# Expected Value

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|---|---|---|---|
| **Lecture A** | **Tiefenbruck** | **MWF 9-9:50am** | **Center 212** |
| **Lecture B** | **Jones** | **MWF 2-2:50pm** | **Center 214** |
| **Lecture C** | **Tiefenbruck** | **MWF 11-11:50am** | **Center 212** |

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# Random Variables Motivation

Sometimes, we are interested in a quantity determined by a random process.
For Example:

The total sum of 2 dice.
The number of heads after flipping n fair coins
The maximum of 2 dice rolls.
The time that a randomized algorithm takes.

# Random Variables

***Rosen p. 460,478***

A **random variable** is a function from the sample space to the real numbers.

The **distribution** of a random variable X is a function from the possible values to [0,1] given by:

$$r \rightarrow P(X = r)$$

# Random Variables Examples:

*Rosen p. 460,478*

Let X be the sum of the pips of two fair dice

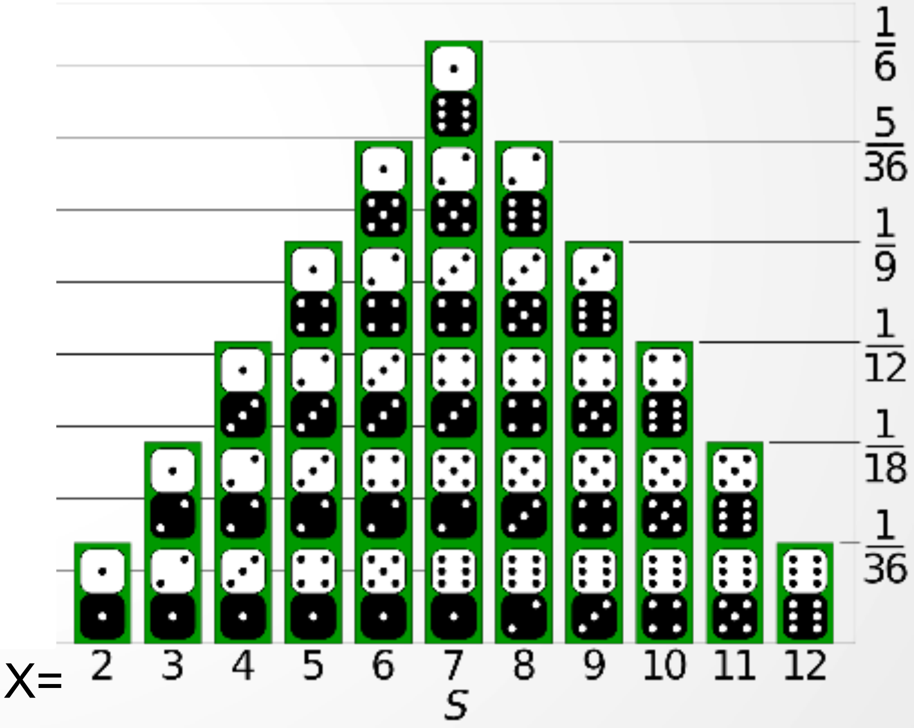
X(5,2)=7

X(3,3) = 6

The distribution is shown as the height of the graph , e.g.

The probability that X=7 is 6/36=1/6

The probability that X=9 is 4/36=1/9

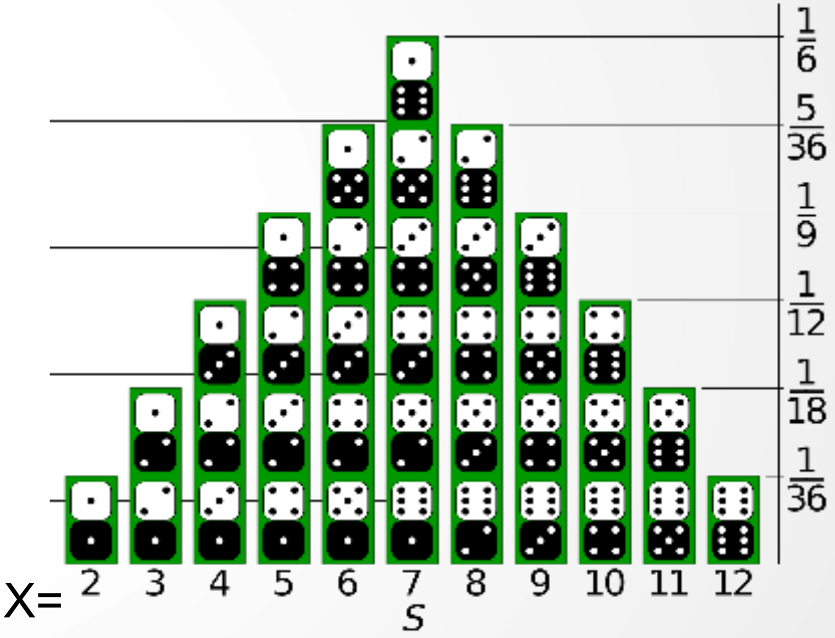


# Expected Value

The **expectation** (average, expected value) of random variable X on sample space S is

$$E(X) = \sum_{s \in S} P(s)X(s)$$
$$= \sum_{r \in X(S)} P(X = r)r$$

For the example of two dice with X being the sum of the pips, we have that the expectation is given by




$$E(X) =$$
$$2\left(\frac{1}{36}\right) + 3\left(\frac{1}{18}\right) + 4\left(\frac{1}{12}\right) + 5\left(\frac{1}{9}\right) + 6\left(\frac{5}{36}\right) + 7\left(\frac{1}{6}\right) + 8\left(\frac{5}{36}\right) + 9\left(\frac{1}{9}\right) + 10\left(\frac{1}{12}\right) + 11\left(\frac{1}{18}\right) + 12\left(\frac{1}{36}\right) = 7$$

# Expected Value Examples

Rosen p. 460,478

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Calculate the expected number of boys in a family with **two** children.

A. 0
B. 1
C. 1.5
D. 2

# Expected Value Examples

***Rosen p. 460,478***

The **expectation** (average, expected value) of random variable X on sample space S is

$$E(X) = \sum_{s \in S} P(s)X(s)$$
$$= \sum_{r \in X(S)} P(X = r)r$$

Calculate the expected number of boys in a family with **three** children.

A. 0
B. 1
C. 1.5
D. 2

# Expected Value Examples

***Rosen p. 460,478***

The **expectation** (average, expected value) of random variable X on sample space S is

$$E(X) = \sum_{s \in S} P(s)X(s)$$

$$= \sum_{r \in X(S)} P(\ldots)r$$

The expected value might not be a possible value of the random variable…
like 1.5 boys!

Calculate the expected number of boys in a family

A. 0
B. 1
C. 1.5
D. 2

# Properties of Expectation

***Rosen p. 460,478***

- E(X) may not be an actually possible value of X.

- But m <= E(X) <= M, where
  - m is minimum value of X and
  - M is maximum value of X.

# Useful trick 1: Case analysis

***Rosen p. 460,478***

The **expectation** can be computed by conditioning on an event and its complement

**Theorem:** For any random variable X and event A,

$$E(X) = P(A)\,E(X \mid A) + P(A^c)\,E(X \mid A^c)$$

where $A^c$ is the complement of A.

Conditional Expectation

# Useful trick 1: Case analysis

**Example**: If X is the number of pairs of consecutive Hs when we flip a fair coin three times, what is the expectation of X?

e.g. X(HHT) = 1
X(HHH) = 2.

# Useful trick 1: Case analysis

**Example**: If X is the number of pairs of consecutive Hs when we flip a fair coin three times, what is the expectation of X?

**Solution**:

*Directly from definition*

$$E(X) = \sum_{s \in S} P(s)X(s)$$

$$= \sum_{r \in X(S)} P(X = r)r$$

For each of eight possible outcomes, find probability and value of X:

HHH (P(HHH)=1/8, X(HHH) = 2) , HHT, HTH, HTT, THH, THT, TTH, TTT etc.

# Useful trick 1: Case analysis

**Example**: If X is the number of pairs of consecutive Hs when we flip a fair coin three times, what is the expectation of X?

**Solution**:

*Using conditional expectation*

Let A be the event "The middle flip is H".

Which subset of S is A?

A. { HHH }
B. { THT }
C. { HHT, THH}
D. { HHH, HHT, THH, THT}
E. None of the above.

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**Solution**:

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Let A be the event "The middle flip is H".

$$E(X) = P(A)\,E(X \mid A) + P(A^c)\,E(X \mid A^c)$$

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**Example**: If X is the number of pairs of consecutive Hs when we flip a fair coin three times, what is the expectation of X?

**Solution**:

*Using conditional expectation*

Let A be the event "The middle flip is H". **$P(A) = 1/2$ , $P(A^c) = 1/2$**

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**$E(X \mid A^c)$** : If middle flip isn't H, there can't be *any* pairs of consecutive Hs

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**$E(X \mid A^c)$** : If middle flip isn't H, there can't be *any* pairs of consecutive Hs
**$E(X \mid A)$** : If middle flip is H, # pairs of consecutive Hs = # Hs in first & last flips

# Useful trick 1: Case analysis

**Example**: If X is the number of pairs of consecutive Hs when we flip a fair coin three times, what is the expectation of X?

**Solution**:

*Using conditional expectation*

Let A be the event "The middle flip is H".

**$P(A) = 1/2$ , $P(A^c) = 1/2$**

$$E(X) = P(A)\, E(X \mid A) + P(A^c)\, E(X \mid A^c)$$

**$E(X \mid A^c) = 0$**
**$E(X \mid A) = \frac{1}{4} * 0 + \frac{1}{2} * 1 + \frac{1}{4} * 2 = 1$**

# Useful trick 1: Case analysis

**Example**: If X is the number of pairs of consecutive Hs when we flip a fair coin three times, what is the expectation of X?

**Solution**:

*Using conditional expectation*

Let A be the event "The middle flip is H". $P(A) = 1/2$ , $P(A^c) = 1/2$

$$E(X) = P(A)\, E(X \mid A) + P(A^c)\, E(X \mid A^c) = \tfrac{1}{2}(1) + \tfrac{1}{2}(0) = 1/2$$

$E(X \mid A^c) = 0$

$E(X \mid A) = \tfrac{1}{4} * 0 + \tfrac{1}{2} * 1 + \tfrac{1}{4} * 2 = 1$

# Useful trick 1: Case analysis

**Examples**: **Ending condition**

- Each time I play solitaire I have a probability p of winning. I play until I win a game.
- Each time a child is born, it has probability p of being left-handed. I keep having kids until I have a left-handed one.

Let X be the number of games OR number of kids until ending condition is met.

What's E(X)?

A. 1.
B. Some big number that depends on p.
C. 1/p.
D. None of the above.

# Useful trick 1: Case analysis

**Ending condition**

Let X be the number of games OR number of kids until ending condition is met.

**Solution**:

*Directly from definition*

Need to compute the sum of all possible **P(X = i) i .**

# Useful trick 1: Case analysis

**Ending condition**

Let X be the number of games OR number of kids until ending condition is met.

**Solution**:

*Directly from definition*

Need to compute the sum of all possible **$P(X = i)\ i$ .**

$P(X = i)$ = Probability that don't stop the first i-1 times and do stop at the $i^{th}$ time

$= (1-p)^{i-1}\ p$

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**Ending condition**

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$P(X = i)$ = Probability that don't stop the first i-1 times and do stop at the $i^{th}$ time
$= (1-p)^{i-1} p$

$$E(x) = \sum_{i=1}^{\infty} i(1-p)^{i-1}p$$

Math 20B?

# Useful trick 1: Case analysis

**Ending condition**

Let X be the number of games OR number of kids until ending condition is met.

**Solution**:

*Using conditional expectation*

$$E(X) = P(A)\,E(X \mid A) + P(A^c)\,E(X \mid A^c)$$

# Useful trick 1: Case analysis

**Ending condition**

Let X be the number of games OR number of kids until ending condition is met.

**Solution**:

*Using conditional expectation*

Let A be the event "success at first try".

$$E(X) = P(A)\,E(X \mid A) + P(A^c)\,E(X \mid A^c)$$

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**Ending condition**

Let X be the number of games OR number of kids until ending condition is met.

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$$\mathbf{E(X) = P(A)\, E(X \mid A) + P(A^c)\, E(X \mid A^c)}$$

$P(A) = p \qquad P(A^c) = 1-p$

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$E(X|A) = 1$ *because stop after first try*

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Let X be the number of games OR number of kids until ending condition is met.

**Solution**:

*Using conditional expectation*

Let A be the event "success at first try".

$$\mathbf{E(X) = P(A)\ E(X \mid A) + P(A^c)\ E(X \mid A^c)}$$

$P(A) = p \qquad P(A^c) = 1-p$

$E(X|A) = 1$

$E(X|A^c) = 1 + E(X)$ *because tried once and then at same situation from start*

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**Ending condition**

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**Solution**:

*Using conditional expectation*

Let A be the event "success at first try".

$$\mathbf{E(X) = P(A)\,E(X \mid A) + P(A^c)\,E(X \mid A^c)}$$

$P(A) = p \qquad P(A^c) = 1-p$

$E(X|A) = 1$
$E(X|A^c) = 1 + E(X)$

$$\mathbf{E(X) = p(1) + (1-p)(1 + E(X))}$$

# Useful trick 1: Case analysis

**Ending condition**

Let X be the number of games OR number of kids until ending condition is met.

**Solution**:

*Using conditional expectation*

Let A be the event "success at first try".

**E(X) = p(1) + ( 1-p ) (1 + E(X) )**

*Solving for E(X) gives:*

$$E(x) = \frac{1}{p}$$

# Useful trick 2: Linearity of expectation

*Rosen p. 477-484*

**Theorem:** If $X_i$ are random variables on S and if a and b are real numbers then

$$E(X_1+\ldots+X_n) = E(X_1) + \ldots + E(X_n)$$

and $E(aX+b) = aE(x) + b$.

# Useful trick 2: Linearity of expectation

**Example:** Expected number of pairs of **consecutive heads** when we flip a fair coin n times?

A. 1.
B. (n-1)/4.
C. n.
D. n/2.
E. None of the above

# Useful trick 2: Linearity of expectation

**Example:** Expected number of pairs **consecutive heads** when we flip a fair coin n times?

**Solution:** Define $X_i$ = 1 if both the $i^{th}$ and $i+1^{st}$ flips are H; $X_i$=0 otherwise.

Looking for E(X) where $X = \sum_{i=1}^{n-1} X_i$.

For each i, what is $E(X_i)$?

A. 0.
B. ¼.
C. ½.
D. 1.
E. It depends on the value of i.

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**Example:** Expected number of **consecutive heads** when we flip a fair coin n times?

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Looking for E(X) where $X = \sum_{i=1}^{n-1} X_i$ .

$$E(X) = \sum_{i=1}^{n-1} E(X_i) = \sum_{i=1}^{n-1} \frac{1}{4} = \frac{n-1}{4}$$

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$$E(X) = \sum_{i=1}^{n-1} E(X_i) = \sum_{i=1}^{n-1} \frac{1}{4} = \frac{n-1}{4}$$

**Indicator variables:**
**1 if pattern occurs, 0 otherwise**

# Useful trick 2: Linearity of expectation

**Example:** Consider the following program:

```
Findmax(a[1…n])
max:=a[1]
for i=2 to n
      if a[i]>max then
            max:=a[i]
return max
```

If the array is in a random order, how many times do we expect max to change?

# Useful trick 2: Linearity of expectation

**Example:** Consider the following program:

```
Findmax(a[1…n])
max:=a[1]
for i=2 to n
	if a[i]>max then
		max:=a[i]
return max
```

Let $X_i = 1$ if a[i] is greater than a[1],..,a[i-1] and $X_i = 0$ otherwise.
Then we change the maximum in the iteration i iff $X_i = 1$
So the quantity we are looking for is the expectation of $X = \sum_{i=2}^{n} X_i$, which by linearity of expectations is $\mathrm{E}(X) = \sum_{i=2}^{n} E(X_i)$.

# Useful trick 2: Linearity of expectation

If the array is random then a[i] is equally likely to be the largest of a[1],..a[i] as all the other values in that range. So

$$E(X_i) = \frac{1}{\mathrm{i}}$$

Thus the expectation of X is

$$\mathrm{E}(X) = \sum_{i=2}^{n} E(X_i) = \sum_{i=2}^{n} \frac{1}{i} \approx \log(n)$$

(the last is because the integral of dx/x is log(x).

# Other functions?

*Rosen p. 460,478*

**Expectation** does **not** in general commute with other functions.

$$E(\ f(X)\ ) \neq f(\ E(X)\ )$$

For example, let X be random variable with P(X = 0) = ½ , P(X =1) = ½

*What's E(X)?*

*What's $E(X^2)$?*

*What's $(\ E(X)\ )^2$?*

# Other functions?

*Rosen p. 460,478*

**Expectation** does **not** in general commute with other functions.

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For example, let X be random variable with $P(X = 0) = ½$ , $P(X = 1) = ½$

*What's E(X)?* $(½)0 + (½)1 = ½$

*What's $E(X^2)$?* $(½)0^2 + (½)1^2 = ½$

*What's $(\ E(X)\ )^2$?* $(½)^2 = ¼$