Basis Images and The Wavelet Transform

Image Processing

CSE 166

Lecture 13

Announcements

- Assignment 4 will be released today
  - Due May 22, 11:59 PM
- Reading
  - Chapter 6: Wavelet and Other Image Transforms
    - Sections 6.5 and 6.10
    - (Sections 6.6-6.9 for details of specific transforms)

Matrix-based transforms

$$T(u) = \sum_{x=0}^{N-1} f(x) r(x, u)$$ Forward transform

$$f(x) = \sum_{u=0}^{N-1} T(u) s(x, u)$$ Inverse transform

where

x is a spatial variable

u is a transform variable

$T(u)$ is the transform of $f(x)$

$f(x)$ is the inverse transform of $T(u)$

$r(x, u)$ is a forward transformation kernel

$s(x, u)$ is an inverse transformation kernel

Matrix-based transforms using orthonormal basis vectors

- In matrix form

$$\begin{bmatrix} T(0) \\ T(1) \\ \vdots \\ T(N-1) \end{bmatrix} = \begin{bmatrix} \mathbf{s}_0^H \\ \mathbf{s}_1^H \\ \vdots \\ \mathbf{s}_{N-1}^H \end{bmatrix} \begin{bmatrix} f(0) \\ f(1) \\ \vdots \\ f(N-1) \end{bmatrix} \text{ for complex vectors}$$

$$\begin{bmatrix} T(0) \\ T(1) \\ \vdots \\ T(N-1) \end{bmatrix} = \begin{bmatrix} \mathbf{s}_0^T \\ \mathbf{s}_1^T \\ \vdots \\ \mathbf{s}_{N-1}^T \end{bmatrix} \begin{bmatrix} f(0) \\ f(1) \\ \vdots \\ f(N-1) \end{bmatrix} \text{ for real vectors}$$

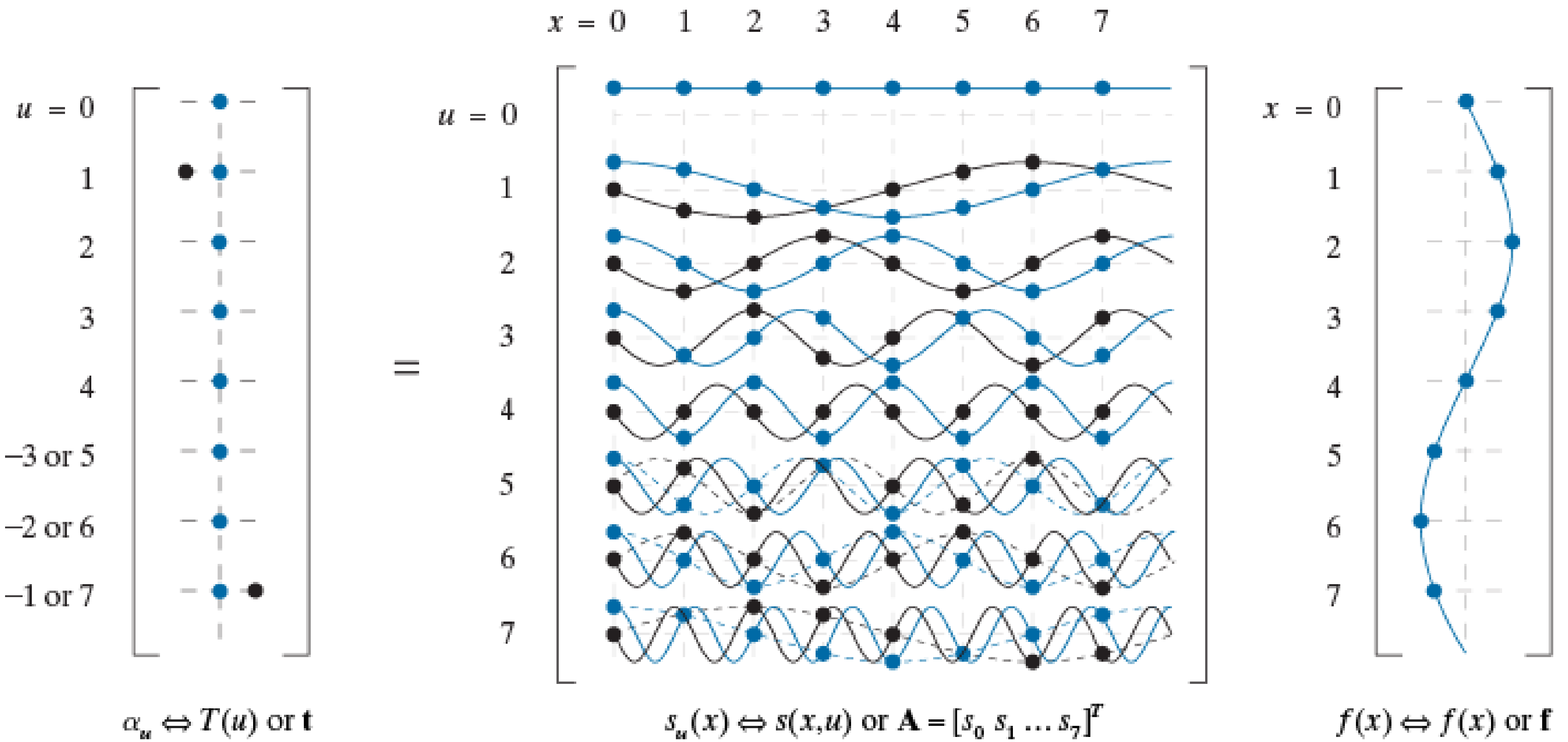
$$\mathbf{t} = \mathtt{A}\mathbf{f}$$

$$\mathbf{f} = \mathtt{A}^H \mathbf{t} \text{ for complex vectors}$$

$$\mathbf{f} = \mathtt{A}^T \mathbf{t} \text{ for real vectors}$$

Matrix-based transform

Example: 8-point DFT of $f(x) = \sin(2\pi x)$



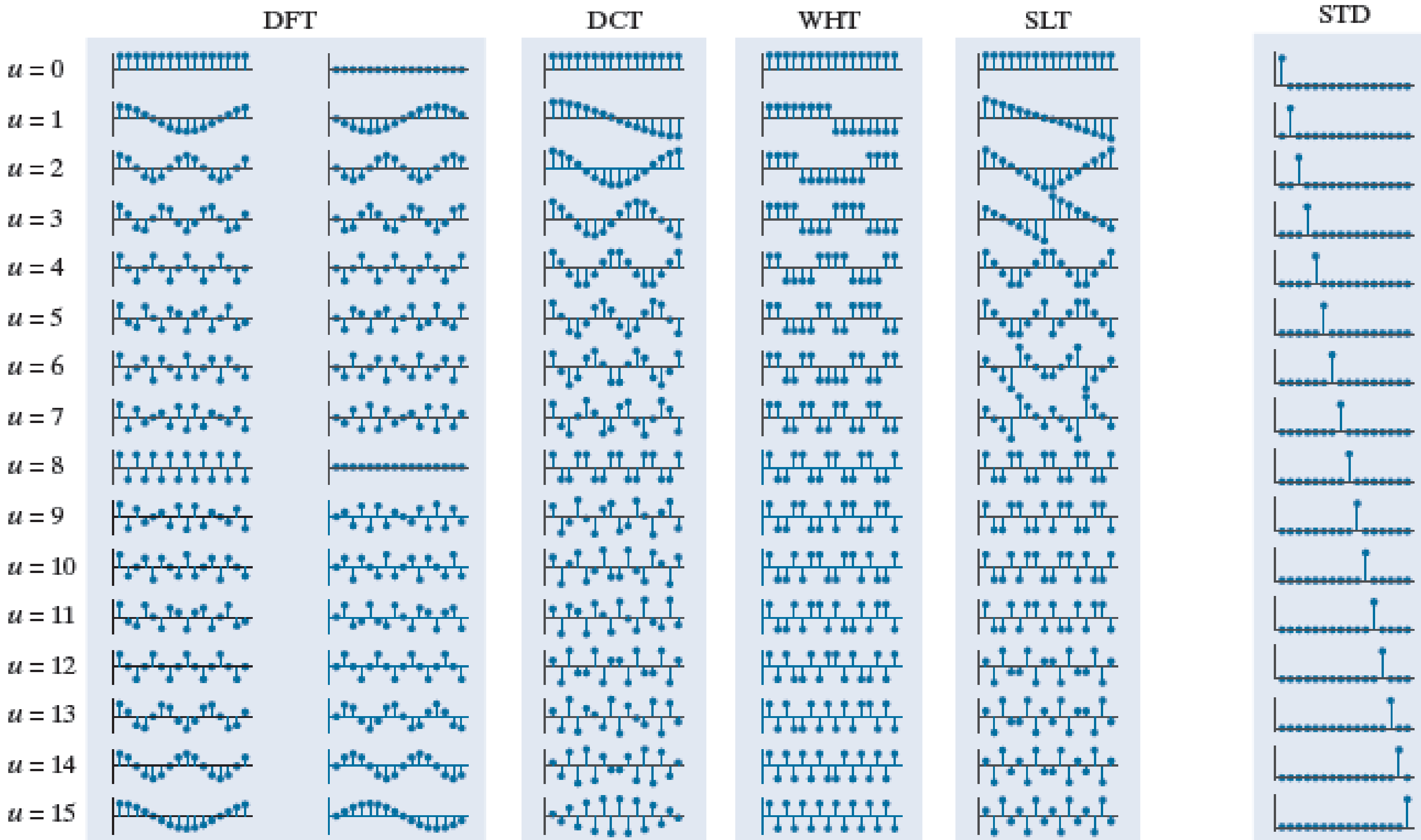
real part + imaginary part

Matrix-based transforms

- Discrete Fourier transform (DFT)
- Discrete Hartley transform (DHT)
- Discrete cosine transform (DCT)
- Discrete sine transform (DST)
- Walsh-Hadamard (WHT)
- Slant (SLT)
- Haar (HAAR)
- Daubechies (DB4)
- Biorthogonal B-spline (BIOR3.1)

Basis vectors of matrix-based 1D transforms

N = 16



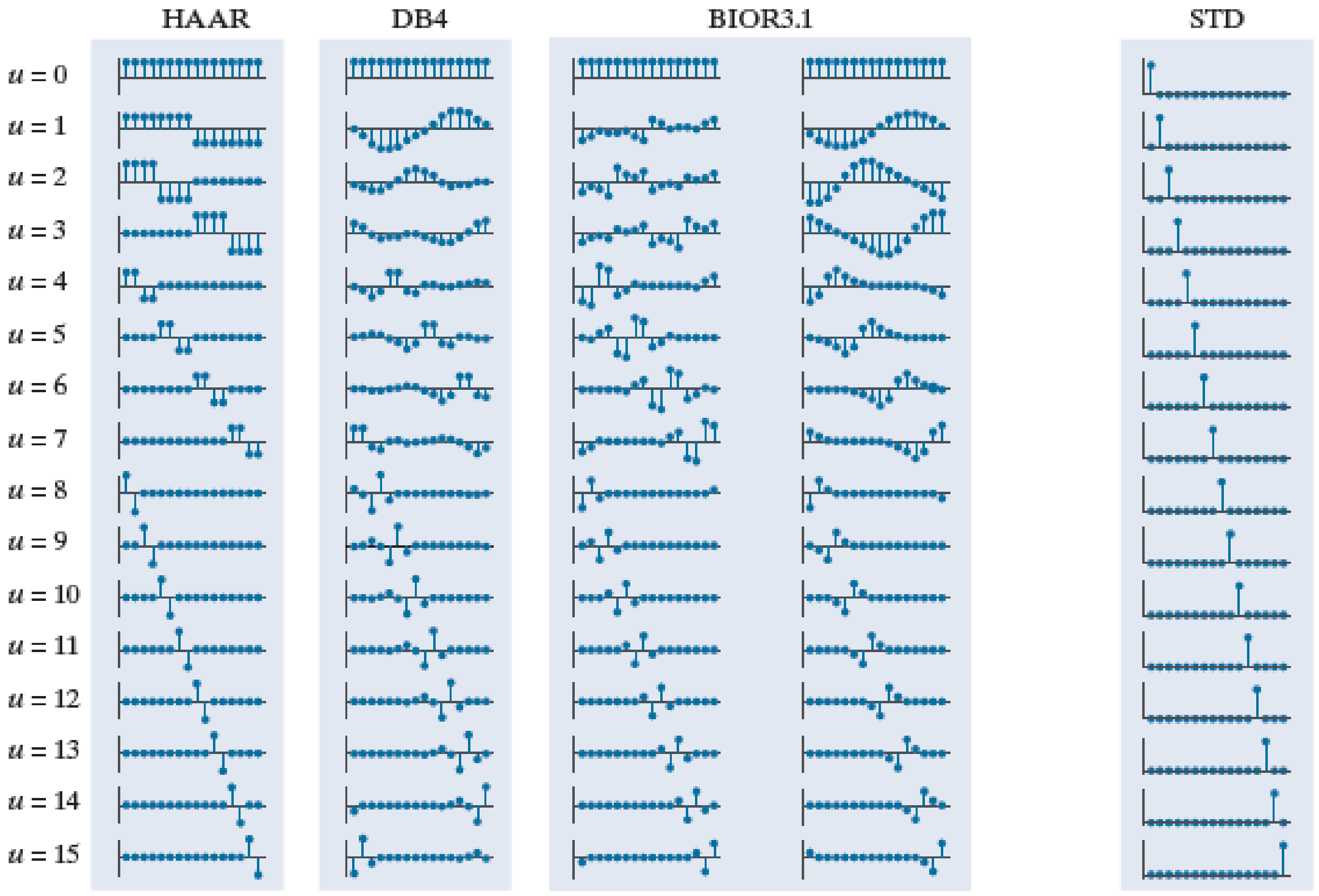
real part

imaginary part

Standard basis (for reference)

Basis vectors of matrix-based 1D transforms

N = 16



basis dual

Standard basis (for reference)

Matrix-based transforms in two dimensions

$$T(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) r(x,y,u,v)$$ Forward transform

$$f(x,y) = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} T(u,v) s(x,y,u,v)$$ Inverse transform

where

x, y are spatial variables

u, v are transform variables

$T(u,v)$ is the transform of $f(x,y)$

$f(x,y)$ is the inverse transform of $T(u,v)$

$r(x,y,u,v)$ is a forward transformation kernel

$s(x,y,u,v)$ is an inverse transformation kernel

Matrix-based transforms in two dimensions using basis images

- Inverse transform

$$f(x,y)=\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}T(u,v)s(x,y,u,v)$$

$$\mathrm{F}=\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}T(u,v)\mathrm{S}_{u,v}$$

where

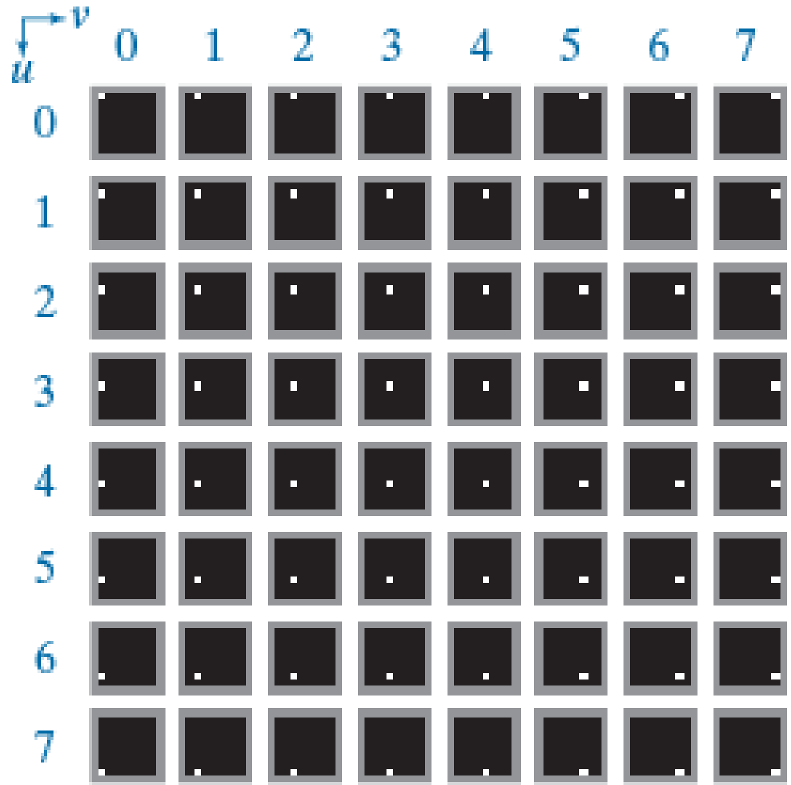
$$\mathrm{F}=\begin{bmatrix} f(0,0) & f(0,1) & \cdots & f(0,N-1) \\ f(1,0) & f(1,1) & \cdots & f(1,N-1) \\ \vdots & \vdots & \ddots & \vdots \\ f(N-1,0) & f(N-1,1) & \cdots & f(N-1,N-1) \end{bmatrix}$$

$$\mathrm{S}_{u,v}=\begin{bmatrix} s(0,0,u,v) & s(0,1,u,v) & \cdots & s(0,N-1,u,v) \\ s(1,0,u,v) & s(1,1,u,v) & \cdots & s(1,N-1,u,v) \\ \vdots & \vdots & \ddots & \vdots \\ s(N-1,0,u,v) & s(N-1,1,u,v) & \cdots & s(N-1,N-1,u,v) \end{bmatrix}$$

Each $S_{u,v}$ is a basis image

Basis images of matrix-based 2D transforms

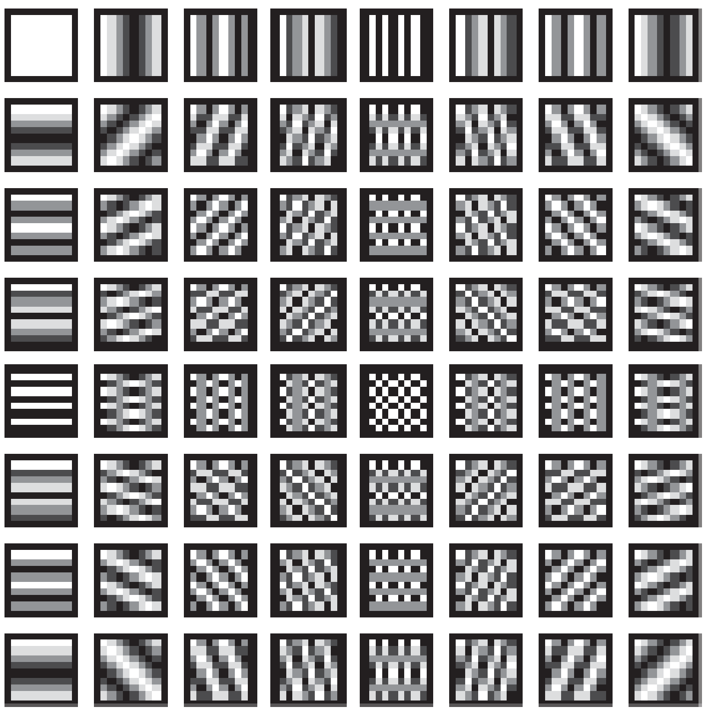
Standard basis images (for reference)



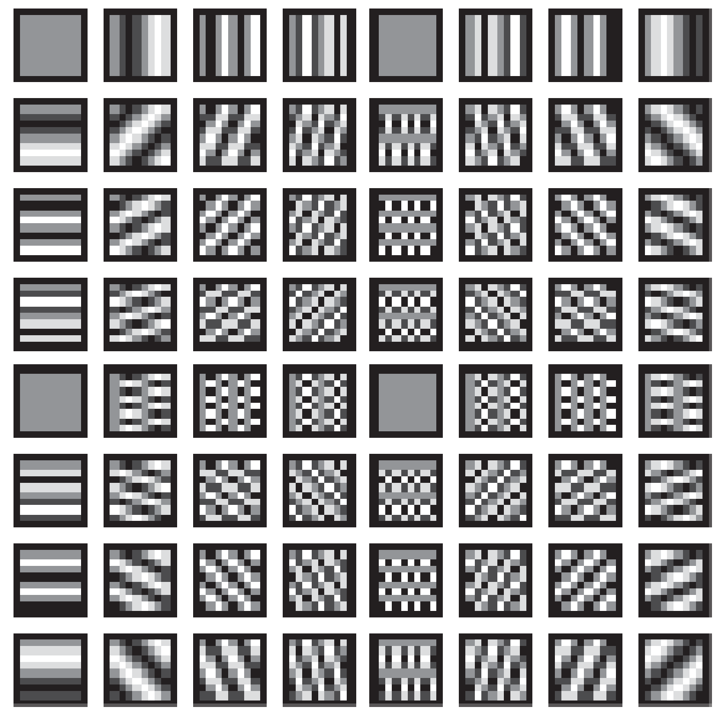
8-by-8
array of
8-by-8
basis images

Basis images of matrix-based 2D transforms

Discrete Fourier transform (DFT) basis images



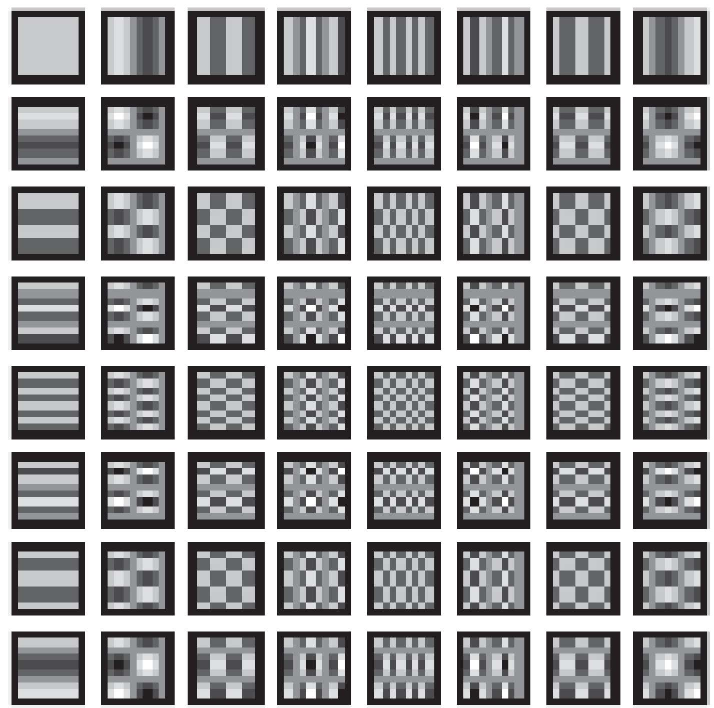
real part



imaginary part

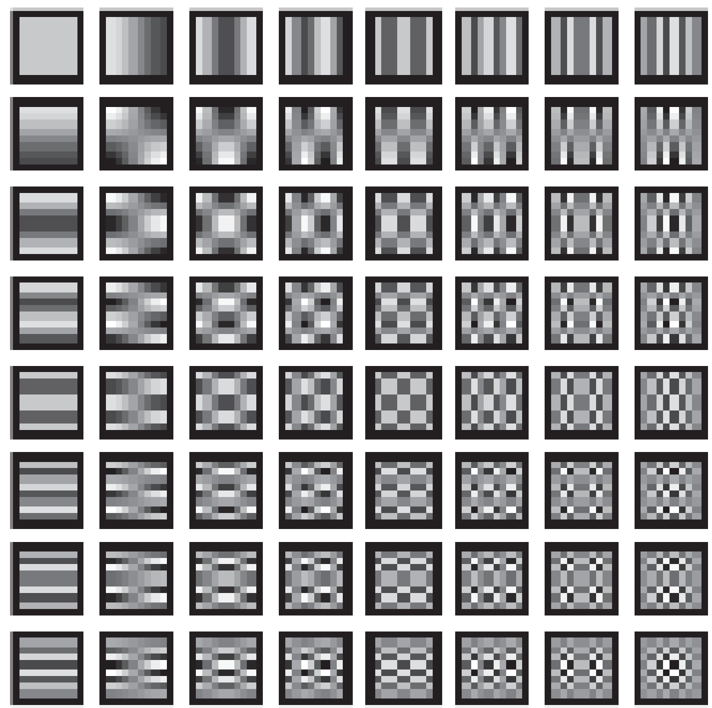
Basis images of matrix-based 2D transforms

Discrete Hartley transform (DHT) basis images



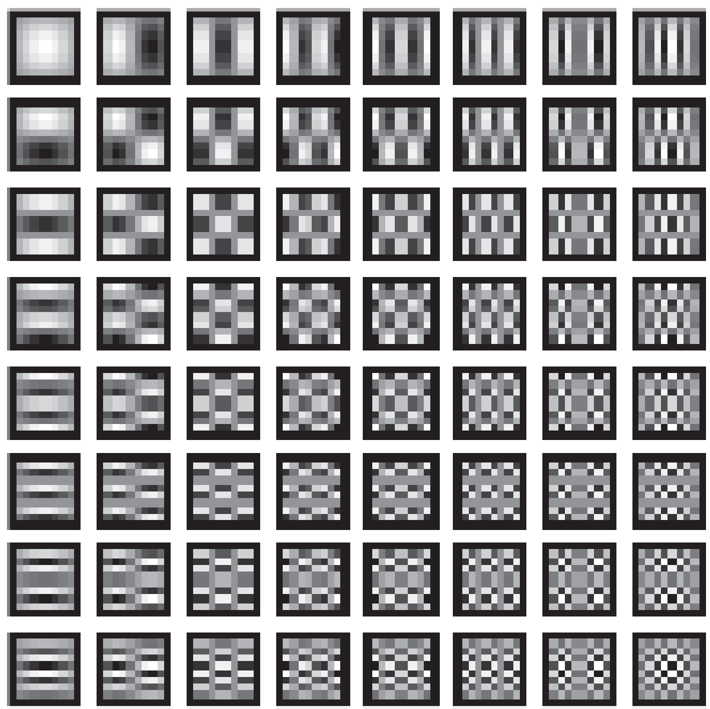
Basis images of matrix-based 2D transforms

Discrete cosine transform (DCT) basis images



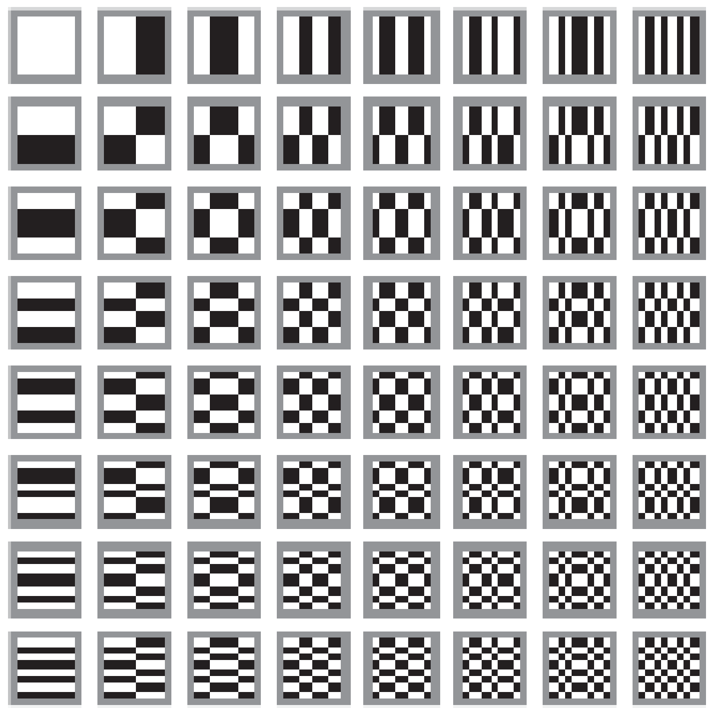
Basis images of matrix-based 2D transforms

Discrete sine transform (DST) basis images



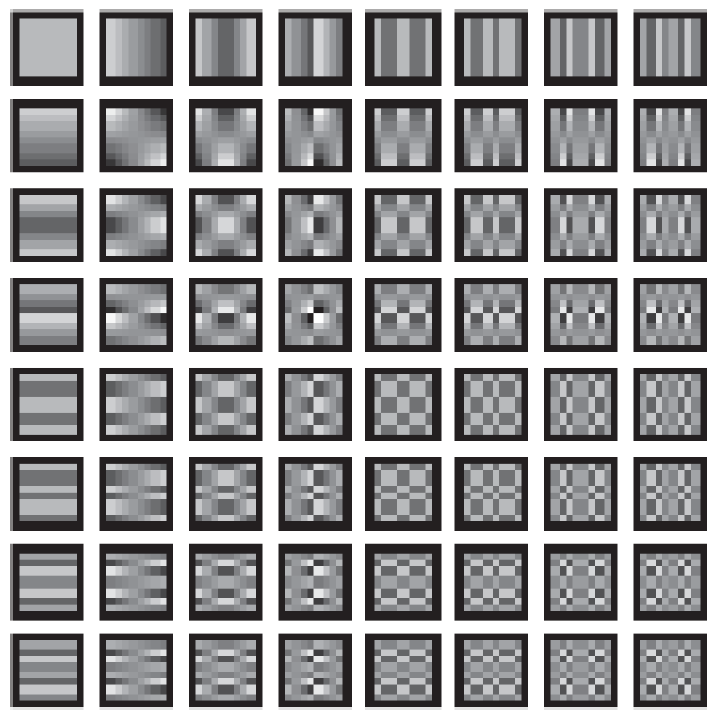
Basis images of matrix-based 2D transforms

Walsh-Hadamard transform (WHT) basis images



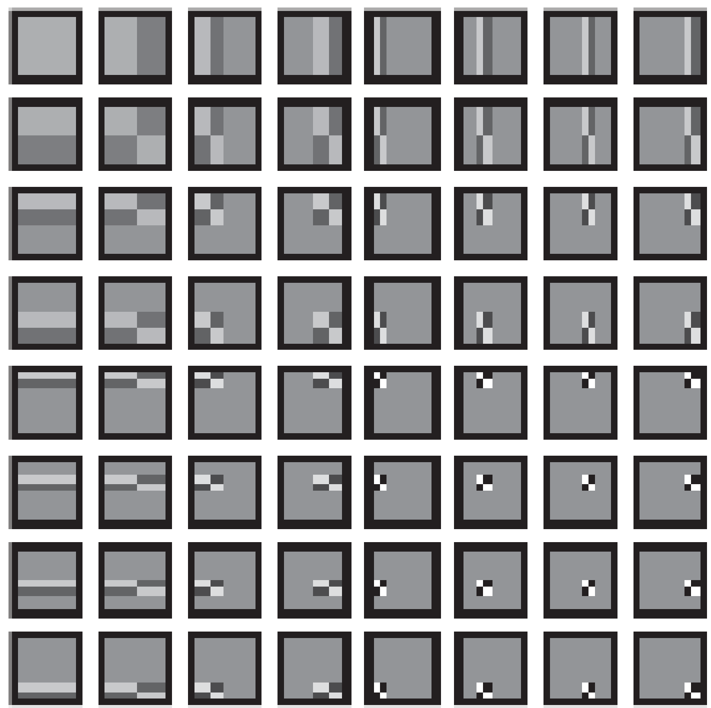
Basis images of matrix-based 2D transforms

Slant transform (SLT) basis images



Basis images of matrix-based 2D transforms

Haar transform (HAAR) basis images



Wavelet transforms

- A scaling function is used to create a series of approximations of a function or image, each differing by a factor of 2 in resolution from its nearest neighboring approximations.
- Wavelet functions (wavelets) are then used to encode the differences between adjacent approximations.
- The discrete wavelet transform (DWT) uses those wavelets, together with a single scaling function, to represent a function or image as a linear combination of the wavelets and scaling function.

Scaling functions and set of basis vectors

- Father scaling function

$$\phi(x)$$

- Set of basis functions

$$\{\phi_{j,k}(x)\}, \text{ where } \phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k) \quad \forall\ j, k \in \mathbb{Z}$$

  – Integer translation *k*

  – Binary scaling *j*

- Basis of the function space spanned by $\phi_{j,k}(x)$ for $j = j_0$ and $k \in \mathbb{Z}$

$$V_{j_0} = \{\phi_{j_0,k}\} \quad \forall\ k \in \mathbb{Z}$$

Scaling function, multiresolution analysis

1. The scaling function is orthogonal to its integer translates
2. The function spaces spanned by the scaling function at low scales are nested within those spanned at higher scales
   $V_{-\infty} \subset \cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots \subset V_\infty$
3. The only function representable at every scale (all V_j) is $f(x) = 0$
4. All measureable, square-integrable functions can be represented as $j \rightarrow \infty$

Wavelet functions

- Given father scaling function $\phi(x)$, there exists a mother wavelet function $\psi(x)$ whose integer translations and binary scalings

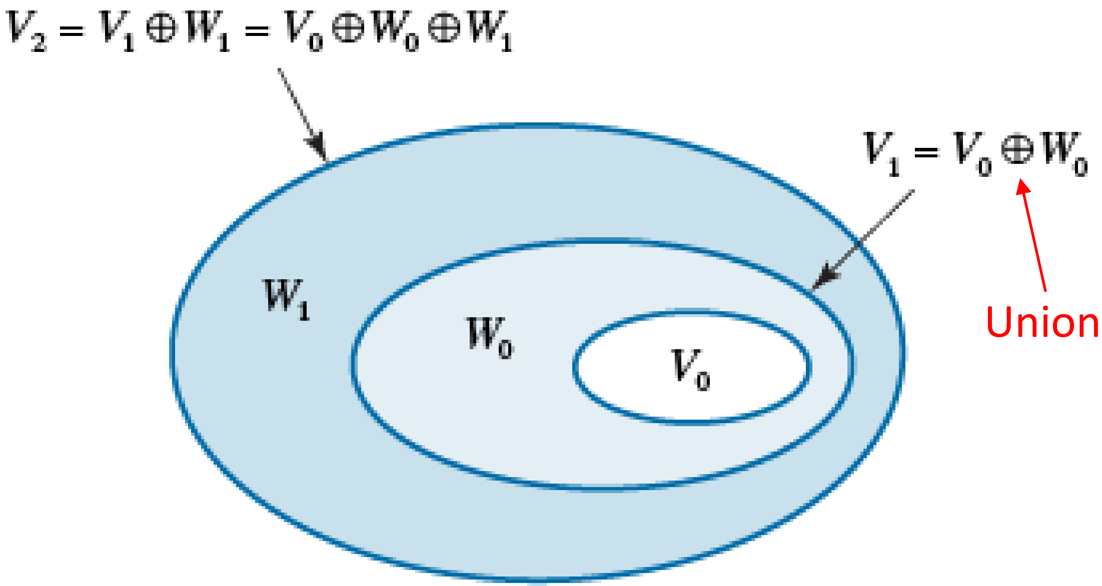
$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k) \quad \forall\ j,k \in \mathbb{Z}$$

  span the difference between any two adjacent scaling spaces
- The orthogonal compliment of V_{j_0} in V_{j_0+1} is W_{j_0}

$$V_{j_0+1} = V_{j_0} \oplus W_{j_0} \text{ for } j = j_0 \text{ and } k \in \mathbb{Z}$$

$$\langle \phi_{j_0,k}(x), \psi_{j_0,l}(x) \rangle = 0 \text{ for } k \neq l$$

Relationship between scaling and wavelet function spaces



Scaling function coefficients and wavelet function coefficients

- Refinement (or dilation) equation

$$\phi(x) = \sum_{k \in \mathbb{Z}} h_\phi(k) \sqrt{2} \phi(2x - k)$$

where $h_\phi(k)$ are scaling function coefficients

- And

$$\psi(x) = \sum_{k} h_\psi(k) \sqrt{2} \psi(2x - k)$$

where $h_\psi(k)$ are wavelet function coefficients

- Relationship

$$h_\psi(k) = (-1)^k h_\phi(1 - k)$$

1D discrete wavelet transform

- Forward

$$T_\phi(0,0) = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} f(x)\phi^*(x)$$ Approximation

$$T_\psi(j,k) = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} f(x)\psi^*_{j,k}(x)$$ Details

- Inverse

$$f(x) = \frac{1}{\sqrt{N}} \left(T_\phi(0,0)\phi(x) + \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} T_\psi(j,k)\psi_{j,k}(x) \right)$$

2D discrete wavelet transform

- Forward

$$T_\phi(j_0, m, n) = \frac{1}{\sqrt{MN}} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) \phi_{j_0,m,n}(x, y)$$ Approximation

$$T_\psi^i(j, m, n) = \frac{1}{\sqrt{MN}} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) \psi_{j,m,n}^i(x, y),\ i = \{H, V, D\}$$ Details

where

$$\phi_{j,m,n}(x, y) = 2^{j/2} \phi(2^j x - m, 2^j y - n)$$

$$\psi_{j,m,n}^i(x, y) = 2^{j/2} \psi^i(2^j x - m, 2^j y - n),\ i = \{H, V, D\}$$

$$\psi^H(x, y) = \psi(x)\phi(y)$$

$$\psi^V(x, y) = \phi(x)\psi(y)$$

$$\psi^D(x, y) = \psi(x)\psi(y)$$

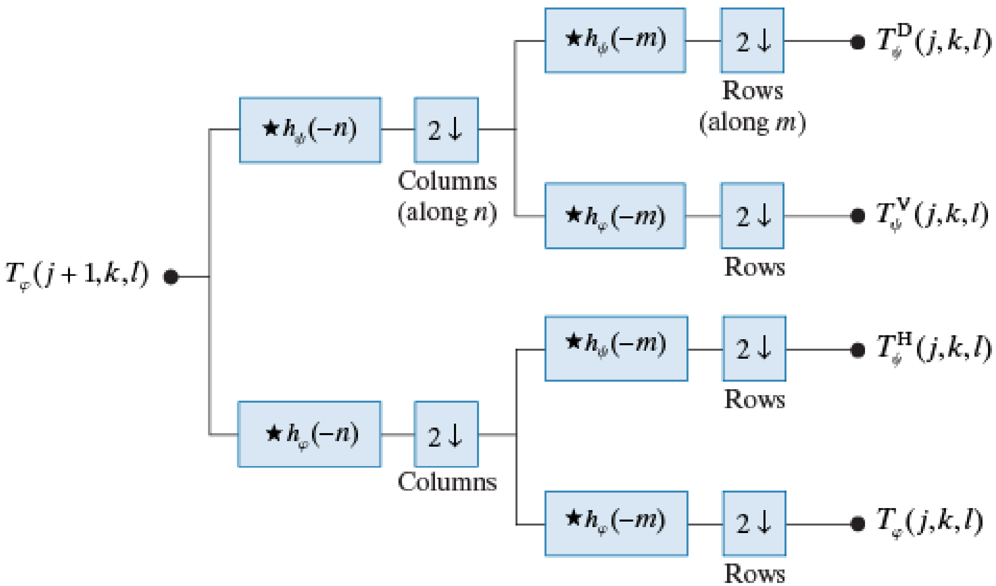
Directional wavelets

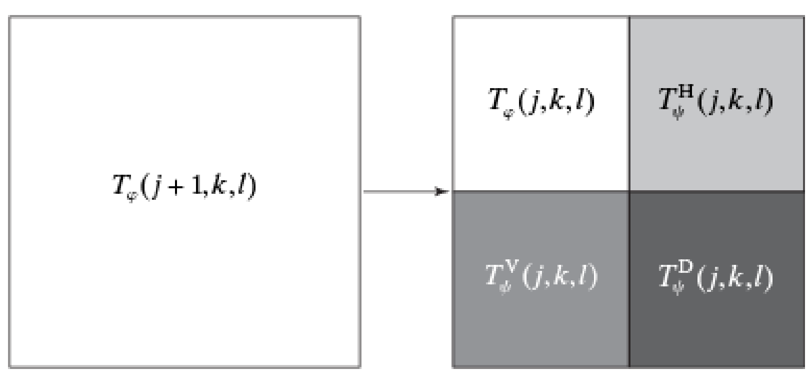
2D discrete wavelet transform

- Inverse

$$f(x,y) = \frac{1}{\sqrt{MN}} \sum_m \sum_n T_\phi(j_0,m,n)\phi_{j_0,m,n}(x,y) + \frac{1}{\sqrt{MN}} \sum_{i=H,V,D} \sum_m \sum_n T_\psi^i(j,m,n)\psi_{j,m,n}^i(x,y)$$

2D discrete wavelet transform

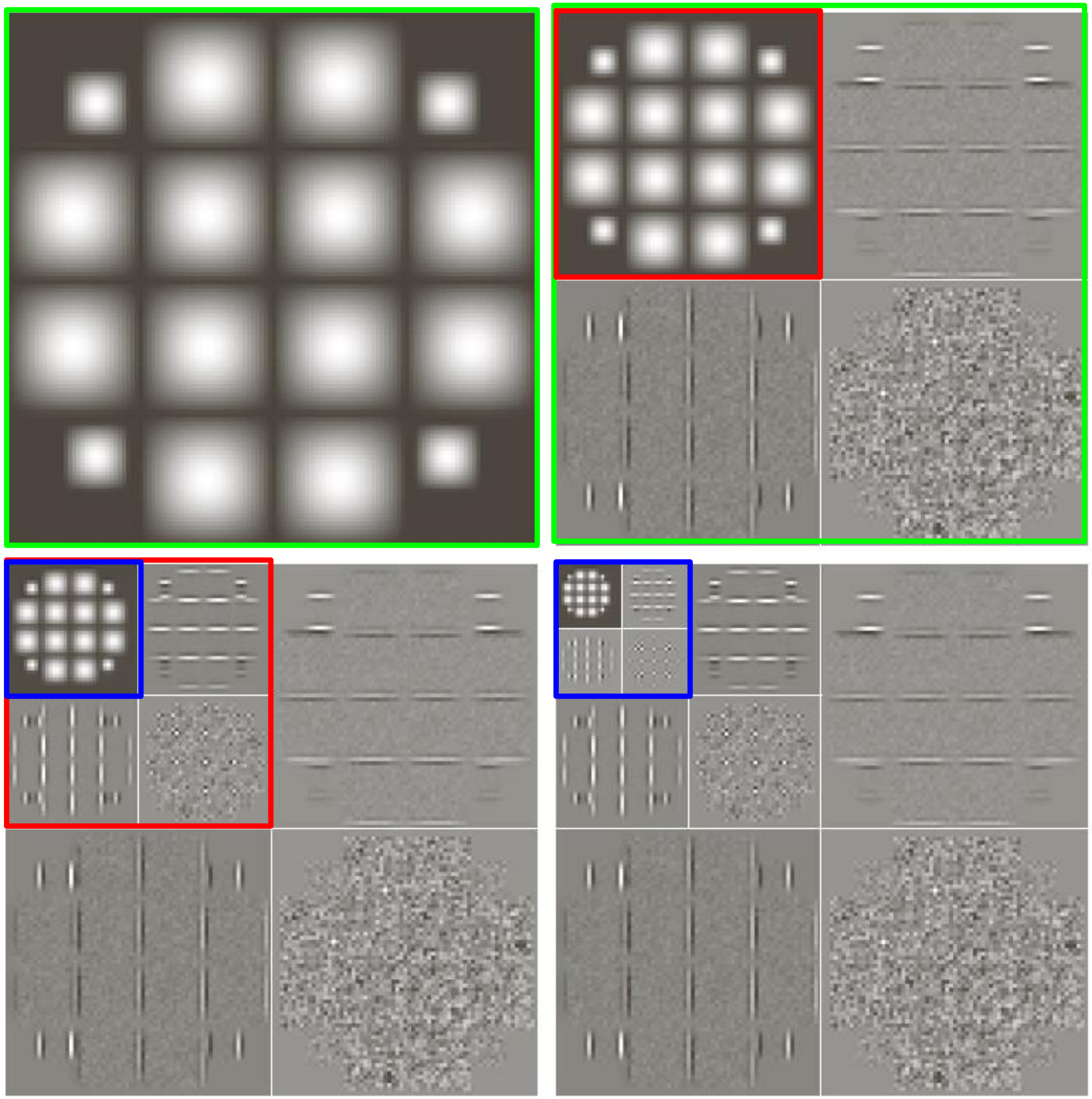




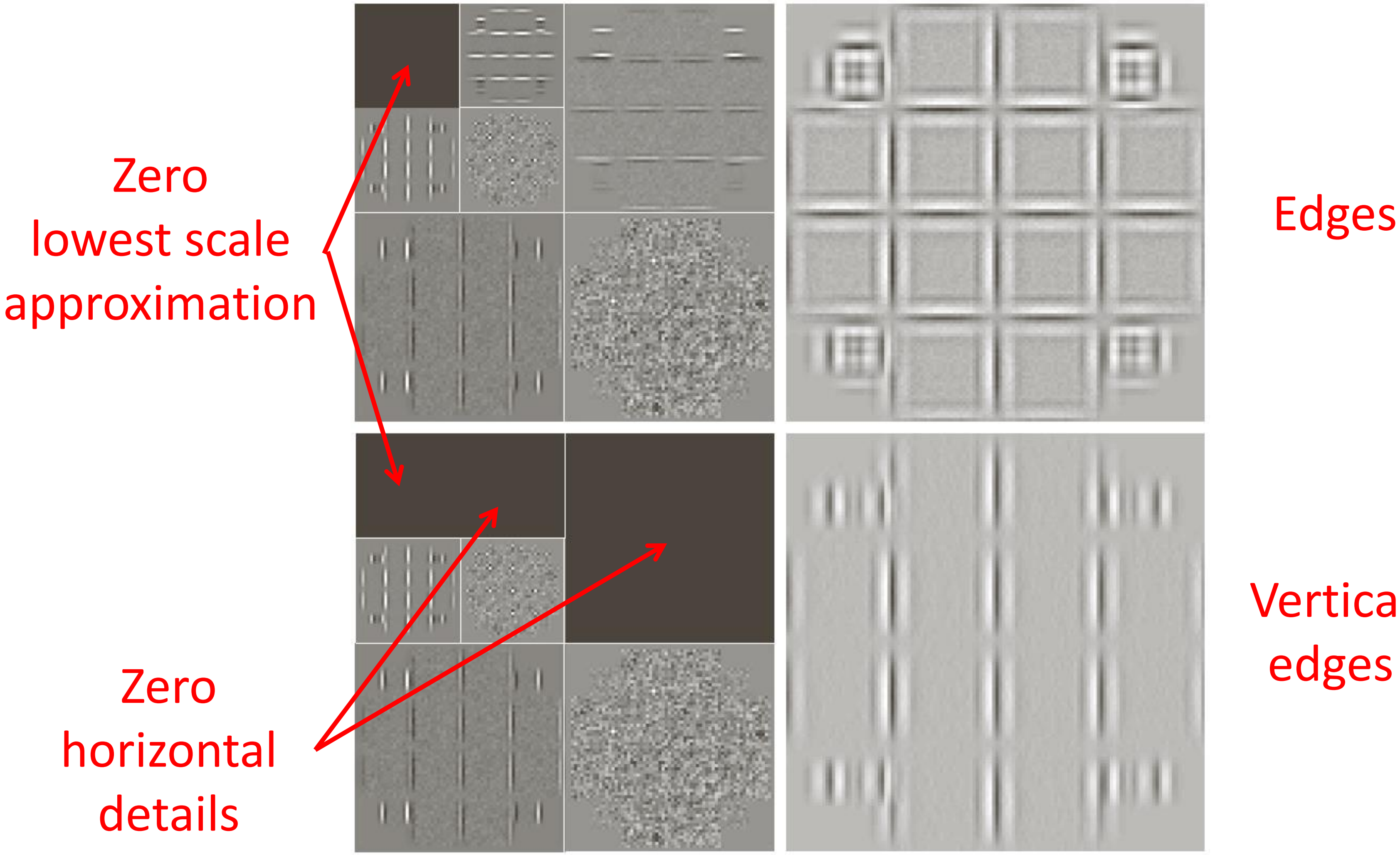
Decomposition

2D discrete wavelet transform

3-level wavelet decomposition



Wavelet-based edge detection



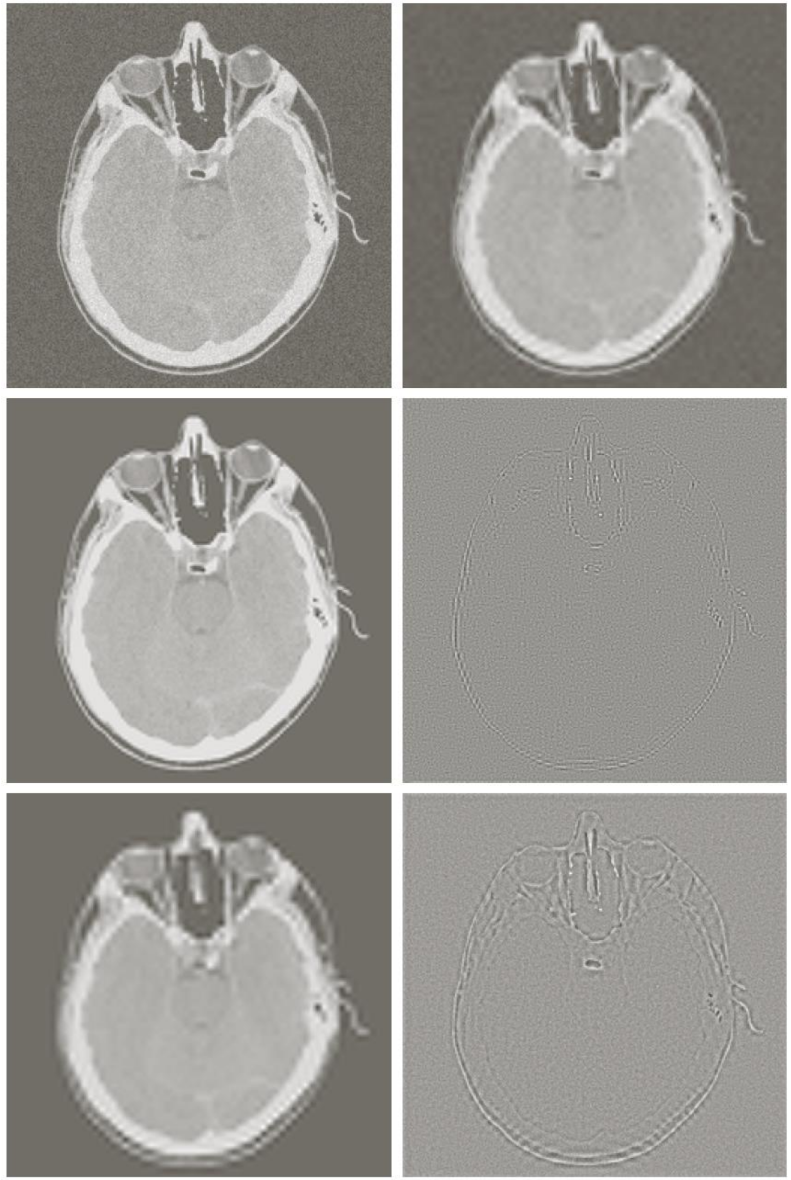
Wavelet-based noise removal

Noisy image

Threshold details

Zero highest resolution details

Zero details for all levels



a b
c d
e f

FIGURE 7.28 Modifying a DWT for noise removal: (a) a noisy CT of a human head; (b), (c) and (e) various reconstructions after thresholding the detail coefficients; (d) and (f) the information removed during the reconstruction of (c) and (e). (Original image courtesy Vanderbilt University Medical Center.)

Next Lecture

- Image compression
- Reading
  - Chapter 8: Image Compression and Watermarking
    - Sections 8.1, 8.9, 8.10, and 8.12