Finite Elements

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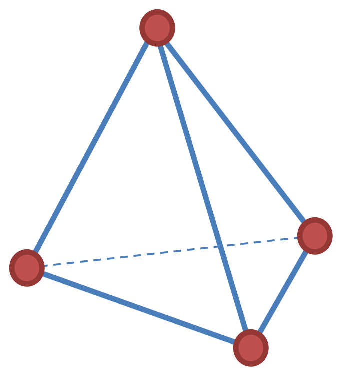
Linear Elastic Simplex Elements

- In the previous lecture, we looked at how to compute linear elastic forces in a single tetrahedron and then used that to compute elastic behavior of objects built with tetrahedral meshes
- This is a example of the *finite element method* (FEM)
- Today, we will look at ways to extend this to add more capabilities

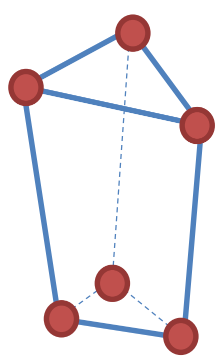


Advanced Material Properties

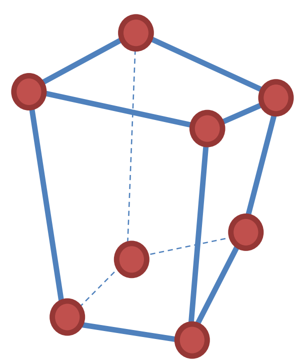
- Elasticity refers to the property of a material to deform under applied forces and return to its original shape when the forces are removed
- In addition to elastic behavior, we can model:
  - Damping
  - Plasticity
  - Fracture
  - Other internal properties (viscoelasticity, hyperelasticity, creep, fatigue, thermodynamics…)
  - Complex materials (composites, laminates, granular materials…)
- We can add collision and contact forces as well, but these are a bit different in that they are not based internal stresses within the material but on externally applied forces

Advanced Elements

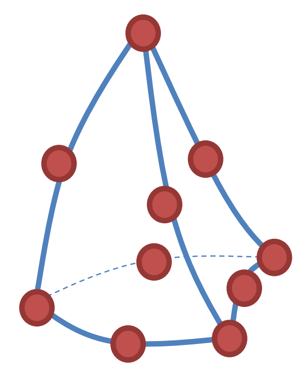
- We can move beyond the basic simplex elements (segment, triangle, tetrahedron) in two main ways:
  - Non-simplex
  - High order elements
- Non-simplex elements include other basic shapes such as 3-prisms and hexahedra
- High order elements use additional vertices to define curved edges and surfaces defined by nonlinear functions (such as with quadratic or cubic functions)



$3 - prism$



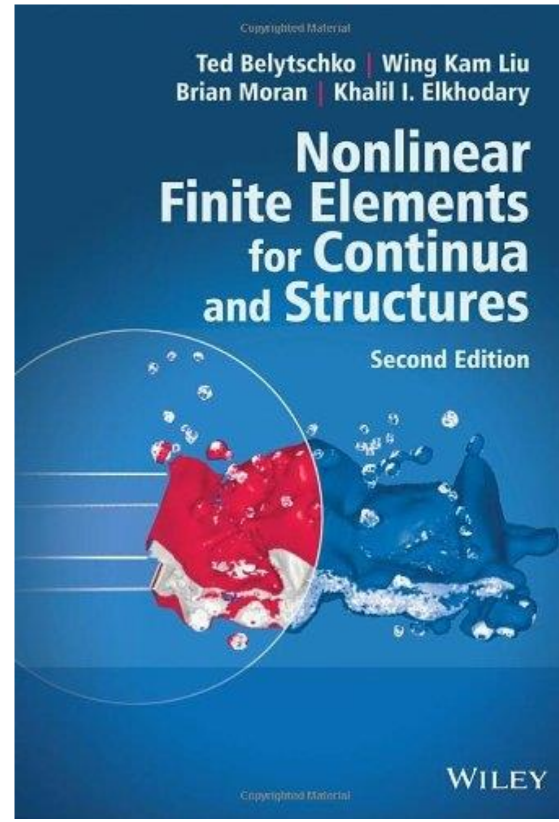
$hexahedron$



$quadratic\ tetrahedron$

Recommended Reading

- “Nonlinear Finite Elements for Continua and Structures”, Second Edition
- Belytschko, Liu, Moran, Elkhodary, 2014



Damping

Conservative Systems

- The 1D Hooke's Law ($f=-kx$) relates a force to a displacement
- The force relates to acceleration through Newton's Second Law ($f=ma$)
- Ultimately, this relates acceleration to position, which leads to a second order ordinary differential equation
- In other words, the second derivative of a value is dependent on the value itself
- If all of the accelerations (forces) in the system can be expressed as functions of only the positions, then we will have a system that conserves energy known as a *conservative system*

Non-Conservative Systems

- In any real system, we will gradually loose mechanical energy to various forms of friction, such as contact friction or internal damping
- These always generate heat as energy is converted from kinetic to thermal forms
- We could model all of the thermal-mechanical interactions and account for the energy, but often times we won't bother and we will just ignore any heat created
- In these cases, we would gradually loose energy in the system, and so we would have a *non-conservative* or *dissipative system*
- The forces of these damping interactions are dependent on velocities rather than positions
- These lead to ODEs where the second derivative of a value is dependent on the first derivative (and the value itself)

Modeling Damping

- To model damping, we will be looking at node velocities rather than positions
- The 1D linear damping model is:

$$f = -k_d v$$

- Where k_d is the damping constant and v is the velocity
- This works just like Hooke's Law except it replaces the displacement x with the first derivative of displacement v (it also uses a different constant)
- Just like we extended *f*=-*kx* to 3D in the last lecture, we can do the exact same thing for damping
- We just have to replace a bunch of position values with velocities and use a different set of constants

Linear Damping

- We can use a simple linear damping model that works almost a exactly like linear elastic model except it uses particle velocities in the equations instead of particle positions
- It also uses an additional set of damping constants that relate velocity to stress. Like elasticity, there could be up to 21 constants for anisotropic materials, but only 2 are needed for isotropic materials
- We used the 6x6 symmetric stiffness tensor $\mathbf{K}$ to store the stiffness constants that relate strains to stresses
- We will define a 6x6 symmetric damping tensor $\mathbf{D}$ to store the damping constants that relate strain rates to stresses

*Strain Rate Tensor

- We will define the strain rate tensor in a similar way as the strain tensor, except based on the particle velocities instead of the positions
- There are different ways to do this that will result in different constants in the D matrix. A reasonable approach to generating the strain rate tensor is just to use the derivative of the strain tensor itself

$$\mathbf{\nu} = \frac{d\boldsymbol{\varepsilon}}{dt}$$

Stress Calculation

- The geometric deformation (measured by the strain tensor) of the material will cause internal elastic stresses
- The velocity gradient (measured by the strain rate tensor) will cause additional viscous stresses
- All together, our linear elastic-damping model looks like:

$$[\boldsymbol{\sigma}] = \mathbf{K} \cdot [\boldsymbol{\varepsilon}] + \mathbf{D} \cdot [\boldsymbol{\nu}]$$

Plasticity

Plasticity

- *Plastic deformation* is the re-arrangement of molecules in a solid material leading to a permanent change of shape
- For example, if we bend a spoon slightly, it will undergo elastic deformation, but if we bend it harder, we will cause a permanent kink resulting from plastic deformation
- The *yield stress* is the internal stress level where the material begins to deform plastically
- This is obviously a crucial value to understand in structural and mechanical engineering because it represents the maximum stress a material can handle before failing

Basic Plasticity Model

- A basic plasticity model can be found in the paper: "Graphical Modeling and Animation of Ductile Fracture" by O'Brien, Bargteil, and Hodgins, 2002
- They extend their brittle fracture paper from 1999 by adding plastic deformation to their model which already included elasticity and fracture
- *Ductile fracture* happens in materials that first undergo plastic deformation before fracturing, leading to permanent deformations around the fractures (such as with metals)
- This is in contrast to *brittle fracture* which happens in materials that don't experience significant plastic deformation before fracturing (such as glass)

Basic Plasticity Model

- They start with the assumption that the strain $\boldsymbol{\varepsilon}$ in any element is the sum of elastic $\boldsymbol{\varepsilon}^e$ and plastic $\boldsymbol{\varepsilon}^p$ components:

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^p + \boldsymbol{\varepsilon}^e$$

- Each element stores a plastic strain tensor $\boldsymbol{\varepsilon}^p$ which represents the amount of permanent deformation the element has undergone. This would typically be initialized to **0**
- The total strain $\boldsymbol{\varepsilon}$ is just the geometric measure of the deformation that we can compute as we did in the previous lecture (i.e., using Green's strain tensor)

Elastic Yielding

- The paper defines an additional material property γ_1 called the *elastic limit*
- This is a geometric (strain-based) tolerance that determines where purely elastic behavior ends and plastic deformation begins
- They compute the *deviation of the elastic strain* as:

$$\boldsymbol{\varepsilon}' = \boldsymbol{\varepsilon}^e - \frac{\mathrm{Tr}(\boldsymbol{\varepsilon}^e)}{3}\mathbf{I}$$

- Where $\mathrm{Tr}(\,)$ is the trace of a matrix (sum of the diagonal elements)
- If the magnitude (Frobenius norm) $\|\boldsymbol{\varepsilon}'\|$ is greater than γ_1, then plastic deformation occurs, otherwise, only purely elastic forces are computed using the elastic strain tensor

Plastic Update

- If yielding occurs, the plastic strain is updated according to:

$$\Delta \boldsymbol{\varepsilon}^{p} = \frac{\|\boldsymbol{\varepsilon}'\| - \gamma_1}{\|\boldsymbol{\varepsilon}'\|}\boldsymbol{\varepsilon}'$$

- They also introduce a material property γ_2 called the *plastic limit*, a geometric (strain-based) limit that restricts the maximum plastic deformation allowed according to:

$$\boldsymbol{\varepsilon}^{p} := (\boldsymbol{\varepsilon}^{p} + \Delta\boldsymbol{\varepsilon}^{p}) \min\left(1, \frac{\gamma_2}{\|\boldsymbol{\varepsilon}^{p} + \Delta\boldsymbol{\varepsilon}^{p}\|}\right)$$

Advanced Plasticity

- Advanced plasticity models can include:
  - *Creep*: the gradual permanent deformation resulting from long-term stresses below the yield stress. Creep tends to increase with heat. There are numerous creep and related models including: dislocation, Nabarro-Herring, solute drag, dislocation clime-glide, Harper-Dorn, sintering…
  - *Viscoelasticity* & *viscoplasticity*: mixtures of elastic, viscous, and plastic behavior in materials. An example of a viscoelastoplastic material is silly putty. When rolled into a ball, silly putty will bounce similar to a rubber ball and undergo very large but rapid forces that result in little permanent deformation. However, applying a small force over a longer time can result in significant permanent deformation

Mesh Deformation

- Under very large plastic deformations, the finite element mesh itself can become very distorted, leading to topological and accuracy problems
- *Adaptive meshing* schemes can dynamically adjust the mesh by splitting and combining nodes and elements based on deformation or error metrics
- *Arbitrary Lagrangian-Eulerian* (ALE) formulations allow for the material coordinates of the nodes to vary over time as well as the world positions. This allows the mesh to remain more regular even under extreme deformations

Nonlinear Materials

Material Models

- We do not need to limit ourselves to simple linear elastic materials
- We can experiment with different nonlinear models to simulate a much wider range of real materials
- These can include effects of plastic deformation, hysteresis, damping, fatigue, and other phenomena

Constitutive Models

- Within the subject of finite elements, the term *constitutive model* refers to the mathematical model defining the key relationships within a particular type of material
- In other words, it would define the type of strain and stress tensors to use as well as the stress-strain relationship, and any other relevant material processes (such as plasticity) and their associated variables
- Throughout FEM literature, there are many different constitutive models used
- The "Nonlinear Finite Elements for Continua and Structures" book has a chapter detailing several popular models

Strain Tensors

- We looked at the simple Green's strain tensor in the last lecture

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left(\mathbf{F}^T \cdot \mathbf{F} - \mathbf{I}\right)$$

- There are several other strain tensors that are in use for different situations
- If we are dealing with very small (or infinitesimal) strains, we can use the infinitesimal strain tensor:

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left(\mathbf{F}^T + \mathbf{F}\right) - \mathbf{I}$$

- For larger deformations, there are a variety of other options such as Cauchy-Green, finger tensor, Cauchy deformation tensor, Green-St. Venant, Green-Lagrange, etc.
- These are all different ways to measure the same geometric deformations (namely scales & shears)
- For a strain tensor to be useful, it must:
  - Vanish to 0 for purely rigid motions
  - Must depend on $\mathbf{F}$ in a continuous, differentiable, monotonic way
  - Reduce to the infinitesimal strain tensor for small $\mathbf{F}$

1D Strain Variations

- If we want, we could use a different way to measure the strain as the ratio of the current length to the rest length:

$$\tilde{x} = \frac{|\mathbf{r}_1 - \mathbf{r}_2|}{l_0} - 1$$

- And we could use a different but similar looking spring equation (not Hooke's Law)

$$f = -k\tilde{x}$$

- It turns out that this should even have some advantages over the previous way we did this, in that this method scales more appropriately to different rest length springs. In other words, if you took a large spring and cut it into two smaller springs, then put them back together, you would expect it to behave like the original spring. This method will lead to that behavior.
- In any case, we are just approximating a smooth function with a straight line (as the first two terms in a Taylor series)
- This is how we can justify Hooke's Law (or variations like the one above) as a reasonable approximation of what is really a nonlinear equation

Stress Variations

- We used the Cauchy stress tensor, which is quite common
- There are also other stress tensors such as Kirchhoff, nominal stress, Piola-Kirchhoff, PK2, Biot, etc.
- The reason we have several strain and stress tensors to choose from is that it is common practice to stick with linear relationships between the strains and stresses in order to keep the number of constants and material properties to a minimum (this also makes it easier to measure real world properties)
- If we change the strain and stress tensors themselves, we can control the nonlinear function that ultimately relates the deformation to the forces
- As we are ultimately just approximating the stress-strain relationship with a linear relationship, this gives us the ability to model several nonlinear relationships while keeping things relatively simple
- Note: https://en.wikipedia.org/wiki/Stress_measures has a good summary of stress measures and a table that converts any one to any other

Corotational Frames

- One common approach to modeling stress-strain relationships that adapts well to large deformations is the use of corotational frames
- With this concept, we think of the deformation gradient $\mathbf{F}$ as being decomposed into a orthonormal pure rotation and a un-rotated pure deformation:

$$\mathbf{F} = \mathbf{Q}\tilde{\mathbf{F}}$$

- This allows us to make larger deformations and more accurately compute node forces resulting from them
- In order to use this, we first compute $\mathbf{F}$ as before, but then we have to use a *polar decomposition* to extract out the rotation $\mathbf{Q}$

Mass Matrix

- Finite element methods evaluate strains and stresses within elements and ultimately produce forces at the nodes
- So far, we've assumed that a node represents a particle where the mass is concentrated at a single point
- This leads to a straightforward *f=ma* relationship for the entire system of particles where each particle has its own mass and we can easily compute 3D particle acceleration $\mathbf{a}$ from a 3D force $\mathbf{f}$ as:

$$\mathbf{a} = \frac{1}{m}\mathbf{f}$$

- We can extend $\mathbf{a}$ and $\mathbf{f}$ to be *n*-dimensional vectors representing the entire system of particles where *n* is 3 times the number of particles and $\mathbf{M}$ is a diagonal matrix of particle masses (x3)

$$\mathbf{a} = \mathbf{M}^{-1}\mathbf{f}$$

- As this matrix is diagonal, we don't really need to invert it or even store it as a matrix, as only the diagonal is non-zero

Lumped vs. Consistent Mass Matrix

$$\mathbf{a} = \mathbf{M}^{-1}\mathbf{f}$$

- If $\mathbf{M}$ is diagonal, then we are explicitly modeling the system as concentrated point masses (particles) connected by massless tetrahedral elements
- This is known as a *lumped* or *diagonal mass matrix*
- This may be accurate enough for many problems, especially if we are already using lots of small elements
- However, we can improve upon this by using a *consistent mass matrix* which attempts to represent the mass as being distributed uniformly through the solid leading to a more accurate stress-acceleration relationship as well as better angular momentum conservation
- There are different methods for computing this based on element type and other variations

Statics, Modal Dynamics, & Thermodynamics

Transient Dynamics

- Most of this course focuses on *transient dynamics of mechanical systems* which refers to non-periodic dynamics of motion in the time domain
- But let's take a quick look at a couple peripheral topics:
  - Statics
  - Modal dynamics
  - Thermodynamics

Statics

- Statics forces within a stable configuration such as a structure
- It is commonly used to analyze buildings and bridges
- From a physics point of view, statics can just refer to a dynamic system that has come to rest through some energy loss process (friction, damping…)
- From a computational point of view, we can actually just model it as dynamic system (maybe with some extra damping) and wait for it to stop moving
- However, we can also use some other approaches

Statics

- Remember that a static object will be feeling the force of gravity, and will therefore experience some deformation
- For buildings and bridges, this deformation will be small, but measurable
- In static simulations, we have to allow some deformation in order for the stresses to develop within the model

*Statics

Modal Dynamics

- *Modal dynamics* refers to the study of periodic motion, such as vibrations in a solid
- We can do some interesting things here without requiring much more than we've already covered
- If we are assuming vibrations, we can usually make the assumption of small deformations, which justifies the use of linear elastic models

1D Vibration

- Let's say we have a 1D spring with length l_0 and stiffness constant k
- The spring has one end fixed and a 1D particle of mass m on the other end
- The particle's position is x and will be 0 when the spring is at rest length. The fixed end of the spring is at $-l_0$, and so the particle position x is also the displacement of the spring
- In this case, the force on the particle is *f=-kx*
- We apply Newton's Second Law (*f=ma*) to this and get the second order ODE:

$$a = \frac{d^2x}{dt^2} = -\frac{k}{m}x$$

Simple Harmonic Oscillator

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

- This is the equation of a simple harmonic oscillator
- This ODE can be solved exactly as:

$$x(t) = A\cos(\omega t + \varphi)$$

- Which is a periodic sinusoid with amplitude A, phase offset φ and frequency ω, where

$$\omega = \sqrt{\frac{k}{m}}$$

3D Vibration

- In the 1D linear elastic case, we can exactly compute the periodic properties
- We can extend this idea into 3D and exactly compute the periodic properties of a tetrahedral element
- We can even do this for entire meshes of elements and analytically (symbolically) compute exact vibration properties of geometrically complex 3D solids
- This allows us to analyze oscillations in everything from structures, to vehicles, to musical instruments

*3D Modal Analysis

Sound Wave Modeling

- One very interesting application of modal dynamics is in modeling sounds of objects
- We can start with a 3D solid model of an object, and then apply a modal analysis to determine the vibration modes
- We can then construct a particular vibration response to an input impulse (i.e., if we hit the object at some location, we can determine the resulting vibration patterns)
- We can then construct the sound waves that would radiate from the vibrating shape, thus determining what sound it will make
- When we hit our virtual object in different locations, we hear different sounds
- This approach has also been extended to handle sounds generated from fluids as well

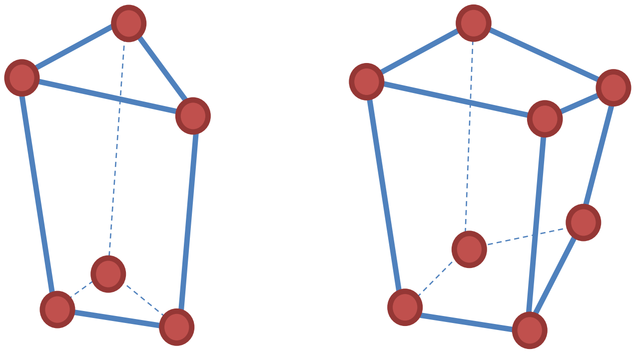
Thermodynamics

- We could add thermal properties to our solids and model additional behaviors
  - Energy lost to damping would lead to warming
  - Thermal diffusion: heat will move throughout solids and gradually approach equilibrium
  - Heat will be gained/lost at the surface due to convective & radiative transfer
  - Temperature changes could affect material constants (Young's modulus, plasticity constants, etc.)
  - Temperature changes could ultimately lead to melting or boiling, but that really requires fluid dynamics which we haven't covered yet
- It would require adding a temperature variable to every particle and some additional material constants
- To model any of these behaviors, one can start with a simple linear model and move on to more advanced models as needed

Element Types

Non-Simplex Elements

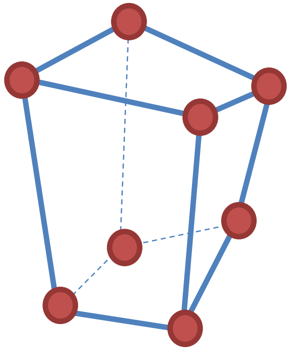
- We can use more elaborate element types than tetrahedra, such as 3-prisms and hexahedra
- Their rest state does not have to be a regular shape (i.e., the hexahedron doesn't have to be a cube when undeformed)
- In practice, these tend to produce better results than just modeling the equivalent out of several tetrahedra
- This occurs for various reasons, but one important reason is that tetrahedra are limited to having constant strain & stress throughout the entire element, while these other types model a smoothly varying stress/strain field, leading to potentially higher accuracy



$3 - prism$ $\quad$ $hexahedron$

Hexahedral Elements

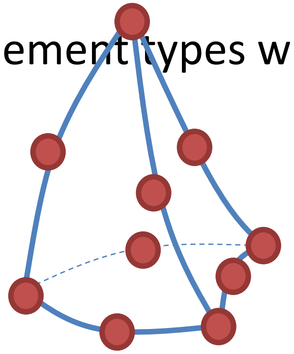
- Consider a hexahedral element, which is a 6-faced element like a deformed box
- We would not have to assume that the stress/strain is constant
- In fact, each corner has enough information to be able to compute a local stress/strain for that region
- We can analyze the deformation of each corner in essentially the same way we did for an entire tetrahedron by using the corner and the 3 connecting points
- The corner stress/strain values can be interpolated through the element in a trilinear fashion, leading to a smoothly (trilinear) varying stress/strain field



$hexahedron$

*High Order Elements

- We can also use quadratic and cubic (etc.) functions to define the deformation of an element
- We create additional points that aren't on the corners, but behave the same as any other particle in the model
- In FEM terminology, these are referred to as *quadrature points*, which is a mathematical term meaning a point where an integral is evaluated
- The quadratic tetrahedron pictured has 10 quadrature points, and the hexahedron from the previous page has 8
- A tetrahedron has only 1 because we effectively evaluate the tensors and stress/strain relationship once and it is constant across the entire element
- There are several different schemes describing a range of element types with a variety of different quadrature options…



quadratic tetrahedron

*Nonlinear Finite Elements

- There are some nice general schemes for combining these different element types