^2\right) & -mr_xr_y & -mr_xr_z \\ -mr_xr_y & m\left({r_x}^2 + {r_z}^2\right) & -mr_yr_z \\ -mr_xr_z & -mr_yr_z & m\left({r_x}^2 + {r_y}^2\right) \end{bmatrix}$$

Rigid Body Rotational Inertia

$$\mathbf{I} = \begin{bmatrix} \int \rho\left(r_y^2 + r_z^2\right) d\Omega & -\int \rho r_x r_y \, d\Omega & -\int \rho r_x r_z \, d\Omega \\ -\int \rho r_x r_y \, d\Omega & \int \rho\left(r_x^2 + r_z^2\right) d\Omega & -\int \rho r_y r_z \, d\Omega \\ -\int \rho r_x r_z \, d\Omega & -\int \rho r_y r_z \, d\Omega & \int \rho\left(r_x^2 + r_y^2\right) d\Omega \end{bmatrix}$$

$$\mathbf{I} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{bmatrix}$$

Rigid Body Rotational Inertia

- The rigid body rotational inertia is a 3x3 symmetric matrix that encodes the distribution of mass around the center of mass
- It is calculated by calculating the integrals on the previous slide by integrating over the volume of the rigid body where **r** indicates the vector from the center of mass to the position of the volume integration element and ρ represents the density at that location
- These integrals can be calculated with analytical formulas for simple shapes like spheres, cylinders, and boxes
- There also exists an *analytical* technique for computing them on triangle meshes as well (Mirtich-Eberly algorithm)

Rotational Inertia Diagonalization

- As the rotational inertia matrix is symmetric, we can diagonalize it and find the orthonormal matrix **A**:

$$\mathbf{I}_0 = \mathbf{A}^T \cdot \mathbf{I} \cdot \mathbf{A}$$

- We are essentially finding the orientation for the rigid body such that its rotational inertia matrix is diagonal
- When it is rotated into this coordinate system, the x, y, and z axes define the *principal axes*
- Typically, we like to model the rigid body such that it is oriented this way (i.e., in local space, the center of mass is at the origin and the principal axes line up with x, y, and z)
- That way, its rotational inertia properties can be represented with three numbers (I , I , and I) and the matrix **A** is the matrix that orients the rigid

Diagonalization of Rotational Inertia

- $$\mathbf{I} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{bmatrix}$$

$$\mathbf{I}_0 = \mathbf{A}^T \cdot \mathbf{I} \cdot \mathbf{A} \quad where \quad \mathbf{I}_0 = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix}$$

Rotational Inertia of a Box

- Fox a box of mass *m* and dimensions *a* x *b* x *c*:

$$I_x = \frac{m}{12}(b^2 + c^2)$$

$$I_y = \frac{m}{12}(a^2 + c^2)$$

$$I_z = \frac{m}{12}(a^2 + b^2)$$

Rotational Inertia of a Sphere

- For a solid sphere of mass *m* and radius *r*:

$$I_x = I_y = I_z = \frac{2mr^2}{5}$$

Rotational Inertia

- If we have modeled the rigid body such that it the origin is at the center of mass and the principal axes line up with x, y, and z, then the values of m, I_x, I_y, and I_z tell us everything we need to know about the mass and its distribution that we need to know
- When we orient our rigid body in space with a matrix **A**, the rotational inertia matrix $\mathbf{I}$ in world space is:

Derivative of Rotational Inertial

$$\frac{d\mathbf{I}}{dt} = \frac{d\left(\mathbf{A}\cdot\mathbf{I}_0\cdot\mathbf{A}^T\right)}{dt} = \frac{d\mathbf{A}}{dt}\cdot\mathbf{I}_0\cdot\mathbf{A}^T + \mathbf{A}\cdot\mathbf{I}_0\cdot\left(\frac{d\mathbf{A}}{dt}\right)^T$$

$$\frac{d\mathbf{I}}{dt} = \boldsymbol{\omega}\times\mathbf{A}\cdot\mathbf{I}_0\cdot\mathbf{A}^T + \mathbf{A}\cdot\mathbf{I}_0\cdot\left(\boldsymbol{\omega}\times\mathbf{A}\right)^T$$

$$\frac{d\mathbf{I}}{dt} = \boldsymbol{\omega}\times\mathbf{I} + \mathbf{A}\cdot\mathbf{I}_0\cdot\left(\hat{\boldsymbol{\omega}}\cdot\mathbf{A}\right)^T$$

$$\frac{d\mathbf{I}}{dt} = \boldsymbol{\omega}\times\mathbf{I} + \mathbf{A}\cdot\mathbf{I}_0\cdot\left(\mathbf{A}^T\cdot\hat{\boldsymbol{\omega}}^T\right) = \boldsymbol{\omega}\times\mathbf{I} + \mathbf{I}\cdot\hat{\boldsymbol{\omega}}^T$$

$$\frac{d\mathbf{I}}{dt} = \boldsymbol{\omega}\times\mathbf{I} - \mathbf{I}\cdot\hat{\boldsymbol{\omega}}$$

Derivative of Angular Momentum

$$\mathbf{L} = \mathbf{I} \cdot \boldsymbol{\omega}$$

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt} = \frac{d\mathbf{I}}{dt} \cdot \boldsymbol{\omega} + \mathbf{I} \cdot \frac{d\boldsymbol{\omega}}{dt}$$

$$\boldsymbol{\tau} = \left(\boldsymbol{\omega} \times \mathbf{I} - \mathbf{I} \cdot \hat{\boldsymbol{\omega}}\right) \cdot \boldsymbol{\omega} + \mathbf{I} \cdot \overline{\boldsymbol{\omega}}$$

$$\boldsymbol{\tau} = \boldsymbol{\omega} \times \mathbf{I} \cdot \boldsymbol{\omega} - \mathbf{I} \cdot \hat{\boldsymbol{\omega}} \cdot \boldsymbol{\omega} + \mathbf{I} \cdot \overline{\boldsymbol{\omega}}$$

$$\boldsymbol{\tau} = \boldsymbol{\omega} \times \mathbf{I} \cdot \boldsymbol{\omega} + \mathbf{I} \cdot \overline{\boldsymbol{\omega}}$$

Newton-Euler Equations

$$\mathbf{f} = m\mathbf{a}$$

$$\boldsymbol{\tau} = \boldsymbol{\omega} \times \mathbf{I} \cdot \boldsymbol{\omega} + \mathbf{I} \cdot \overline{\boldsymbol{\omega}}$$

Applied Forces & Torques

$$\mathbf{f}_{cg} = \sum \mathbf{f}_i$$

$$\boldsymbol{\tau}_{cg} = \sum \left(\mathbf{r}_i \times \mathbf{f}_i \right)$$

$$\mathbf{a} = \frac{1}{m}\mathbf{f}$$

$$\overline{\boldsymbol{\omega}} = \mathbf{I}^{-1} \cdot \left(\boldsymbol{\tau} - \boldsymbol{\omega} \times \mathbf{I} \cdot \boldsymbol{\omega} \right)$$

Properties of Rigid Bodies

m	$\mathbf{I}$
$\mathbf{x}$	$\mathbf{A}$
$\mathbf{v}$	$\boldsymbol{\omega}$
$\mathbf{a}$	$\overline{\boldsymbol{\omega}}$
$\mathbf{p} = m\mathbf{v}$	$\mathbf{L} = \mathbf{I} \cdot \boldsymbol{\omega}$
$\mathbf{f} = m\mathbf{a}$	$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{f} = \boldsymbol{\omega} \times \mathbf{I} \cdot \boldsymbol{\omega} + \mathbf{I} \cdot \overline{\boldsymbol{\omega}}$

Rigid Body Simulation

```
RigidBody {
    void Update(float time);
    void ApplyForce(Vector3 &f,Vector3 &pos);
private:
    // constants
    float Mass;
    Vector3 RotInertia;     // Ix, Iy, & Iz from diagonal inertia

    // variables
    Matrix34 Mtx;           // contains position & orientation
    Vector3 Momentum,AngMomentum;

    // accumulators
    Vector3 Force,Torque;
};
```

Rigid Body Simulation

```
RigidBody::ApplyForce(Vector3 &f,Vector3 &pos) {
    Force += f;
    Torque += (pos-Mtx.d) x f
}
```

Rigid Body Simulation

```
RigidBody::Update(float time) {
    // Update position
    Momentum += Force * time;
    Mtx.d += (Momentum/Mass) * time;    // Mtx.d = position

    // Update orientation
    AngMomentum += Torque * time;
    Matrix33 I = Mtx·I0·MtxT          // A·I0·AT
    Vector3 ω = I-1·L
    float angle = |ω| * time;         // magnitude of ω
    Vector3 axis = ω;
    axis.Normalize();
    Mtx.RotateUnitAxis(axis,angle);
}
```

Where: $I = Mtx \cdot I_0 \cdot Mtx^T$ ($A \cdot I_0 \cdot A^T$), $\omega = I^{-1} \cdot L$