Fluid Dynamics

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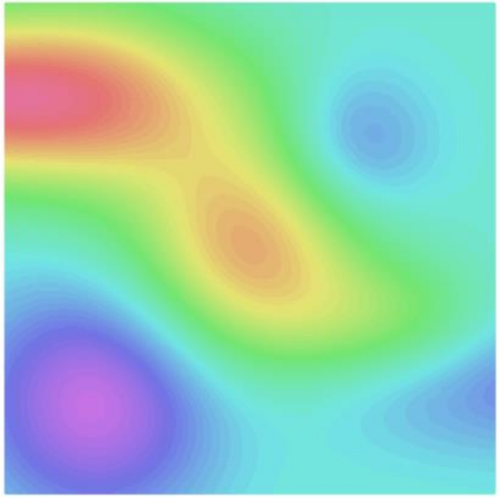
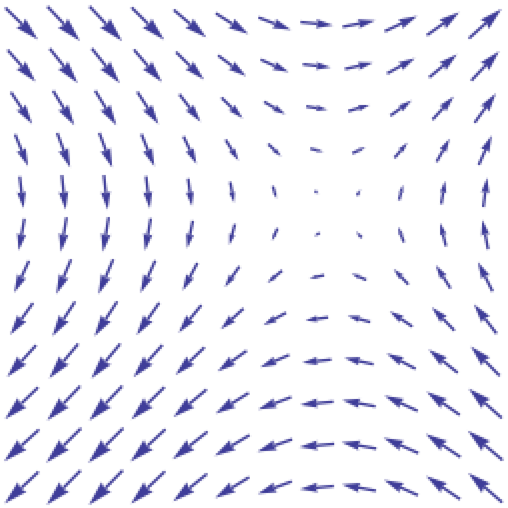
Fluid Dynamics

- *Fluid dynamics* refers to the physics of fluid motion
- The *Navier-Stokes equation* describes the motion of fluids and can appear in many forms
- Note that ‘fluid’ can mean both liquids and gasses, as both are described by the same equations
- *Computational fluid dynamics* (CFD) refers to the large body of computational techniques involved in simulating fluid motion. CFD is used extensively in engineering for aerodynamic design and analysis of vehicles and other systems. Some of the techniques have been borrowed by the computer graphics community
- We can use fluid dynamics to simulate smoke, fire, water, liquids, viscous fluids, and even semi-solid materials

Fields



- A *field* is a function of position **x** and may vary over time *t*
- A *scalar field* such as *s*(**x**,*t*) assigns a scalar value to every point in space. A good example of a scalar field would be the temperature in a room
- A *vector field* such as **v**(**x**,*t*) assigns a vector to every point in space. An example of a vector field would be the velocity of the air



Del Operations

- Del: $\nabla \quad = \left[\frac{\partial}{\partial x} \quad \frac{\partial}{\partial y} \quad \frac{\partial}{\partial z}\right]^T$
- Gradient: $\nabla s \quad = \left[\frac{\partial s}{\partial x} \quad \frac{\partial s}{\partial y} \quad \frac{\partial s}{\partial z}\right]^T$
- Divergence: $\nabla \cdot \mathbf{v} \quad = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$
- Curl: $\nabla \times \mathbf{v} \quad = \left[\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \quad \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \quad \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right]^T$
- Laplacian: $\nabla^2 s \quad = \frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2} + \frac{\partial^2 s}{\partial z^2}$

Frame of Reference

- When describing fluid motion, it is important to be consistent with the frame of reference
- In fluid dynamics, there are two main ways of addressing this
- With the ***Eulerian* frame of reference**, we describe the motion of the fluid from some fixed point in space
- With the ***Lagrangian* frame of reference**, we describe the motion of the fluid from the point of view moving with the fluid itself
- Eulerian simulations typically use a fixed grid or similar structure and store velocities at every point in the grid
- Lagrangian simulations typically use particles that move with the fluid itself. Velocities are stored on the particles that are irregularly spaced throughout the domain
- We will stick with an Eulerian point of view today, but we will look at Lagrangian methods in the next lecture when we discuss particle based fluid simulation
- Note: we can also use the *arbitrary Lagrangian-Eulerian* (ALE) frame of reference, which is a mix between the two. This is sometimes used in solid simulations with very large plastic deformations

Velocity Field

- We will describe the equations of motion for a basic incompressible fluid (such as air or water)
- To keep it simple, we will assume uniform density and temperature
- The main field that we are interested in therefore, is the velocity $\mathbf{v}(\mathbf{x}, t)$
- We assume that our field is defined over some domain (such as a rectangle or box) and that we have some numerical representation of the field (such as a uniform grid of velocity vectors)
- We will effectively be applying Newton's second law by computing a force everywhere on the grid, and then converting it to an acceleration by $\mathbf{f} = m\mathbf{a}$, however, as we are assuming uniform density (mass/volume), then the m term is always constant, and we can assume that it is just 1.0
- Therefore, we are really just interested in computing the acceleration $\frac{d\mathbf{v}}{dt}$ at every point on the grid

Transport Operations

Transport Equations

- In order to understand the equations of fluid dynamics, we will first look at some simpler examples of *transport equations* as well as some related concepts:
  - Advection
  - Convection
  - Diffusion
  - Viscosity
  - Pressure gradient
  - Incompressibility

Advection

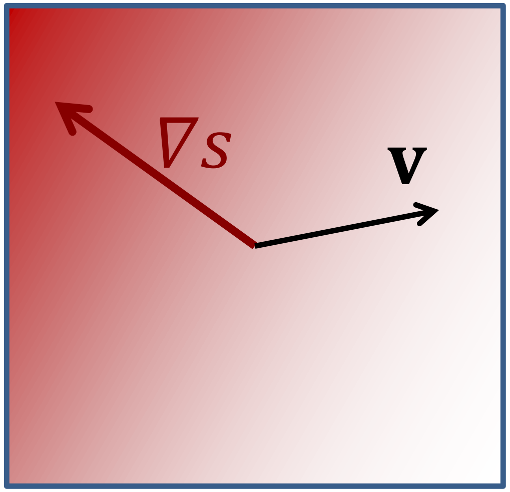
- *Advection* is the transport of a fluid property through the macroscopic motion of the fluid itself (i.e., through the velocity field **v**)
- Let us assume that we have a velocity vector field **v**(**x**,*t*) and we have a scalar field *s*(**x**,*t*) that represents some scalar quantity that is being transported through the velocity field
- For example, **v** might be the velocity of air in the room and *s* might be temperature, or the concentration of some pigment or smoke, etc.
- As the fluid moves around, it will transport the scalar field with it. We say that the scalar field is *advected* by the fluid
- The advection equation specifies a scalar field $\frac{ds}{dt}$ which is the rate of change of the scalar field *s* that is being advected by the velocity field **v**:

$$\frac{ds}{dt} = -\mathbf{v} \cdot \nabla s$$

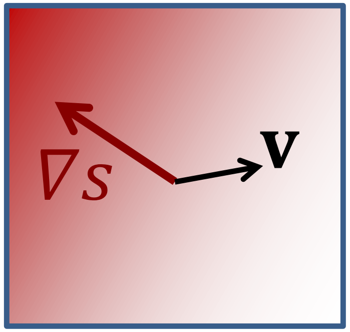
Advection

- Remember that the gradient ∇s of a scalar field s is a vector field pointing in the direction that s is increasing
- The advection is the rate of change of s **at a fixed location** based on the gradient ∇s and the velocity $\mathbf{v}$

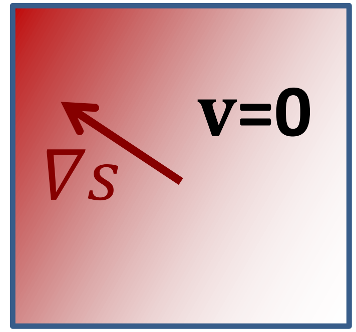
$$\frac{ds}{dt} = -\mathbf{v} \cdot \nabla s$$



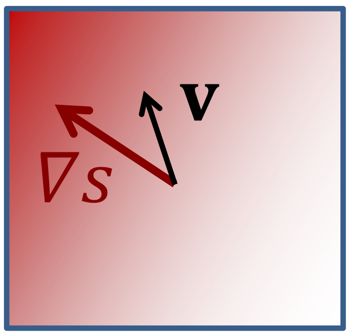
Advection



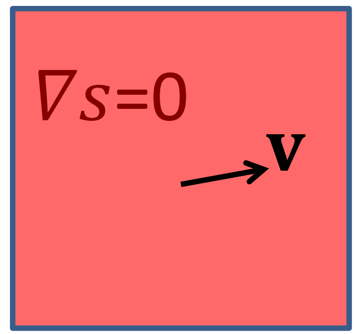
$$\frac{ds}{dt} > 0$$



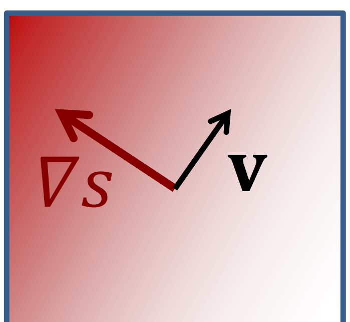
$$\frac{ds}{dt} = 0$$



$$\frac{ds}{dt} < 0$$



$$\frac{ds}{dt} = 0$$



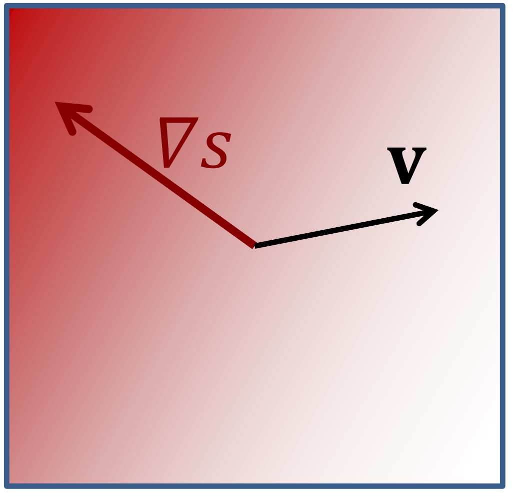
$$\frac{ds}{dt} = 0$$

$$\boxed{\frac{ds}{dt} = -\mathbf{v} \cdot \nabla s}$$

Advection

- Let's say our scalar field *s* represents 'redness'
- The gradient ∇s is in units of redness/meter
- The velocity $\mathbf{v}$ is in units of meters/second
- The advection $\frac{ds}{dt}$ is in units of redness/second

$$\frac{ds}{dt} = -\mathbf{v} \cdot \nabla s$$



Convection

- The velocity field **v** describes the movement of the fluid down to the molecular level
- Therefore, all properties of the fluid at a particular point should be advected by the velocity field
- This includes the property of velocity itself!
- The advection of velocity through the velocity field is called *convection*

$$\frac{d\mathbf{v}}{dt} = -\mathbf{v} \cdot \nabla \mathbf{v}$$

- Remember that $d\mathbf{v}/dt$ is an acceleration, and since $\mathbf{f}=m\mathbf{a}$, we are really describing a force

Convection

$$\frac{d\mathbf{v}}{dt} = -\mathbf{v} \cdot \nabla \mathbf{v}$$

- Convection is the transport of velocity by the velocity field
- In other words, it just carries the motion of the fluid forward by Newton’s First Law (a body in motion will stay in motion and a body at rest will stay at rest unless acted upon by some force)
- It is a fundamental property of fluids and must be present in some form in order for a fluid to be physically valid
- By itself, it represents the behavior of a hypothetical fluid made up of particles that never collide with each other

Diffusion

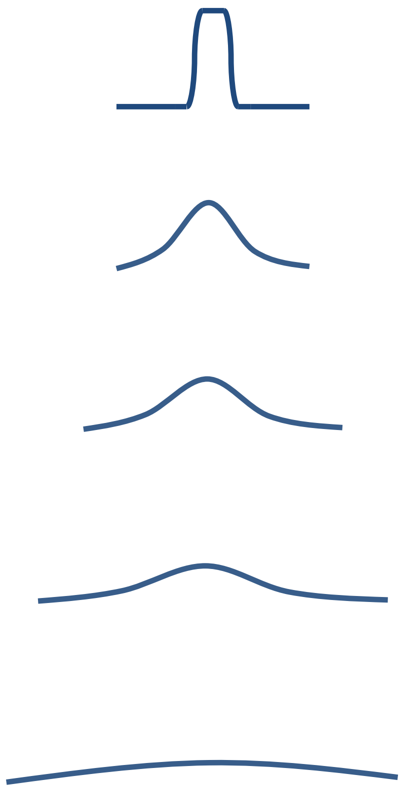
- In real fluids, however, particles do collide with each other
- Lets say that we put a drop of red food coloring in a motionless (**v**=**0** everywhere) water tank. Due to random molecular motion, the red color will still gradually diffuse throughout the tank until it reaches equilibrium
- This is known as a *diffusion* process and is described by the diffusion equation

$$\frac{ds}{dt} = k\nabla^2 s$$

- The constant *k* describes the rate of diffusion
- Heat diffuses through solids and fluids through a similar process and is described by a diffusion equation

Diffusion

$$\frac{ds}{dt} = k\nabla^2 s$$



- By itself, diffusion causes a gradual blurring of a scalar field over time

Second Derivatives

- Remember that the Laplacian operator ∇^2 is a type of spatial second derivative
- A positive Laplacian indicates that the surrounding field is higher on average and a negative Laplacian indicates it is lower
- This is the essence of the diffusion process. If the surrounding field is higher (or lower) in some value, it will cause the value to increase (or decrease) towards the average
- This will gradually lead to an averaging out of the field over time

Viscosity

- *Viscosity* is the diffusion of velocity in a fluid and is described by a diffusion equation as well:

$$\frac{d\mathbf{v}}{dt} = \mu \nabla^2 \mathbf{v}$$

- The constant μ is the *coefficient of viscosity* and describes how viscous the fluid is. Air and water have low values, whereas something like syrup would have a relatively higher value
- Some materials like modeling clay or silly putty are extremely viscous fluids and can behave similar to solids
- Like convection, viscosity is a force. It resists relative motion and tries to smooth out the velocity field

Pressure Gradient

- Fluids flow from high pressure regions to low pressure regions in the opposite direction of the *pressure gradient*

$$\frac{d\mathbf{v}}{dt} = -\nabla p$$

- The difference in pressure acts as a force in the direction from high to low (thus the minus sign)

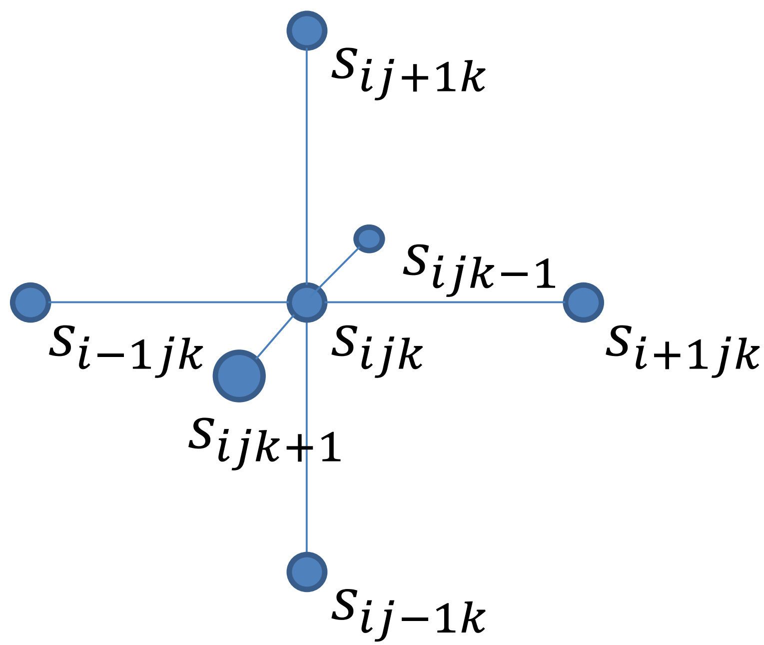
Transport Equations

- Advection: $\frac{ds}{dt} = -\mathbf{v} \cdot \nabla s$
- Convection: $\frac{d\mathbf{v}}{dt} = -\mathbf{v} \cdot \nabla \mathbf{v}$
- Diffusion: $\frac{ds}{dt} = k \nabla^2 s$
- Viscosity: $\frac{d\mathbf{v}}{dt} = \mu \nabla^2 \mathbf{v}$
- Pressure: $\frac{d\mathbf{v}}{dt} = -\nabla p$

Finite Difference Transport Equations

Finite Difference Spatial Derivatives

- In the previous lecture, we learned about the spatial derivative operators and how to generate finite difference approximations on uniform grids



Finite Difference Operations

- Gradient: $\nabla s \approx \frac{1}{2h}\begin{bmatrix} s_{i+1jk} - s_{i-1jk} \\ s_{ij+1k} - s_{ij-1k} \\ s_{ijk+1} - s_{ijk-1} \end{bmatrix}$

- Divergence: $\nabla \cdot \mathbf{v} \approx \frac{1}{2h}\left(v_{x_{i+1jk}} - v_{x_{i-1jk}} + v_{y_{ij+1k}} - v_{y_{ij-1k}} + v_{z_{ijk+1}} - v_{z_{ijk-1}}\right)$

- Curl: $\nabla \times \mathbf{v} \approx \frac{1}{2h}\begin{bmatrix} \left(v_{z_{ij+1k}} - v_{z_{ij-1k}}\right) - \left(v_{y_{ijk+1}} - v_{y_{ijk-1}}\right) \\ \left(v_{x_{ijk+1}} - v_{x_{ijk-1}}\right) - \left(v_{z_{i+1jk}} - v_{z_{i-1jk}}\right) \\ \left(v_{y_{i+1jk}} - v_{y_{i-1jk}}\right) - \left(v_{x_{ij+1k}} - v_{x_{ij-1k}}\right) \end{bmatrix}$

- Laplacian: $\nabla^2 s \approx \frac{1}{h^2}\left(s_{i+1jk} + s_{i-1jk} + s_{ij+1k} + s_{ij-1k} + s_{ijk+1} + s_{ijk-1} - 6s_{ijk}\right)$

Finite Difference Transport Equations

- We can use the same process to generate finite difference versions of the transport equations as well

Finite Differencing Gradients

- In the previous lecture, we looked at representing fields on uniform grids and computing spatial derivatives using finite differencing
- We saw that we can finite difference the gradient operator on a scalar field s with:

$$\nabla s \approx \frac{1}{2h}\begin{bmatrix} s_{i+1jk} - s_{i-1jk} \\ s_{ij+1k} - s_{ij-1k} \\ s_{ijk+1} - s_{ijk-1} \end{bmatrix}$$

- Where s_{ijk} is the value of s at grid cell ijk and h is the spacing between cells

Finite Differencing Advection

- To finite difference the advection $-\mathbf{v} \cdot \nabla s$, we just compute the gradients ∇s and compute the dot product with the velocity $\mathbf{v}_{ijk}$

$$\nabla s \approx \frac{1}{2h} \begin{bmatrix} s_{i+1jk} - s_{i-1jk} \\ s_{ij+1k} - s_{ij-1k} \\ s_{ijk+1} - s_{ijk-1} \end{bmatrix}$$

$$-\mathbf{v} \cdot \nabla v_x \approx$$

$$\frac{-1}{2h}\left(v_{x_{ijk}}\left(s_{i+1jk} - s_{i-1jk}\right) + v_{y_{ijk}}\left(s_{ij+1k} - s_{ij-1k}\right) + v_{z_{ijk}}\left(s_{ijk+1} - s_{ijk-1}\right) \right)$$

Finite Differencing Convection

- To finite difference the convection $-\mathbf{v} \cdot \nabla \mathbf{v}$, we just compute 3 scalar gradients ∇v_x, ∇v_y, and ∇v_z separately and compute the dot product of each of them with the velocity $\mathbf{v}_{ijk}$

$$\nabla v_x \approx \frac{1}{2h} \begin{bmatrix} v_{x_{i+1jk}} - v_{x_{i-1jk}} \\ v_{x_{ij+1k}} - v_{x_{ij-1k}} \\ v_{x_{ijk+1}} - v_{x_{ijk-1}} \end{bmatrix}$$

$$-\mathbf{v} \cdot \nabla v_x \approx \frac{-1}{2h} \left(v_{x_{ijk}} \left(v_{x_{i+1jk}} - v_{x_{i-1jk}} \right) + v_{y_{ijk}} \left(v_{x_{ij+1k}} - v_{x_{ij-1k}} \right) + v_{z_{ijk}} \left(v_{x_{ijk+1}} - v_{x_{ijk-1}} \right) \right)$$

$$-\mathbf{v} \cdot \nabla \mathbf{v} \approx \frac{-1}{2h} \begin{bmatrix} v_{x_{ijk}} \left(v_{x_{i+1jk}} - v_{x_{i-1jk}} \right) + v_{y_{ijk}} \left(v_{x_{ij+1k}} - v_{x_{ij-1k}} \right) + v_{z_{ijk}} \left(v_{x_{ijk+1}} - v_{x_{ijk-1}} \right) \\ v_{x_{ijk}} \left(v_{y_{i+1jk}} - v_{y_{i-1jk}} \right) + v_{y_{ijk}} \left(v_{y_{ij+1k}} - v_{y_{ij-1k}} \right) + v_{z_{ijk}} \left(v_{y_{ijk+1}} - v_{y_{ijk-1}} \right) \\ v_{x_{ijk}} \left(v_{z_{i+1jk}} - v_{z_{i-1jk}} \right) + v_{y_{ijk}} \left(v_{z_{ij+1k}} - v_{z_{ij-1k}} \right) + v_{z_{ijk}} \left(v_{z_{ijk+1}} - v_{z_{ijk-1}} \right) \end{bmatrix}$$

Finite Differencing Laplacians

- In the previous lecture, we looked at how to compute the Laplacian ∇^2 of a scalar field s on a uniform 3D grid using finite differencing:

$$\nabla^2 s \approx \frac{1}{h^2}\left(s_{i+1jk} + s_{i-1jk} + s_{ij+1k} + s_{ij-1k} + s_{ijk+1} + s_{ijk-1} - 6s_{ijk}\right)$$

Finite Differencing Diffusion

- Diffusion can be finite differenced easily by taking the Laplacian and just multiplying by the diffusion coefficient k:

$$k\nabla^2 s \approx \frac{k}{h^2}\left(s_{i+1jk} + s_{i-1jk} + s_{ij+1k} + s_{ij-1k} + s_{ijk+1} + s_{ijk-1} - 6s_{ijk}\right)$$

Finite Differencing Viscosity

- To finite difference the viscosity term $\mu\nabla^2\mathbf{v}$, we just compute the Laplacian for each of the components of $\mathbf{v}$

$$\nabla^2 v_x \approx \frac{1}{h^2}\left(v_{x_{i+1jk}} + v_{x_{i-1jk}} + v_{x_{ij+1k}} + v_{x_{ij-1k}} + v_{x_{ijk+1}} + v_{x_{ijk-1}} - 6v_{x_{ijk}}\right)$$

$$\mu\nabla^2\mathbf{v} \approx \frac{\mu}{h^2}\begin{bmatrix} v_{x_{i+1jk}} + v_{x_{i-1jk}} + v_{x_{ij+1k}} + v_{x_{ij-1k}} + v_{x_{ijk+1}} + v_{x_{ijk-1}} - 6v_{x_{ijk}} \\ v_{y_{i+1jk}} + v_{y_{i-1jk}} + v_{y_{ij+1k}} + v_{y_{ij-1k}} + v_{y_{ijk+1}} + v_{y_{ijk-1}} - 6v_{y_{ijk}} \\ v_{z_{i+1jk}} + v_{z_{i-1jk}} + v_{z_{ij+1k}} + v_{z_{ij-1k}} + v_{z_{ijk+1}} + v_{z_{ijk-1}} - 6v_{z_{ijk}} \end{bmatrix}$$

Finite Difference Transport Operations

- Advection:

$$-\mathbf{v}\cdot\nabla s \approx \frac{-1}{2h}\left(v_{x_{ijk}}\left(s_{i+1jk}-s_{i-1jk}\right)+v_{y_{ijk}}\left(s_{ij+1k}-s_{ij-1k}\right)+v_{z_{ijk}}\left(s_{ijk+1}-s_{ijk-1}\right)\right)$$

- Convection:

$$-\mathbf{v}\cdot\nabla\mathbf{v} \approx \frac{-1}{2h}\begin{bmatrix} v_{x_{ijk}}\left(v_{x_{i+1jk}}-v_{x_{i-1jk}}\right)+v_{y_{ijk}}\left(v_{x_{ij+1k}}-v_{x_{ij-1k}}\right)+v_{z_{ijk}}\left(v_{x_{ijk+1}}-v_{x_{ijk-1}}\right) \\ v_{x_{ijk}}\left(v_{y_{i+1jk}}-v_{y_{i-1jk}}\right)+v_{y_{ijk}}\left(v_{y_{ij+1k}}-v_{y_{ij-1k}}\right)+v_{z_{ijk}}\left(v_{y_{ijk+1}}-v_{y_{ijk-1}}\right) \\ v_{x_{ijk}}\left(v_{z_{i+1jk}}-v_{z_{i-1jk}}\right)+v_{y_{ijk}}\left(v_{z_{ij+1k}}-v_{z_{ij-1k}}\right)+v_{z_{ijk}}\left(v_{z_{ijk+1}}-v_{z_{ijk-1}}\right) \end{bmatrix}$$

- Diffusion:

$$\nabla^2 s \approx \frac{1}{h^2}\left(s_{i+1jk}+s_{i-1jk}+s_{ij+1k}+s_{ij-1k}+s_{ijk+1}+s_{ijk-1}-6s_{ijk}\right)$$

- Viscosity:

$$\mu\nabla^2\mathbf{v} \approx \frac{\mu}{h^2}\begin{bmatrix} v_{x_{i+1jk}}+v_{x_{i-1jk}}+v_{x_{ij+1k}}+v_{x_{ij-1k}}+v_{x_{ijk+1}}+v_{x_{ijk-1}}-6v_{x_{ijk}} \\ v_{y_{i+1jk}}+v_{y_{i-1jk}}+v_{y_{ij+1k}}+v_{y_{ij-1k}}+v_{y_{ijk+1}}+v_{y_{ijk-1}}-6v_{y_{ijk}} \\ v_{z_{i+1jk}}+v_{z_{i-1jk}}+v_{z_{ij+1k}}+v_{z_{ij-1k}}+v_{z_{ijk+1}}+v_{z_{ijk-1}}-6v_{z_{ijk}} \end{bmatrix}$$

Navier-Stokes Equations

Navier-Stokes Equation

- The complete *Navier-Stokes equation* describes the strict conservation of mass, energy, and momentum within a fluid
- Energy can be converted between potential, kinetic, and thermal states
- The full equation accounts for fluid flow, convection, viscosity, sound waves, shock waves, thermal buoyancy, and more
- However, simpler forms of the equation can be derived for specific purposes. Fluid simulation, for example, typically uses a limited form known as the incompressible flow equation

Incompressibility

- Real fluids always have some degree of compressibility. Gasses are very compressible and even liquids can be compressed a little
- Sound waves in a fluid are caused by compression, as are supersonic shocks, but for now, we are not interested in modeling these phenomena
- We will therefore assume that the fluid is *incompressible* and we will enforce this as a constraint
- Incompressibility requires that there is zero divergence of the velocity field everywhere

$$\nabla \cdot \mathbf{v} = 0$$

- This also implies the density will remain constant everywhere if it is constant at the start
- Incompressibility is actually a reasonable approximation, as compression has a negligible affect on fluids moving well below the speed of sound
- We are effectively assuming that the speed of sound is infinite (or much faster than the velocities that we are interested in simulating)

Navier-Stokes Equation

- The *incompressible Navier-Stokes equation* describes the forces on a fluid as the sum of convection, viscosity, and pressure terms:

$$\frac{d\mathbf{v}}{dt} = -\mathbf{v} \cdot \nabla \mathbf{v} + \mu \nabla^2 \mathbf{v} - \nabla p$$

- In addition, we also have the incompressibility constraint:

$$\nabla \cdot \mathbf{v} = 0$$

Navier-Stokes Equation

- The incompressible Navier-Stokes equation describes the acceleration of a fluid at each point as being the sum of convection, viscosity, and pressure terms

$$\frac{d\mathbf{v}}{dt} = -\mathbf{v} \cdot \nabla\mathbf{v} + \mu\nabla^2\mathbf{v} - \nabla p$$

- The convection term ($-\mathbf{v} \cdot \nabla\mathbf{v}$) causes the fluid motion to continue along according to Newton’s First Law
- The viscosity term ($\mu\nabla^2\mathbf{v}$) causes the fluid motion to smooth out either gradually (low μ) or quickly (high μ)
- The pressure term ($-\nabla p$) enforces the incompressibility by causing potential changes in local density to be counteracted by a pressure force from high pressure (high density) areas to low pressure (low density) areas

Incompressible Navier-Stokes Equation

$$\frac{d\mathbf{v}}{dt} = -\mathbf{v} \cdot \nabla\mathbf{v} + \mu\nabla^2\mathbf{v} - \nabla p$$

$$\nabla \cdot \mathbf{v} = 0$$

- This is about as simple of an equation that we can use to simulate fluid motion
- OK, we can actually make it a little simpler by assuming zero viscosity and dropping the $\mu\nabla^2\mathbf{v}$ term. This is actually reasonable for low viscosity fluids like air and water, since the process of numerical simulation tends to add artificial viscosity as a result of the finite sampling in space and time

Incompressible Navier-Stokes Equation

$$\frac{d\mathbf{v}}{dt} = -\mathbf{v} \cdot \nabla \mathbf{v} + \mu \nabla^2 \mathbf{v} - \nabla p$$

$$\nabla \cdot \mathbf{v} = 0$$

- Assuming we keep the viscosity, we still have several things we're ignoring:
  - Variable temperature (thermal buoyancy, thermal diffusion…)
  - Variable density (mixed oil & water…)
  - Surface tension
  - Compression waves (sound waves)
  - Supersonic flows, supersonic shocks
  - Phase changes (melting, evaporating…)
  - Chemical reactions
  - Combustion
  - Non-Newtonian fluids (fluids with more complex viscosity behavior)
  - And more
- Some of these require modeling compression, but others can be integrated into an incompressible solver

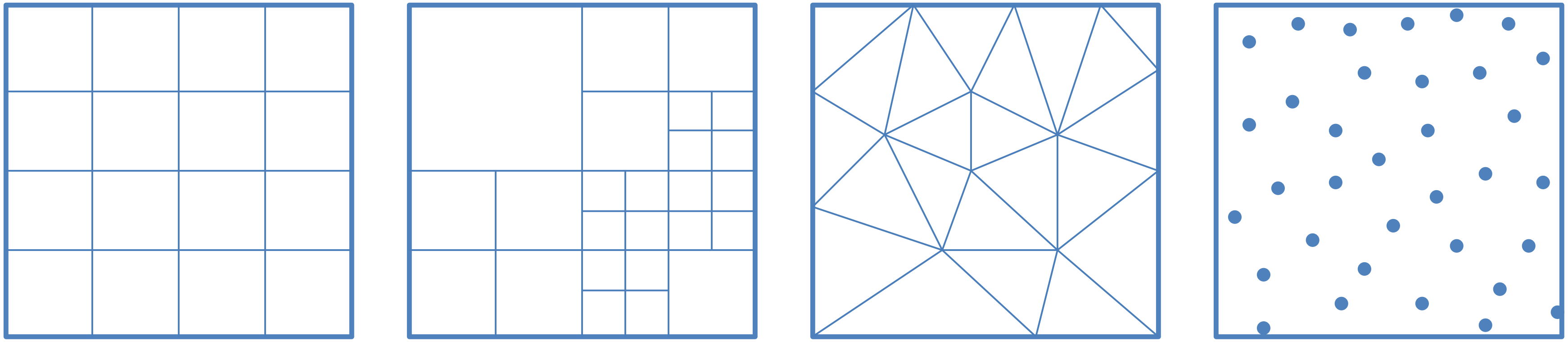
Computational Fluid Dynamics

Computational Fluid Dynamics

- Computational fluid dynamics (CFD) refers to the large collection of techniques for modeling fluids on computers
- It usually involves solving the Navier-Stokes equations in some form
- It has been studied since the earliest days of computers and even before. Some early practical methods for 3D flow simulation date back to around 1955

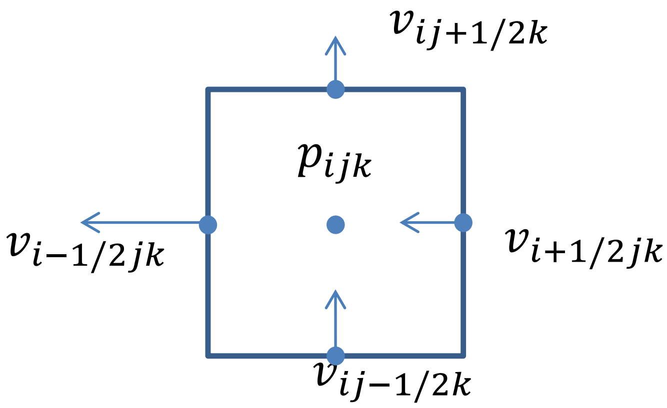
Field Representations

- We have several choices for representing fields
- Each method uses its own way of sampling the field at some interval
- Each method requires a way to interpolate the field between sample points
- Each method requires a way to compute the different spatial derivatives ($\nabla, \nabla \cdot, \nabla \times, \nabla^2$)



Staggered Grids

- We are mainly interested in modeling the vector velocity field
- We will use a staggered uniform grid, where we store the *x*-component of velocity on the *x*-face of each cell, and the *y*-component on the *y*-face, etc.
- Pressures are computed at the cell centers



Boundary Conditions

- We will assume that the velocity fluid doesn’t flow through the boundaries, so we can force the velocity (and its derivatives) to 0
- We could use the same approach to tag certain grid cells as solid to model the interaction of a fluid with complex shapes

Navier-Stokes Fluid Simulation

- The incompressible Navier-Stokes equation describes the derivative of the fluid velocity at every point as:

$$\frac{d\mathbf{v}}{dt} = -\mathbf{v} \cdot \nabla\mathbf{v} + \mu\nabla^2\mathbf{v} - \nabla p$$

- Along with the kinematic constraint that the fluid must have zero divergence everywhere:

$$\nabla \cdot \mathbf{v} = 0$$

Forward Euler Integration

- To simulate a fluid, we want to advance the velocity field each time step. If we used a forward Euler integration, we would want to do something like:

$$\mathbf{v}_1 = \mathbf{v}_0 + \Delta t \frac{d\mathbf{v}}{dt}$$

$$= \mathbf{v}_0 + \Delta t(-\mathbf{v}_0 \cdot \nabla \mathbf{v}_0 + \mu \nabla^2 \mathbf{v}_0 - \nabla p)$$

- We have two problems: first we haven't discussed how to compute the pressure gradient ∇p, and even if we did, we would still have the problem that the final velocity field $\mathbf{v}_1$ would violate the zero divergence constraint due to errors introduced by using a finite time step Δt:

$$\nabla \cdot \mathbf{v}_1 \neq 0$$

Pressure Projection Method

- Due to violation of the divergence constraint, we can **not** simply do the following:

$$\mathbf{v}_1 = \mathbf{v}_0 + \Delta t(-\mathbf{v}_0 \cdot \nabla \mathbf{v}_0 + \mu \nabla^2 \mathbf{v}_0 - \nabla p)$$

- Instead, we will use a two-step pressure projection method, where we split the integration into two steps:

$$\mathbf{v}^* = \mathbf{v}_0 + \Delta t(-\mathbf{v}_0 \cdot \nabla \mathbf{v}_0 + \mu \nabla^2 \mathbf{v}_0)$$
$$\mathbf{v}_1 = \mathbf{v}^* + \Delta t(-\nabla p)$$

- The catch is that in between the two steps, we will solve for a pressure field p such that the final solution $\mathbf{v}_1$ obeys the divergence constraint

Pressure Projection

$$\mathbf{v}_1 = \mathbf{v}^* + \Delta t(-\nabla p)$$

- We need to find a pressure field p such that the final velocity field $\mathbf{v}_1$ is non-divergent:

$$\nabla \cdot \mathbf{v}_1 = 0$$

$$\nabla \cdot \left(\mathbf{v}^* + \Delta t(-\nabla p)\right) = 0$$

$$\nabla \cdot \mathbf{v}^* + \nabla \cdot \left(\Delta t(-\nabla p)\right) = 0$$

$$\nabla \cdot \mathbf{v}^* - \Delta t(\nabla \cdot \nabla p) = 0$$

$$\nabla \cdot \mathbf{v}^* - \Delta t(\nabla^2 p) = 0$$

$$\nabla^2 p = \frac{1}{\Delta t} \nabla \cdot \mathbf{v}^*$$

Pressure Field

- We need to find a pressure field p such that:

$$\nabla^2 p = \frac{1}{\Delta t} \nabla \cdot \mathbf{v}^*$$

- This is known as a Poisson equation
- Several options exist for solving these systems
  - Direct solution
  - Iterative relaxation scheme
  - Conjugate gradient solver
  - Multi-grid solver
- Solving the Poisson equation is really the key computational step in fluid dynamics… however… we won't get into the details today

Pressure Projection Method

At the beginning of the time step, we have a valid (non-divergent) velocity field $\mathbf{v}_0$. To advance the velocity field forward in time by Δt:

1. Compute partial (divergent) velocity field $\mathbf{v}^*$ by finite differencing the convection and viscosity:

$$\mathbf{v}^* = \mathbf{v}_0 + \Delta t(-\mathbf{v}_0 \cdot \nabla \mathbf{v}_0 + \mu \nabla^2 \mathbf{v}_0)$$

2. Solve Poisson equation to get pressure field p:

$$\nabla^2 p = \frac{1}{\Delta t} \nabla \cdot \mathbf{v}^*$$

3. Compute final (non-divergent) velocity field $\mathbf{v}_1$ by adding pressure gradient term:

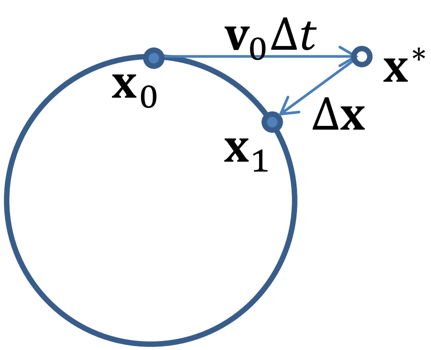
$$\mathbf{v}_1 = \mathbf{v}^* + \Delta t(-\nabla p)$$

Projection Method

- This technique is a form of *kinematic projection method*
- This means that we first compute a partial solution that may violate some set of kinematic constraints
- We then project the partial solution to the nearest point in the space of legal solutions

Projection Method

- As a basic example of a kinematic projection method, consider a particle that has a kinematic constraint that it must stay on a circle of radius *r*
- Let's say it's position at time t_0 is $\mathbf{x}_0$ and its velocity is $\mathbf{v}_0$
- To compute the position $\mathbf{x}_1$ at time Δt later, we can start by computing a partial solution $\mathbf{x}^* = \mathbf{x}_0 + \mathbf{v}_0 \Delta t$ that moves in a straight line and thus will violate the constraint
- We then "solve" for our correction $\Delta\mathbf{x}$ and add it to the partial solution to compute the final legal solution $\mathbf{x}_1$



1. $\mathbf{x}^* = \mathbf{x}_0 + \mathbf{v}_0 \Delta t$ (Compute partial solution by advancing with derivative)

2. $\Delta\mathbf{x} = r\frac{\mathbf{x}^*}{|\mathbf{x}^*|} - \mathbf{x}^*$ (Compute projection factor using kinematic (geometric) rule)

3. $\mathbf{x}_1 = \mathbf{x}^* + \Delta\mathbf{x}$ (Add correction to get final legal solution)