

# CSE140 HW1 Solution (100 Points)

## 1 Introduction

The purpose of this assignment is three-fold. First, it aims to help you practice the application of Boolean Algebra theorems to transform and reduce Boolean expressions. The second goal is to help you learn how to go from the world of Boolean expressions to the world of digital circuits. The final goal is to help you translate a problem described in words to a Boolean algebraic expression. We hope you can think of why each of these exercises is useful when designing digital circuits.

## 2 Application of Boolean Algebra Theorems (15 Points - graded for completion only)

### 2.1 Prove the DeMorgans Law using Boolean algebra (5 Points)

**Prove** the DeMorgans law, i.e.  $(AB)' = A' + B'$ :

**Solution:** Based on the uniqueness of complement, one way to prove the equation  $X' = Y$  is to show that Y is the complement of X, that is,  $X.Y = 0$  and  $X + Y = 1$ . Here, letting  $X = AB$  and  $Y = A' + B'$ , we show that  $AB.(A' + B') = 0$  and  $AB + (A' + B') = 1$ .

**Prove**  $AB.(A' + B') = 0$ :

$$\begin{aligned} AB.(A' + B') &= (AB).A' + (AB).B' && \text{(Distributive)} \\ &= (AA').B + A.(BB') && \text{(Associative)} \\ &= 0.B + A.0 && \text{(Complement)} \\ &= 0 + 0 && \text{(Null element)} \\ &= 0 && \text{(Identity)} \end{aligned}$$

**Prove**  $AB + (A' + B') = 1$ :

$$\begin{aligned} AB + (A' + B') &= (A + (A' + B')).(B + (A' + B')) && \text{(Distributive)} \\ &= ((A + A') + B').((B + B') + A') && \text{(Associative)} \\ &= (1 + B').(1 + A') && \text{(Complement)} \\ &= 1.1 && \text{(Null element)} \\ &= 1 && \text{(Identity)} \end{aligned}$$

## 2.2 Prove the following equations using Boolean algebra (5 Points)

A.  $a'd + bc'd + abc' = a'd + abc'$

B.  $(a' + d)(b + c' + d)(a + b + c') = (a' + d)(a + b + c')$

**Solution:**

**Prove**  $a'd + bc'd + abc' = a'd + abc'$ :

$$\begin{aligned}
 a'd + bc'd + abc' &= a'd + 1.bc'd + abc' && \text{(Identity)} \\
 &= a'd + (a + a').bc'd + abc' && \text{(Complement)} \\
 &= a'd + abc'd + a'bc'd + abc' && \text{(Distributive)} \\
 &= (a'd + a'dbc') + (abc'd + abc') && \text{(Associative)} \\
 &= (a'd.1 + a'dbc') + (abc'd + abc'.1) && \text{(Identity)} \\
 &= a'd(1 + bc') + abc'(d + 1) && \text{(Distributive)} \\
 &= a'd(1) + abc'(1) && \text{(Null element)} \\
 &= a'd + abc' && \text{(Identity)}
 \end{aligned}$$

**Prove**  $(a' + d)(b + c' + d)(a + b + c') = (a' + d)(a + b + c')$ :

$$\begin{aligned}
 (a' + d)(b + c' + d)(a + b + c') &= (a' + d)(0 + b + c' + d)(a + b + c') && \text{(Identity)} \\
 &= (a' + d)((aa') + b + c' + d)(a + b + c') && \text{(Complement)} \\
 &= (a' + d)(a + b + c' + d)(a' + b + c' + d)(a + b + c') && \text{(Distributive)} \\
 &= [(a' + d)(a' + d + c' + b)] && \\
 &\quad .[(a + b + c' + d)(a + b + c')] && \text{(Associative)} \\
 &= [(a' + d + 0)(a' + d + c' + b)] && \\
 &\quad .[(a + b + c' + d)(a + b + c' + 0)] && \text{(Identity)} \\
 &= [(a' + d) + (0.(c' + b))] && \\
 &\quad .[(a + b + c') + (d.0)] && \text{(Distributive)} \\
 &= [(a' + d) + (0)] && \\
 &\quad .[(a + b + c') + (0)] && \text{(Null element)} \\
 &= (a' + d)(a + b + c') && \text{(Identity)}
 \end{aligned}$$

Are the above equations related to the consensus theorem?

**Solution:** Yes.

Letting  $X = a'$ ,  $Y = d$ , and  $Z = bc'$  in the consensus theorem  $XY + YZ + X'Z = XY + X'Z$  gives equation A.

Letting  $X = a'$ ,  $Y = d$ , and  $Z = b + c'$  in the consensus theorem  $(X + Y)(Y + Z)(X' + Z) = (X + Y)(X' + Z)$  gives equation B.

Indeed, the same sequence of Boolean algebra laws we used to prove the equations can be used to prove the consensus theorems.

### 2.3 Prove the above two equations using Shannon's expansion (5 Points)

**Solution:**

**A. Prove**  $a'd + bc'd + abc' = a'd + abc'$ :

$$\begin{aligned} a'd + bc'd + abc' &= f(a, b, c, d) \\ &= a.f(1, b, c, d) + a'.f(0, b, c, d) && \text{(Shannon's expansion)} \\ &= a(0.d + bc'd + 1.bc') + a'(1.d + bc'd + 0.bc') \\ &= a(bc'd + bc') + a'(d + bc'd) && \text{(Null element + Identity)} \\ &= a(bc') + a'(d) && \text{(Absorption)} \\ &= a'd + abc' && \text{(Commutative)} \end{aligned}$$

**B. Prove**  $(a' + d)(b + c' + d)(a + b + c') = (a' + d)(a + b + c')$ :

$$\begin{aligned} (a' + d)(b + c' + d)(a + b + c') &= f(a, b, c, d) \\ &= (a + f(0, b, c, d))(a' + f(1, b, c, d)) && \text{(Shannon's expansion)} \\ &= (a + (1 + d)(b + c' + d)(0 + b + c')) \\ &\quad .(a' + (0 + d)(b + c' + d)(1 + b + c')) \\ &= (a + (b + c' + d)(b + c')) \\ &\quad .(a' + (d)(b + c' + d)) && \text{(Null element + Identity)} \\ &= (a + (b + c'))(a' + (d)) && \text{(Absorption)} \\ &= (a' + d)(a + b + c') && \text{(Commutative)} \end{aligned}$$

### 3 From Problem to Boolean Expression (30 Points)

A majority voting machine inputs four binary bits  $(x_3, x_2, x_1, x_0)$ , and outputs  $y = 0$  when the more than half of the input bits are 1. Otherwise, the output is  $y = 1$ .

i. Write the truth table. (10 Points)

| Index | $x_3$ | $x_2$ | $x_1$ | $x_0$ | $y$ |
|-------|-------|-------|-------|-------|-----|
| 0     | 0     | 0     | 0     | 0     | 1   |
| 1     | 0     | 0     | 0     | 1     | 1   |
| 2     | 0     | 0     | 1     | 0     | 1   |
| 3     | 0     | 0     | 1     | 1     | 1   |
| 4     | 0     | 1     | 0     | 0     | 1   |
| 5     | 0     | 1     | 0     | 1     | 1   |
| 6     | 0     | 1     | 1     | 0     | 1   |
| 7     | 0     | 1     | 1     | 1     | 0   |
| 8     | 1     | 0     | 0     | 0     | 1   |
| 9     | 1     | 0     | 0     | 1     | 1   |
| 10    | 1     | 0     | 1     | 0     | 1   |
| 11    | 1     | 0     | 1     | 1     | 0   |
| 12    | 1     | 1     | 0     | 0     | 1   |
| 13    | 1     | 1     | 0     | 1     | 0   |
| 14    | 1     | 1     | 1     | 0     | 0   |
| 15    | 1     | 1     | 1     | 1     | 0   |

ii. Write the functions in sum-of-products canonical form. (10 Points)

$$F(x_3, x_2, x_1, x_0) = \sum m(0, 1, 2, 3, 4, 5, 6, 8, 9, 10, 12) = x'_3x'_2x'_1x'_0 + x'_3x'_2x'_1x_0 + x'_3x'_2x_1x'_0 + x'_3x'_2x_1x_0 + x'_3x_2x'_1x'_0 + x'_3x_2x'_1x_0 + x'_3x_2x_1x'_0 + x_3x'_2x'_1x'_0 + x_3x'_2x'_1x_0 + x_3x'_2x_1x'_0 + x_3x_2x'_1x'_0$$

iii. Write the functions in product-of-sums canonical form. (10 Points)

$$F(x_3, x_2, x_1, x_0) = \prod M(7, 11, 13, 14, 15) = (x_3 + x'_2 + x'_1 + x'_0)(x'_3 + x_2 + x'_1 + x'_0)(x'_3 + x'_2 + x_1 + x'_0)(x'_3 + x'_2 + x'_1 + x_0)(x'_3 + x'_2 + x'_1 + x'_0)$$

## 4 Boolean Algebra and Implementation (50 Points - only part B graded)

- Simplify each of the following two Boolean equations. **(30 Points)**
- Sketch a reasonably simple combinational circuit implementing the simplified equation. **(10 Points)**
- Compare the numbers of literals and operators versus the numbers of gates, nets, and pins in the schematic diagrams. **(10 Points)**

A.  $abc'd + ab'c + bc'd + ab'c' + acd + a'bcd$

B.  $(b' + ac)'(a'b' + c)(a + b)$

**Solution:**

**A.**  $abc'd + ab'c + bc'd + ab'c' + acd + a'bcd$ :

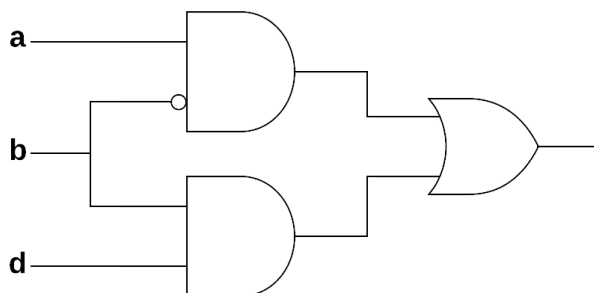
$$\begin{aligned}
 abc'd + ab'c + bc'd + ab'c' + acd + a'bcd &= (abc'd + bc'd) + (ab'c + ab'c') + (acd + a'bcd) && \text{(Associative)} \\
 &= (bc'd) + (ab'c + ab'c') + (acd + a'bcd) && \text{(Absorption)} \\
 &= bc'd + ab'(c + c') + cd(a + a'b) && \text{(Distributive)} \\
 &= bc'd + ab'(1) + cd(a + a'b) && \text{(Complement)} \\
 &= bc'd + ab' + cd(a + a'b) && \text{(Identity)} \\
 &= bc'd + ab' + cd(a + b + a'b) && \text{(Consensus)} \\
 &= bc'd + ab' + cd(a + b) && \text{(Absorption)} \\
 &= bc'd + ab' + acd + bcd && \text{(Distributive)} \\
 &= (bc'd + bcd) + ab' + acd && \text{(Associative)} \\
 &= bd(c' + c) + ab' + acd && \text{(Distributive)} \\
 &= bd(1) + ab' + acd && \text{(Complement)} \\
 &= bd + ab' + acd && \text{(Identity)} \\
 &= bd + ab' + ad + acd && \text{(Consensus)} \\
 &= bd + ab' + ad && \text{(Absorption)} \\
 &= bd + ab' && \text{(Consensus)}
 \end{aligned}$$

Note: You might be able to replace some operations with consensus theorem.

#Variables = 3

#Literals = 4

#Operators = 3 (2 AND and 1 OR)



**#Gates** = 3 (2 AND and 1 OR) = **#Operators**

**#Nets** = 6 = **#Variables** + **#Operators**

**#Pins** = 9 = **#Literals** + 2 \* **#Operators** - 1

**B.**  $(b' + ac)'(a'b' + c)(a + b)$ :

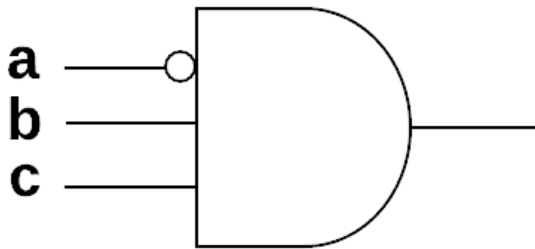
$$\begin{aligned}(b' + ac)'(a'b' + c)(a + b) &= (b' + ac)'[(a'b')(a + b) + c(a + b)] && \text{(Distributive)} \\ &= (b' + ac)'[c(a + b)] && \text{(Complement + Identity)} \\ &= (b' + ac)'(ac + bc) && \text{(Distributive)} \\ &= (b.(ac)')(ac + bc) && \text{(DeMorgan's law)} \\ &= b.(ac)'.(ac) + b.(ac)'.(bc) && \text{(Distributive)} \\ &= b.(ac)'.(bc) && \text{(Complement + Null element + Identity)} \\ &= (a' + c').(bc) && \text{(DeMorgan's law + Idempotent)} \\ &= a'bc + bcc' && \text{(Distributive)} \\ &= a'bc && \text{(Complement + Null element + Identity)}\end{aligned}$$

Note: You might be able to replace some operations with consensus theorem.

**#Variables** = 3

**#Literals** = 3

**#Operators** = 1

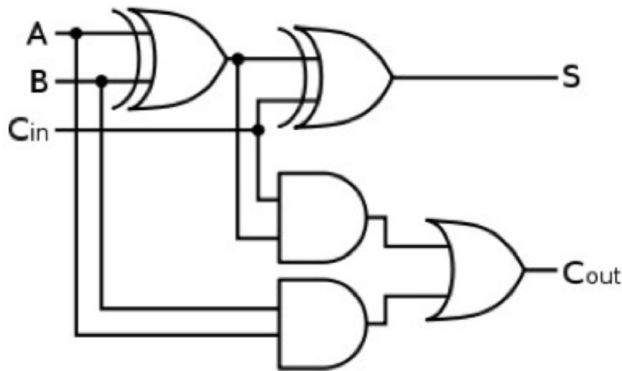


**#Gates** = 1 = **#Operators**

**#Nets** = 4 = **#Variables** + **#Operators**

**#Pins** = 4 = **#Literals** + 2 \* **#Operators** - 1

## 5 BSV Adder (5 Points - graded for completion only)



Complete the following BSV implementation of a full adder (shown above) using `&`, `|`, and `^` for logic AND, OR, and XOR, respectively. Please refer to *BSV by Example* for any syntax questions.

```
function Bit#(2) fa (Bit#(1) a, Bit#(1) b, Bit#(1) c_in);

    Bit#(1) s = _____;

    Bit#(1) c_out = _____;

    return {c_out, s};
endfunction
```

**Solution:**

```
function Bit#(2) fa (Bit#(1) a, Bit#(1) b, Bit#(1) c_in);
    Bit#(1) s = (a ^ b) ^ c_in;
    Bit#(1) c_out = (a & b) | (c_in & (a ^ b));
    return {c_out,s};
endfunction
```