CSE 20
DISCRETE MATH

SPRING 2016

http://cseweb.ucsd.edu/classes/sp16/cse20-ac/

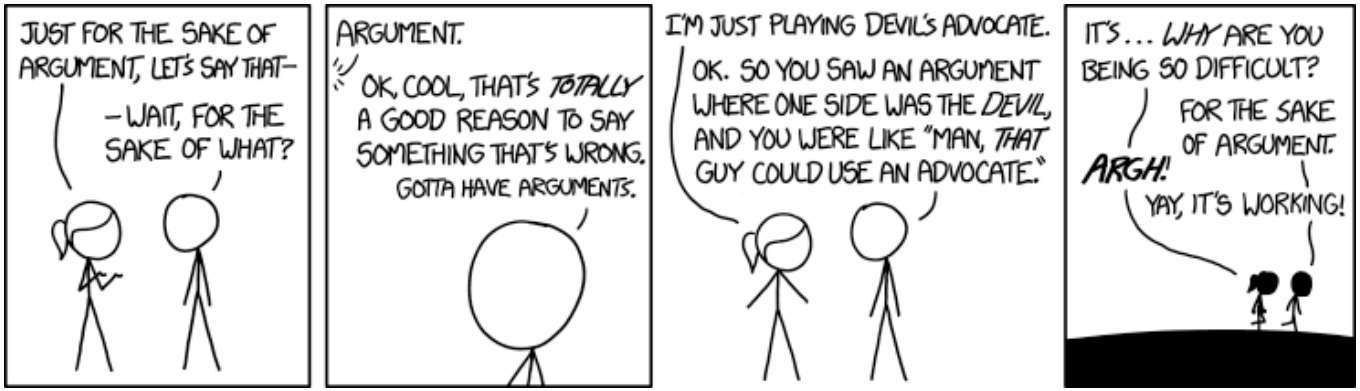
Today's learning goals

- Distinguish between a theorem, an axiom, lemma, a corollary, and a conjecture.
- Recognize direct proofs
- Recognize proofs by contraposition
- Recognize proofs by contradiction
- Recognize fallacious “proofs”
- Evaluate which proof technique(s) is appropriate for a given proposition
  - Direct proof
  - Proofs by contraposition
  - Proofs by contradiction
  - Proof by cases
  - Constructive existence proofs
- Correctly prove statements using appropriate style conventions, guiding text, notation, and terminology

Textbook references: Sections 1.7-1.8 (with reference to 1.6)

For the sake of argument

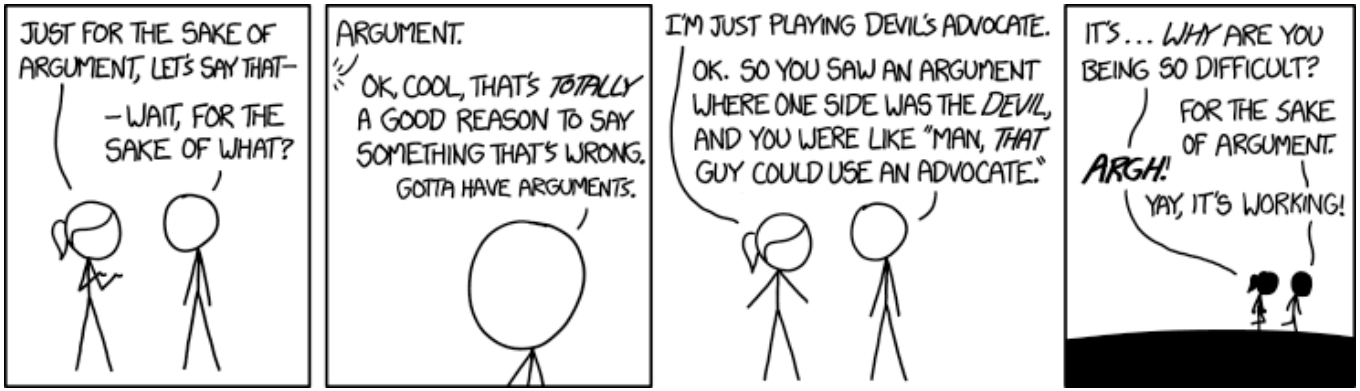
Rosen p. 81



"A proof is a valid argument that establishes the truth of a statement"

For the sake of argument

Rosen p. 69



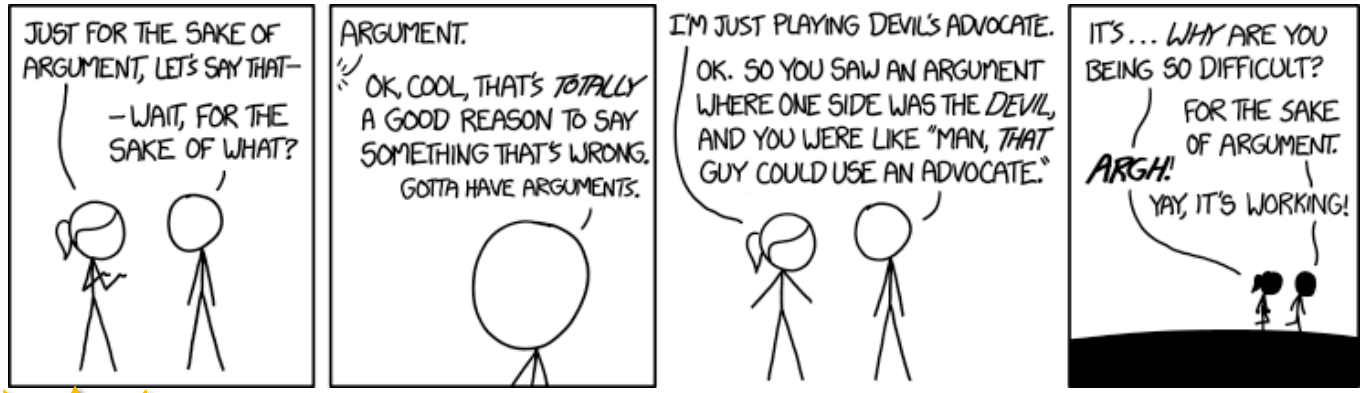
"A proof is a valid argument that establishes the truth of a statement"

Valid: truth of the conclusion (final statement) is guaranteed by the truth of the premises (preceding statements)

Argument: sequence of statements ending with a conclusion

For the sake of argument

Rosen p. 81



Axioms: statements assumed to be true

Rules of inference +definitions

Theorem
Proposition
Fact
Result
Lemma
Corollary

What rules of inference are allowed?

Rosen 1.6

Argument form is valid if no matter which propositions are substituted in, the conclusion is true if all the premises are true

What about quantifiers?

Rosen 1.6

Which of the following is **not** a valid argument?

A. All outdoor art on campus is intact after the storm. Therefore, the stone bear is intact after the storm.

B. The stone bear is intact after the storm. Therefore, all outdoor art on campus is intact after the storm.

C. The stone bear is intact after the storm. Therefore, some outdoor art on campus is intact after the storm.

D. The stone bear is intact after the storm. Therefore, not all outdoor art on campus was destroyed by the storm.

E. I don't understand the meaning of the concept "valid argument".

Fancy names

Rosen 1.6

- Modus ponens
- Modus tollens
- Hypothetical syllogism …
- Universal instantiation
- Universal generalization
- Existential instantiation
- Existential generalization

Translating to proof strategy

Rosen p. 82

- Modus ponens **Direct proof**

To prove that "If P, then Q"

- Assume P is true.
- Use rules of inference, axioms, definitions to…
- … conclude Q is true.

Translating to proof strategy

Rosen p. 83

- Modus tollens

Proof by contraposition

To prove that "If P, then Q"

- Assume Q is false.
- Use rules of inference, axioms, definitions to…
- … conclude P is also false.

Both modus ponens and modus tollens apply when proving a conditional statement

Main connective

What's the main logical connective in the following sentence?

$$n^2 + 1 \geq 2^n \text{ when } n \text{ is a positive integer with } 1 \leq n \leq 4$$

A. Conditional (p → q)
B. Universal statement
C. Conjunction ("AND")
D. Inequality (>=)
E. What's a "main logical connective"?

Main connective

What's the main logical connective in the following sentence?

$$n^2 + 1 \geq 2^n \text{ when } n \text{ is a positive integer}$$

A. Conditional (p → q)
B. Universal statement
C. Conjunction ("AND")
D. Inequality (>=)
E. What's a "main logical connective"?

Main connective

Are these statements true?

I. $n^2 + 1 \geq 2^n$ when n is a positive integer with $1 \leq n \leq 4$
II. $n^2 + 1 \geq 2^n$ when n is a positive integer

A. Both are true.
B. Both are false.
C. I is true and II is false.
D. II is true and I is false.
E. I don't know how to evaluate the truth value of these sentences.

Universal conditionals

$n^2 + 1 \geq 2^n$ when n is a positive integer with $1 \leq n \leq 4$

$n^2 + 1 \geq 2^n$ when n is a positive integer

can be translated to

$$\forall n \ (\ldots \ \rightarrow \ \ldots)$$

Universal conditionals

Rosen p. 88

$$\forall n\ (\ldots \rightarrow \ldots)$$

To prove this statement is **false** :

To prove this statement is **true** :

Universal conditionals

Rosen p. 88, 82

$$\forall n\ (\ldots \rightarrow \ldots)$$

To prove this statement is **false** :

Find a **counterexample**.

To prove this statement is **true** :

Select a **general element** of the domain, use rules of inference (e.g direct proof, proof by contrapositive, etc.) to prove that the conditional statement is true of this element, c conclude that it holds of all members of the domain.

Universal conditionals

Claim: $n^2 + 1 \geq 2^n$ when n is a positive integer is false.

Proof:

Universal conditionals

Claim: $n^2 + 1 \geq 2^n$ when n is a positive integer is false.

Proof: Since this is a **universal** conditional statement, to prove that it is false, it's enough to find one counterexample. That is, we need a specific value of *n* which is a positive integer and for which

A. $n^2 + 1 > 2^n$
B. $n^2 + 1 < 2^n$
C. $n^2 + 1 <= 2^n$
D. $n^2 + 1 >= 2^n$
E. What's a universal conditional?

Universal conditionals

Claim: $n^2 + 1 \geq 2^n$ when n is a positive integer is false.

Proof: Since this is a **universal** conditional statement, it's enough to find one counterexample. That is, we need a specific value of *n* which is a positive integer and for which $n^2 + 1 < 2^n$. Consider *n* =

A. $n = 0$
B. $n = 1$
C. $n = 4$
D. $n = 6$
E. None of the above

Universal conditionals

Claim: $n^2 + 1 \geq 2^n$ when n is a positive integer is false.

Proof: Since this is a universal conditional statement, it's enough to find one counterexample. That is, we need a specific value of *n* which is a **positive integer** and for which $n^2 + 1 < 2^n$. Consider $n = 5$. Then, for this example, the LHS of the inequality evaluates to $5^2+1 = 26$ and the RHS evaluates to $2^5=32$. Therfore, **LHS < RHS**, contradicting the statement. Thus, there is (at least) one counterexample to the universal statement and hence the universal statement is false.

Universal conditionals

Claim: $n^2 + 1 \geq 2^n$ when n is a positive integer with $1 \leq n \leq 4$ is true.

Proof: WTS $\forall n\ (1 \leq n \leq 4 \rightarrow n^2 + 1 \geq 2^n)$

Let *n* be a positive integer (a general element of the domain). For a direct proof, assume

A. $n \geq 1$
B. $n^2 + 1 \geq 2^n$
C. $n \leq 4$
D. $n^2 + 1 < 2^n$
E. None of the above

Universal conditionals

Claim: $n^2 + 1 \geq 2^n$ when n is a positive integer with $1 \leq n \leq 4$ is true.

Proof: WTS $\forall n \, (1 \leq n \leq 4 \rightarrow n^2 + 1 \geq 2^n)$

Let *n* be a positive integer (a general element of the domain). For a direct proof, assume that $1 \leq n \leq 4$.

Goal : WTS that $n^2 + 1 \geq 2^n$

How can we use the hypothesis?

Universal conditionals

Claim: $n^2 + 1 \geq 2^n$ when n is a positive integer with $1 \leq n \leq 4$ is true.

Proof: WTS $\forall n \, (1 \leq n \leq 4 \rightarrow n^2 + 1 \geq 2^n)$

Let *n* be a positive integer (a general element of the domain). For a direct proof, assume that $1 \leq n \leq 4$.

Goal : WTS that $n^2 + 1 \geq 2^n$

Since we assume that *n* is an **integer** between 1 and 4, there are only four possible cases. We check that the conclusion is true in each one.

Universal conditionals

Claim: $n^2 + 1 \geq 2^n$ when n is a positive integer with $1 \leq n \leq 4$ is true.

Proof: WTS $\forall n\ (1 \leq n \leq 4 \rightarrow n^2 + 1 \geq 2^n)$

Let *n* be a positive integer (a general element of the domain). For a direct proof, assume that 1 <= *n* <= 4.

Goal : WTS that $n^2 + 1 >= 2^n$

Case 1: Assume *n*=1. Then LHS is $1^2+1 = 2$; RHS is $2^1=2$ so LHS >=RHS. ☺

Case 2: Assume *n*=2. Then LHS is $2^2+1 = 5$; RHS is $2^2=4$ so LHS >=RHS. ☺

Case 3: Assume *n*=3. Then LHS is $3^2+1 = 10$; RHS is $2^3=8$ so LHS >=RHS. ☺

Case 4: Assume *n*=4. Then LHS is $4^2+1 = 17$; RHS is $2^4=16$ so LHS >=RHS. ☺

Universal conditionals

Claim: $n^2 + 1 \geq 2^n$ when [illegible] ue.

Exhaustive proof
(cf. page 93)

Proof: WTS $\forall n$ [illegible]

Let *n* be a positive integer (a general element of the domain). For a direct proof, assume that 1 <= *n* <= 4.

Goal : WTS that $n^2 + 1 >= 2^n$

Case 1: Assume *n*=1. Then LHS is $1^2+1 = 2$; RHS is $2^1=2$ so LHS >=RHS. ☺

Case 2: Assume *n*=2. Then LHS is $2^2+1 = 5$; RHS is $2^2=4$ so LHS >=RHS. ☺

Case 3: Assume *n*=3. Then LHS is $3^2+1 = 10$; RHS is $2^3=8$ so LHS >=RHS. ☺

Case 4: Assume *n*=4. Then LHS is $4^2+1 = 17$; RHS is $2^4=16$ so LHS >=RHS. ☺

Reminder: perfect squares

Rosen p. 83

An integer a is a **perfect square** if there is some integer b such that

$$a = b^2$$

Which of the following is **not** a perfect square?

A. 0
B. 1
C. 2
D. 4
E. More than one of the above

Reminder: perfect squares

Rosen p. 83

An integer a is a **perfect square** if there is some integer b such that

$$a = b^2$$

Which of the following is **not** true?

A. If n is a perfect square, so is n^2.
B. If n^2 is a perfect square, so is n.
C. If is n^2 is not a perfect square, neither is n.
D. n^2 is a perfect square.
E. More than one of the above

Reminder: perfect squares

Rosen p. 83

Theorem: For integers k>1, then 2^k-1 is not a perfect square.

Proof:

What would we assume in a direct proof?

A. k > 1
B. k <= 1
C. 2^k-1 is not a perfect square
D. 2^k-1 is a perfect square
E. More than one of the above

Reminder: perfect squares

Rosen p. 83

Theorem: For integers k>1, then 2^k-1 is not a perfect square.

Proof:

What would we assume in a proof by contraposition?

A. k > 1
B. k <= 1
C. 2^k-1 is not a perfect square
D. 2^k-1 is a perfect square
E. More than one of the above

Reminder: perfect squares

Rosen p. 83

Theorem: For integers k>1, then 2^k-1 is not a perfect square.

Proof: ???

Proof by contradiction

Rosen p. 86

Idea: To prove P, instead, we prove that the conditional

$$(\neg P) \rightarrow F$$

is true. **But**, the only way for a conditional to be true if its conclusion is false is for **its hypothesis to be false too**.

Conclude: $(\neg P)$ is false, i.e. **P is true**!

Back to: perfect squares

Rosen p. 83

Theorem: For integers k>1, then 2^k-1 is not a perfect square.

Proof:

What would we assume in a proof by contradiction?

A. k > 1
B. k <= 1
C. 2^k-1 is not a perfect square
D. 2^k-1 is a perfect square
E. More than one of the above

Reminder: perfect squares

Rosen p. 83

Theorem: For integers k>1, then 2^k-1 is not a perfect square.

Proof: Let k be an integer. Assume that

- k >1 **and that**
- 2^k-1 is a perfect square.

Goal: look for a contradiction that is now guaranteed.

Keep going …

Overall strategy

- Do you believe the statement?
  - Try some small examples.
- Determine logical structure + main connective.
- Determine relevant definitions.
- Map out possible proof strategies.
  - For each strategy: what can we **assume**, what is the **goal**?
  - Start with simplest, move to more complicated if/when get stuck.

Next up

- Using proofs in the context of sets

Reminder: divisibility

Rosen p. 238-239

For integers *a*, *b* with *a* nonzero

$$\exists c(b = ac)$$

means

$$a|b$$

"a divides b"

$$\frac{b}{a} \in \mathbb{Z}$$

"b is an integer multiple of a"