# CSE 105

## THEORY OF COMPUTATION

Spring 2018

http://cseweb.ucsd.edu/classes/sp18/cse105-ab/

# Today's learning goals

Sipser Section 1.4

- Apply the Pumping Lemma in proofs of nonregularity
- Identify some nonregular sets

- **Exam 1** Wednesday April 25, in class
  - Request special seats via Google form by end of class today!
  - Seat assignments posted on Piazza tomorrow
  - Review: in-class on Monday and Monday evening 7-9pm  GH 242

# Pumping Lemma

*Sipser p. 78 Theorem 1.70*

If $A$ is a regular language, then there is a number $p$ (*the pumping length*) where, if $s$ is any string in $A$ of length at least $p$, then $s$ may be divided into three pieces, $s = xyz$ such that

- $|y| > 0$, and
- for each $i \geq 0$, $xy^iz \in A$,
- $|xy| \leq p$.

# states in DFA recognizing A

Transition labels along loop

# Pumping Lemma

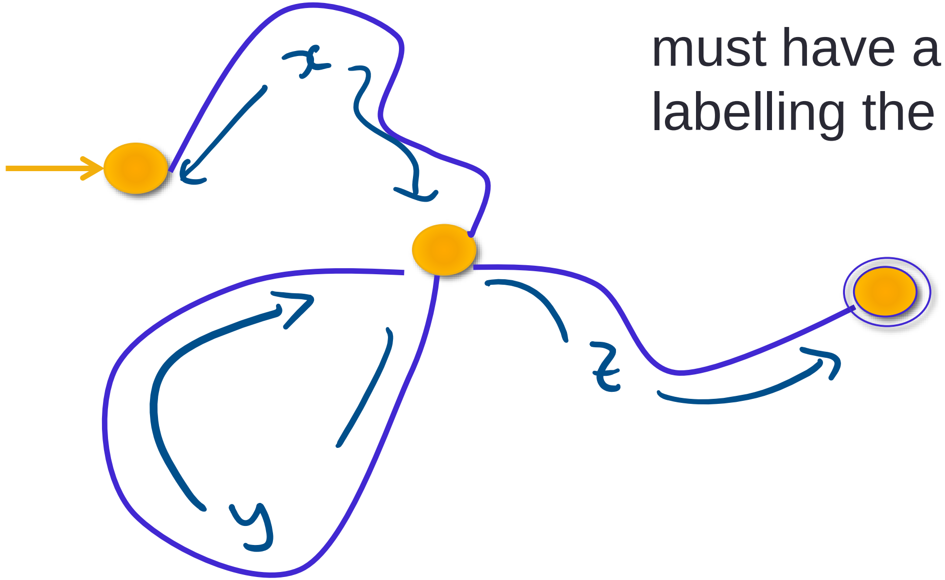
If A is recognized by DFA M with state diagram below,

the computation of M on any string s of length >= p = |Q|

must have a **loop**. Divide s into the strings labelling the path before the loop x ,

the loop itself y , and

from the loop to the accept state z



# Pumping Lemma

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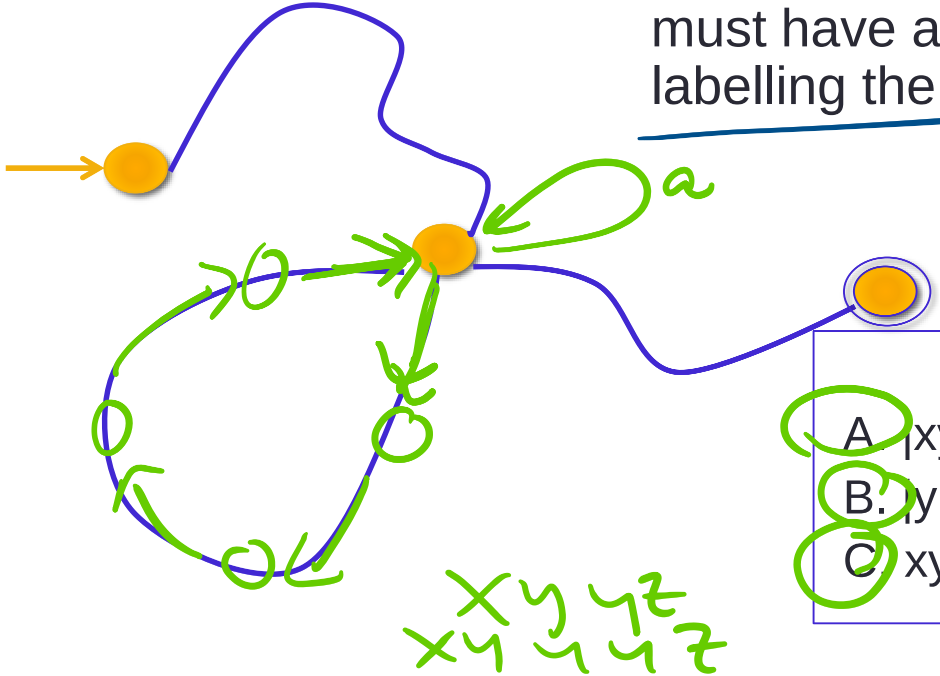

Which of the following is true?

A. |xy| <= p

B. |y| > 0

C. $xy^iz$ is accepted by M for all i

D. All of A,B,C

E. None of them

# Pumping Lemma

- True for **all** (but not only) regular sets.

  - Can't be used to prove that a set ***is*** regular   Ex 1.49
  - Can be used to a prove that a set ***is not*** regular … how?

# Negation

flash-back to CSE 20 ☺

- Pumping lemma ``**There is** p, where p is a pumping length for L"

- Given a specific number p, it being a pumping length for L means

$$\forall s\left(\left(|s| \geq p \wedge s \in L\right) \rightarrow \exists x \exists y \exists z\left(s = xyz \wedge |y| > 0 \wedge |xy| \leq p \wedge \forall i(xy^i z \in L)\right)\right)$$

- So p **not** being a pumping length of L means

$$\exists s\left(|s| \geq p \wedge s \in L \wedge \forall x \forall y \forall z\left(\left(s = xyz \wedge |y| > 0 \wedge |xy| \leq p\right) \rightarrow \exists i(xy^i z \notin L)\right)\right)$$

all ways of cutting s fail

# Proof strategy

To prove that a language L is **not** regular

- Consider arbitrary positive integer p.
- Prove that p isn't a pumping length for L.

  Find s!

- Conclude that L does not have any pumping length and is therefore not regular.

# Pumping Lemma

*Sipser p. 78 Theorem 1.70*

If $A$ is a regular language, then there is a number $p$ (*the pumping length*) where, if $s$ is any string in $A$ of length at least $p$, then $s$ may be divided into three pieces, $\boxed{s = x\ y\ z}$ such that

- $|y| > 0$, and
- for each $i \geq 0$, $xy^iz \in A$,
- $|xy| \leq p$.

How does this apply to some known regular languages, e.g. the set of all strings, or {a, ab}

# Using the Pumping Lemma

**Claim:** The set $L = \{0^n1^n \mid n \geq 0\}$ is not regular.

Pf: Let p arbitrary pos int.
WTS p is not pumping length for L.

$$s = 0^p 1^p$$

* in L
* $|s| = 2p > p$

# Using the Pumping Lemma

**Claim:** The set $L = \{0^n 1^n \mid n \geq 0\}$ is not regular.

**Proof:** Consider an arbitrary positive integer. WTS p is not a pumping length for L.

*How?* Want to show that there is some string that *should* be pump'able but isn't.

Conclude that L does not have any pumping length and is therefore not regular.

# Using the Pumping Lemma

L = $\{0^n1^n \mid n \geq 0\}$ **CLAIM:** p is not a pumping length for L.

*How would you prove the claim?*

A. Find a string with length >= p that is not in L.

B. Find a string with length <p that is in L.

C. None of the above.

$$\exists s\left(|s| \geq p \wedge s \in L \wedge \forall x \forall y \forall z\left((s = xyz \wedge |y| > 0 \wedge |xy| \leq p) \rightarrow \exists i(xy^iz \notin L)\right)\right)$$

# Using the Pumping Lemma

$L = \{0^n1^n \mid n \geq 0\}$ **CLAIM:** p is not a pumping length for L.

*WTS*

$$\exists s \left(|s| \geq p \land s \in L \land \forall x \forall y \forall z \left((s = xyz \land |y| > 0 \land |xy| \leq p) \rightarrow \exists i (xy^i z \notin L)\right)\right)$$

Find a string s such that

1. |s| >= p
2. s is in L
3. No matter how we cut s into three (viable) pieces, some related string obtained by repeating the middle part falls out of L.

# Using the Pumping Lemma

**Claim:** The set L = **$\{0^n1^n \mid n \geq 0\}$** is not regular.

**Proof:** Consider an arbitrary positive integer. WTS p is not a pumping length for L.

Consider the string

$$s = 0^p1^p.$$

1. |s| >= p ?
2. s is in L ?
3. No matter how we cut s into three (viable) pieces, some related string obtained by repeating the middle part falls out of L ?

# Using the Pumping Lemma

parameterized by p

**Claim:** The set L = **$\{0^n1^n \mid n \geq 0\}$** is not regular.

**Proof:** Consider an arbitrary positive integer. WTS p is not a pumping length for L. Consider the string $s = 0^p1^p$. Then, s is in L and $|s| = 2p >= p$. Consider any division of s into three parts

$$s = xyz \text{ with } |y|>0, |xy| \leq p.$$

Since $|xy| \leq p$, $x=0^k$ , $y = 0^m$ , $z = 0^r1^p$ with $k+m+r = p$, and since $|y| > 0$, $m>0$. Picking i=0: $xy^iz = xz = 0^k0^r1^p = 0^{k+r}1^p$ , which is not in L because $k+r < p$. Thus, no p can be a pumping length for L and L is not regular.

y missing!

unbalanced!

# Proof strategy

To prove that a language L is **not** regular

- Consider arbitrary positive integer p.
- Prove that p isn't a pumping length for L.

- Conclude that L does not have any pumping length and is therefore not regular.

# Picking s

To complete proofs with Pumping Lemma, we will need to build (useful) examples of strings with **length >= p** that are **in** a given language.

- L1 = $\{a^n b^m a^n \mid m,n \geq 0\}$
- L2 = { ww | w is a string over {0,1} }
- L3 = { $ww^R$ | w is a string over {0,1} }

# Another example

**Claim:** The set $\{w\, w^R \mid w$ is a string over $\{0,1\}\}$ is not regular.

**Proof**: … You must pick s carefully: we want $|s| \geq p$ and s in L and s "can't be pumped" … **Consider i=…**

Which s and i let us complete the proof?

A. $s = 0^p0^p$, i=2 ~~B.~~ $s = 0110$, i=0 ~~C.~~ $s = 0^p110^p$, i=1 D. $s = 1^p001^p$, i=3

E. I don't know

not longer then p-1

$s \in L$

$s = xyz$

$s = xy^iz \notin L$

# How do we choose i?

**Claim:** The set $\{0^j 1^k \mid j,k \geq 0 \text{ and } j \geq k\}$ is not regular.

**Proof**: … You must pick s carefully: we want $|s| \geq p$ and s in L and s "can't be pumped" … **Consider i=…**

Which s and i let us complete the proof?

A. $s = 0^p1^p$, i=2 B. $s = 0^p1^p$, i=p C. $s = 0^p1^p$, i=1 D. $s = 0^p1^p$, i=0

E. I don't know

# Another example

**Claim:** The set $\{a^n b^m a^n \mid m,n \geq 0\}$ is not regular.

**Proof**: … You must pick s carefully: we want $|s| \geq p$ and s in L and s "can't be pumped"

Which choices of s could we have used in the proof?

A. $s = a^p b^p$ B. $s = aba$ C. $s = a^p b^p a^p$ D. $s = b^p$

E. None of the above

# Do we always need Pumping Lemma?

**Claim**: The set

{w | w has different #s of 0s and 1s OR has a 1 before a 0}

is nonregular.

**Proof:**

# Regular sets: not the end of the story

- Many **nice / simple / important** sets are not regular
- Limitation of the finite-state automaton model
  - Can't "count"
  - Can only remember finitely far into the past
  - Can't backtrack
  - Must make decisions in "real-time"
- We know computers are more powerful than this model…

*Which conditions should we relax?*

# For next time

- Work on Practice questions for Exam 1

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