CSE 105
THEORY OF COMPUTATION

Spring 2018

http://cseweb.ucsd.edu/classes/sp18/cse105-ab/

Today's learning goals

Sipser Section 1.4

- Explain the limits of the class of regular languages
- Justify why the Pumping Lemma is true
- Apply the Pumping Lemma in proofs of nonregularity
- Identify some nonregular sets

Proving nonregularity

How can we prove that a set is non-regular?

A. Try to design a DFA that recognizes it and, if the first few attempts don't work, conclude there is none that does.
B. Prove that it's a strict subset of some regular set.
C. Prove that it's the union of two regular sets.
D. Prove that its complement is not regular.
E. I don't know.

Counting languages

How many languages over {0,1} are there?

A. Finitely many because {0,1} is finite.
B. Finitely many because strings are finite.
C. Countably infinitely many because {0,1}* is countably infinite.
D. Uncountably many because languages are in the power set of {0,1}*.
E. None of the above.

Counting regular languages over {0,1}

| { regular languages} | ≤ | { regular expressions } |

Each regular expression is a finite string over the alphabet

{ 0 , 1 , ε , ø , (,) , U , * }

The set of strings over an alphabet is countably infinite.

Conclude: countably infinitely many regular languages.

Where we stand

Fact 1: There exist nonregular languages.

Fact 2: If we know some languages are nonregular, we can conclude others must be too. (cf. Discussion)

But, we don't have any specific examples of nonregular languages.

Yet.

Bounds on DFA

- in DFA, memory = states

- Automata can only "remember"…
  - …finitely far in the past
  - …finitely much information

- If a computation path visits the same state more than once, the machine can't tell the difference between the first time and future times it visited that state.

Example!

$$\{ 0^n 1^n \mid n \geq 0 \}$$

What are some strings in this set?

What are some strings not in this set?

Is this set finite or infinite?

Compare to $L(0^*1^*)$

Design a DFA? NFA?

Example!

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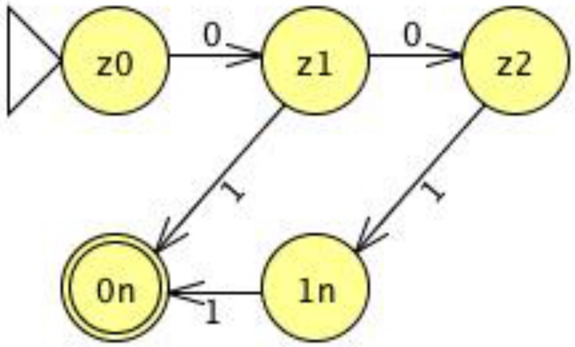
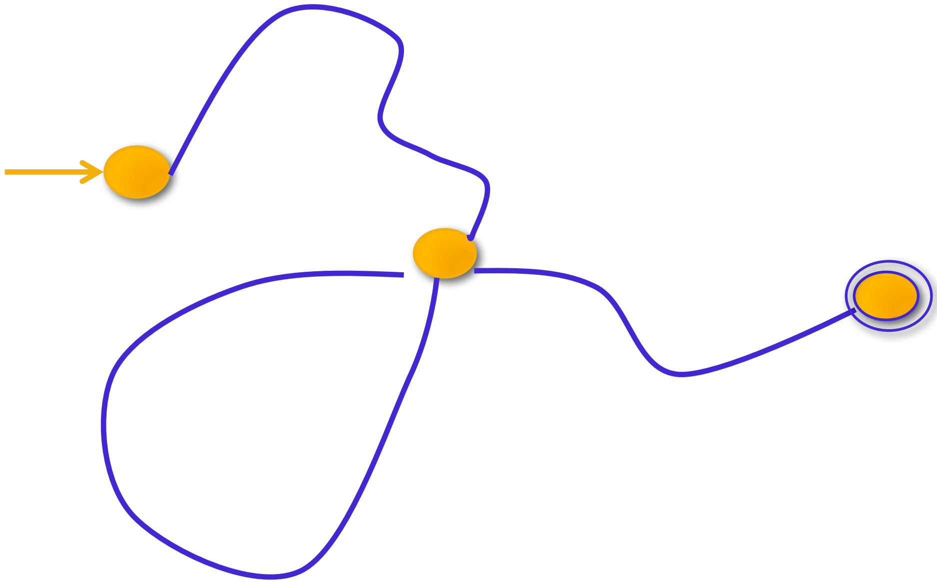
Design a DFA? NFA?



Pumping

- Focus on computation path through DFA



Pumping

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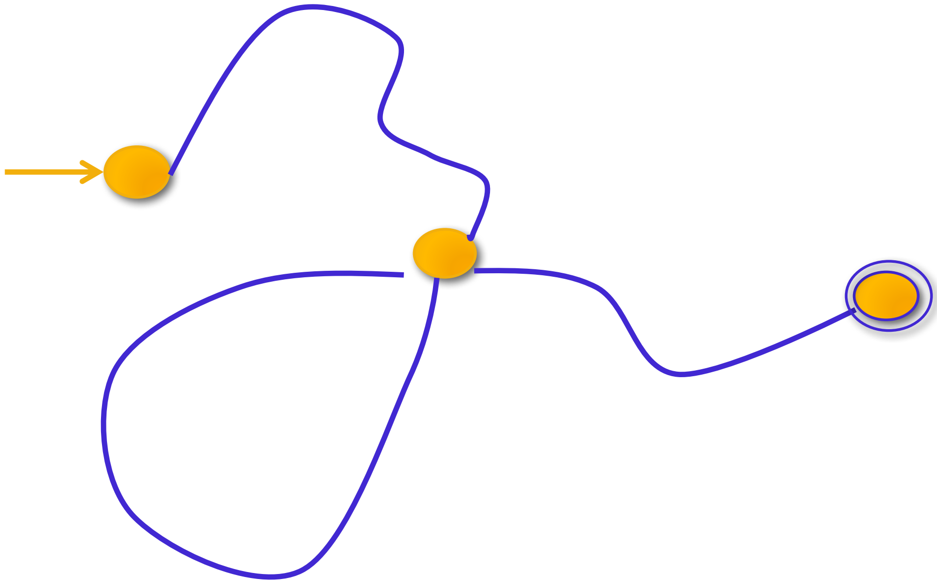

Pumping

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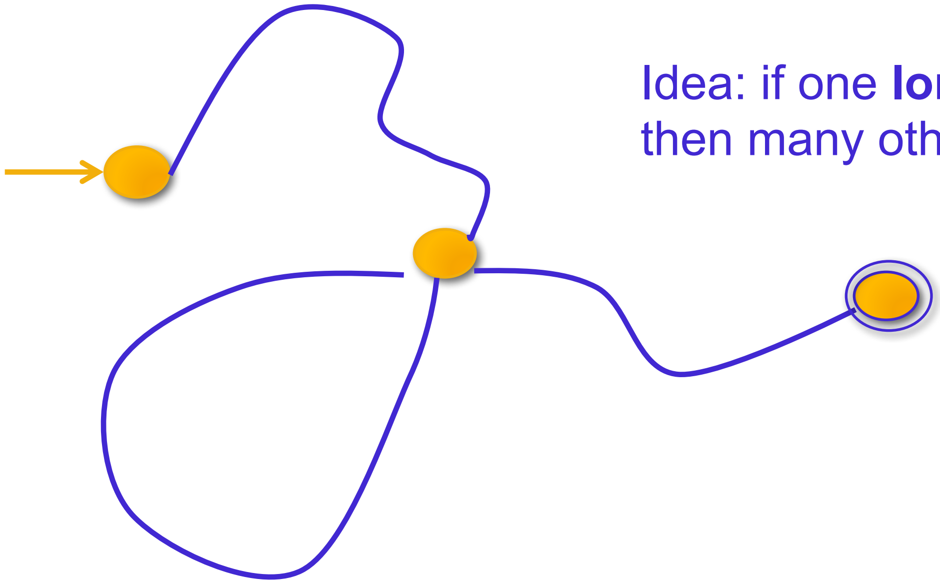


Idea: if one **long** string is accepted,
then many other similar strings have to be accepted too

Pumping Lemma

Sipser p. 78 Theorem 1.70

If A is a regular language, then there is a number p (*the pumping length*) where, if s is any string in A of length at least p, then s may be divided into three pieces, $\boxed{s = x\ y\ z}$ such that

- $|y| > 0$, and
- for each $i \geq 0$, $xy^iz \in A$,
- $|xy| \leq p$.

Pumping Lemma

Sipser p. 78 Theorem 1.70

If A is a regular language, then there is a number p (*the pumping length*) where, if s is any string in A of length at least p, then s may be divided into three pieces, $s = xyz$ such that

states in DFA recognizing A

- $|y| > 0$, and
- for each $i \geq 0$, $xy^iz \in A$,
- $|xy| \leq p$.

Transition labels along loop

Pumping Lemma

If A is recognized by DFA M with state diagram below,

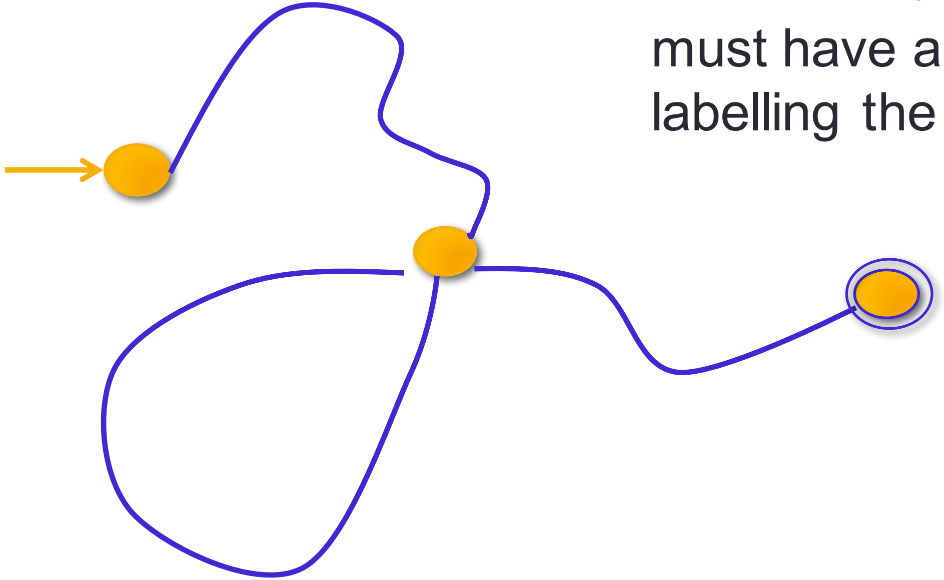
the computation of M on any string s of length >= p = |Q|

must have a **loop**. Divide s into the strings

labelling the path before the loop x ,

the loop itself y , and

from the loop to the accept state z



Pumping Lemma

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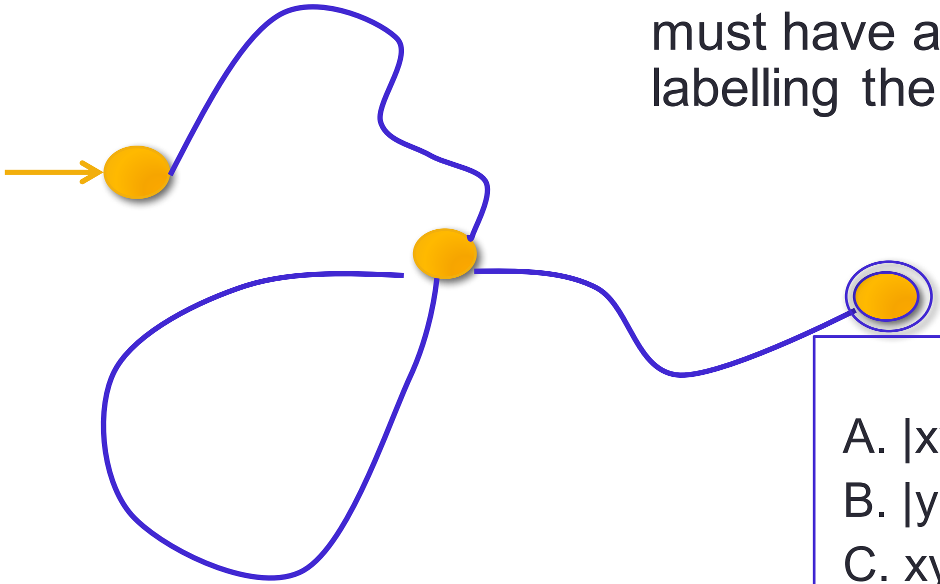

Which of the following is true?

A. |xy| <= p

B. |y| > 0

C. xy^iz is accepted by M for all i

D. All of A,B,C

E. None of them

Pumping Lemma

- True for **all** (but not only) regular sets.
  - Can't be used to prove that a set ***is*** regular   Ex 1.49
  - Can be used to a prove that a set ***is not*** regular … how?

Negation

flash-back to CSE 20 ☺

- Pumping lemma ``**There is** p, where p is a pumping length for L"

- Given a specific number p, it being a pumping length for L means

$$\forall s \left((|s| \geq p \wedge s \in L) \rightarrow \exists x \exists y \exists z \left(s = xyz \wedge |y| > 0 \wedge |xy| \leq p \wedge \forall i (xy^i z \in L) \right) \right)$$

- So p **not** being a pumping length of L means

$$\exists s \left(|s| \geq p \wedge s \in L \wedge \forall x \forall y \forall z \left((s = xyz \wedge |y| > 0 \wedge |xy| \leq p) \rightarrow \exists i (xy^i z \notin L) \right) \right)$$

Proof strategy

To prove that a language L is **not** regular

- Consider arbitrary positive integer p.
- Prove that p isn't a pumping length for L.

- Conclude that L does not have any pumping length and is therefore not regular.

For next time

- Work on Group Homework 2 **due Saturday**

Pre class-reading for Friday: Examples 1.75, 1.77.

Using the Pumping Lemma

Claim: The set $L = \{0^n 1^n \mid n \geq 0\}$ is not regular.

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Proof: Consider an arbitrary positive integer. WTS p is not a pumping length for L.

How? Want to show that there is some string that *should* be pump'able but isn't.

Using the Pumping Lemma

L = $\{0^n 1^n \mid n \geq 0\}$ **CLAIM:** p is not a pumping length for L.

How would you prove the claim?

A. Find a string with length >= p that is not in L.

B. Find a string with length <p that is in L.

C. None of the above.

$$\exists s \left(|s| \geq p \wedge s \in L \wedge \forall x \forall y \forall z \left((s = xyz \wedge |y| > 0 \wedge |xy| \leq p) \rightarrow \exists i (xy^i z \notin L) \right) \right)$$

Using the Pumping Lemma

$L = \{0^n 1^n \mid n \geq 0\}$ **CLAIM:** p is not a pumping length for L.

WTS

$$\exists s \left(|s| \geq p \land s \in L \land \forall x \forall y \forall z \left((s = xyz \land |y| > 0 \land |xy| \leq p) \rightarrow \exists i (xy^i z \notin L)\right)\right)$$

Find a string s such that

1. |s| >= p
2. s is in L
3. No matter how we cut s into three (viable) pieces, some related string obtained by repeating the middle part falls out of L.

Using the Pumping Lemma

Claim: The set L = **$\{0^n1^n \mid n \geq 0\}$** is not regular.

Proof: Consider an arbitrary positive integer. WTS p is not a pumping length for L.

Consider the string

$$s = 0^p1^p.$$

1. |s| >= p ?
2. s is in L ?
3. No matter how we cut s into three (viable) pieces, some related string obtained by repeating the middle part falls out of L ?

Using the Pumping Lemma

Claim: The set L = **$\{0^n1^n \mid n \geq 0\}$** is not regular.

Proof: Consider an arbitrary positive integer. WTS p is not a pumping length for L. Consider the string $s = 0^p1^p$. Then, s is in L and |s| = 2p >= p. Consider any division of s into three parts

$$s = xyz \text{ with } |y|>0, |xy| \leq p.$$

Since $|xy| \leq p$, $x=0^k$, $y = 0^m$, $z = 0^r1^p$ with $k+m+r = p$, and since $|y| > 0$, $m>0$. Picking i=0: $xy^iz = xz = 0^k0^r1^p = 0^{k+r}1^p$, which is not in L because $k+r < p$. Thus, no p can be a pumping length for L and L is not regular.