
The second exam for this class is on Wednesday May 23. The exam covers sections 1.4, 2.1, 2.2, 2.3, 3.1, 3.2, 3.3, 4.1 of Sipser.

This practice exam has some problems that would be good practice for the exam. You will get the most out of this practice exam if you attempt the problems on your own first, then ask for help from your classmates and the instructional staff if you get stuck.

The formatting of questions in the actual exam may be different from the questions below, and will be shorter than this practice exam. These practice questions are designed to help in your studying.

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1. **Context-Free Languages**

   Let L be the set of strings over $\{a, b\}$ that contain at least one b in the second half of the string. For strings of odd length, we do not consider the middle symbol to be part of the second half. For example, $aba \notin L$.

   (a) Draw the state transition diagram of a PDA recognizing L.

   (b) Design and specify a CFG generating L.

   (c) Is L regular? Prove your answer.

2. **Closure** Prove or disprove the following claims:

   (a) The set of decidable languages over some fixed alphabet Σ is closed under the operation of taking subsets (that is, if L_1 is decidable and $L_2 \subseteq L_1$, then L_2 is decidable.)

   (b) The set of context-free languages over some fixed alphabet Σ is closed under Kleene star.

   (c) The set of Turing-recognizable languages over some fixed alphabet Σ is closed under intersection.

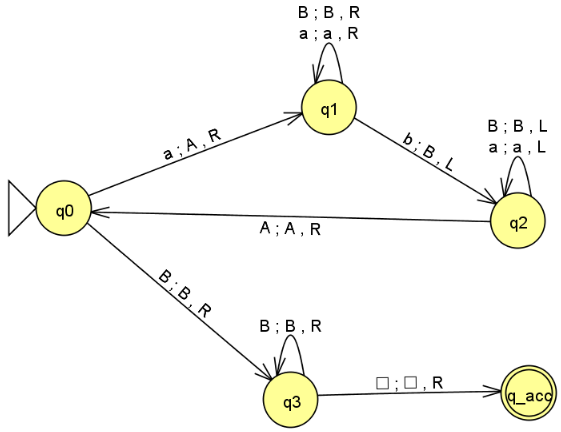
   (d) Let A and B be sets of languages over some fixed alphabet Σ, with $A \subseteq B$. If A is closed under some operation P, then B is also closed under P.

   (e) There exist sets A, B, C, D, E all over the alphabet $\{a, b, c\}$ such that

   - A is not context-free
   - B is context-free and nonregular
   - C is regular
   - D is non-regular
   - E is regular and not $\{a, b, c\}^*$
   - $A \subsetneq B \subsetneq C \subsetneq D \subsetneq E$.

3. **Turing Machines**

   (a) Consider the Turing machine over the alphabet $\Sigma = \{a, b\}$ with tape alphabet $\Gamma = \{a, b, A, B, \square\}$, given by the following state diagram (using the convention that the reject state is omitted from the diagram and any missing transitions are assumed to be sent to it).



(i) Write the formal definition of this machine.

(ii) Write an implementation-level description of this machine.

(iii) Trace the computation of this machine (by listing out the configurations of the machine) on input (1) ab , (2) $aaabb$, and (3) ε.

(iv) What is the language recognized by this machine?

(v) This machine is a decider. Modify the state diagram slightly to obtain a new machine that recognizes the same language but is not a decider.

(b) Consider the language

$$L = \{w = w_1 w_2 \dots w_n \in \{0,1\}^* \mid w_i \in \{0,1\}, n \geq 0, w_1 + w_2 + \cdots + w_n = 4k \text{ for some integer } k \geq 0\}.$$

(i) Give an implementation-level description of a Turing machine that decides L.

(ii) Give a formal definition (i.e. by drawing and labelling the state diagram) for a Turing machine that decides L. In your state diagram, you may follow our usual conventions: you do not need to draw the reject state nor any incoming transitions to it, nor outgoing transitions from the accept state.

(c) Prove that the set of Turing-decidable languages is closed under reversal. *Use high-level descriptions in your proof.*

4. **Church-Turing thesis** A Turing machine with left reset is similar to an ordinary Turing machine, but the transition function has the form

$$\delta : Q \times \Gamma \to Q \times \Gamma \times \{\text{R}, \text{RESET}\}$$

If $\delta((q, a)) = (r, b, \text{RESET})$, when the machine is in state q reading an a, the machine's head jumps to the left-hand end of the tape after it writes b on the tape and enters state r. Show that each Turing machine with left reset is equivalent to some ordinary Turing machine.

5. **Decidable problems** For this question, the alphabet is $\{0, 1\}$. Show that the following languages are decidable.

(a)

$$L_{nondet} = \{\langle N \rangle \mid N \text{ is a NFA and the state diagram of } N \text{ is not the state diagram of a DFA}\}$$

(b)

$$L_{11} = \{\langle R \rangle \mid R \text{ is a regular expression and } L(R) \text{ has at least one string with two consecutive ones}\}$$

(c)

$$L_{rec} = \{\langle M \rangle \mid M \text{ is a Turing machine and } L(M) \text{ is Turing-recognizable}\}$$