CSE 20
DISCRETE MATH

Spring 2016

http://cseweb.ucsd.edu/classes/sp16/cse20-ac/

Reminders

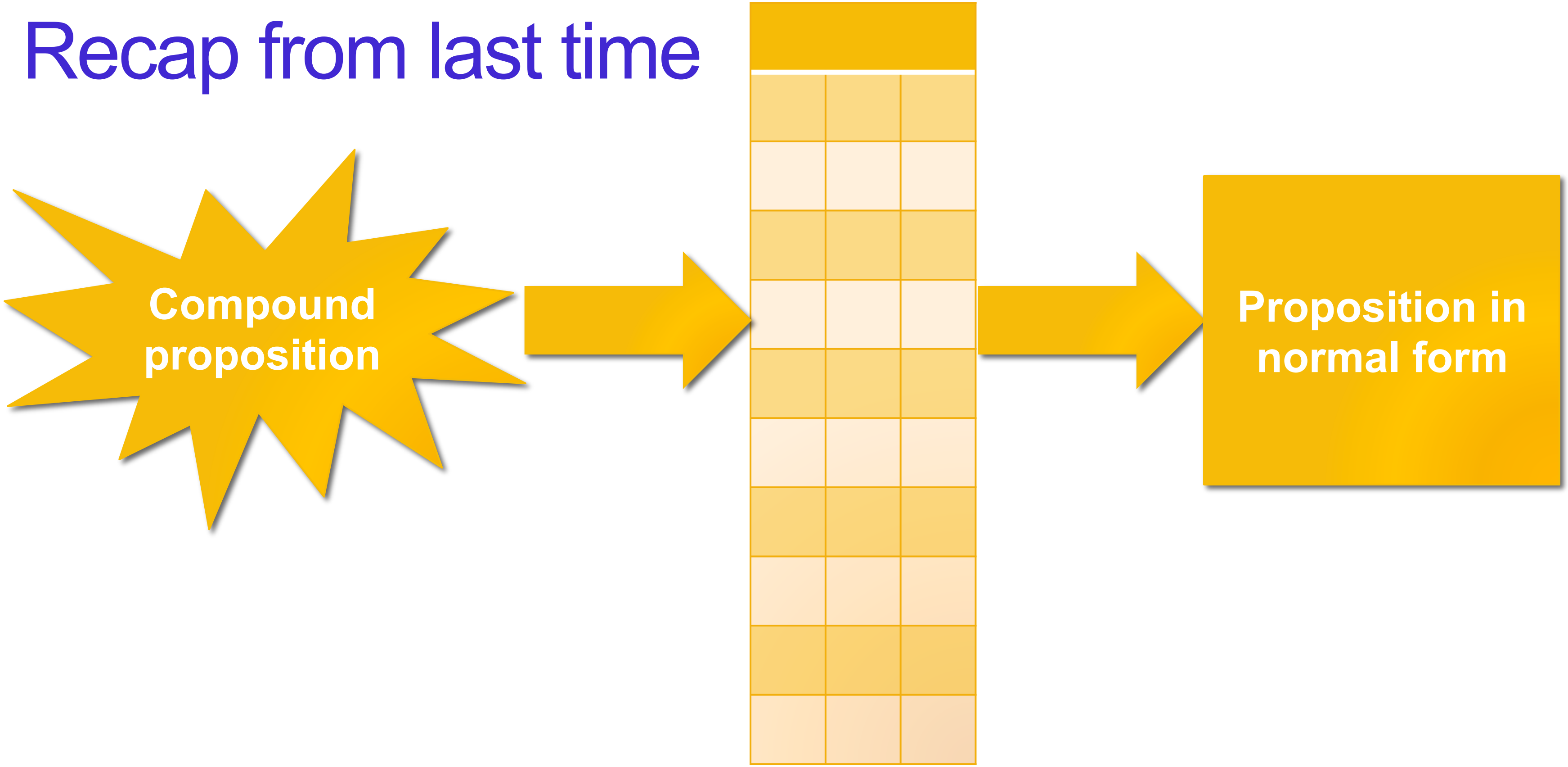
- Exam 1 on Tuesday April 19
  - One note sheet ok
  - Review sessions Saturday / Sunday TBA
  - Assigned seats + practice exam: on class website
- HW 3 due tomorrow
- Office hours

Today's learning goals

- Relate boolean operations to applications

  Combinatorial circuits Logic puzzles
- Identify when and prove that a statement is a tautology or contradiction
- Prove propositional equivalences using truth tables
- Prove propositional equivalences using other known equivalences.
- Compute the CNF and DNF of a given compound proposition.
- Identify when and prove that a statement is satisfiable or unsatisfiable, and when a set of statements is consistent or inconsistent.
- Determine the truth value of predicates for specific values of their arguments
- Determine the truth sets of predicates
- Define the universal and existential quantifiers
- Translate sentences from English to predicate logic using appropriate predicates and quantifiers

Recap from last time



Implement a proposition

What combinatorial logic circuit (with AND, OR, NOT, XOR gates) implements the compound proposition

$$(p \to q) \leftrightarrow (r \to p)$$?

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Strategy: Find CNF or DNF equivalent and then implement.

- Via logical equivalences
- Via truth table algorithm

Via logical equivalences

$$
\begin{aligned}
&(p \to q) \leftrightarrow (r \to p) \\
\equiv&(\neg p \vee q) \leftrightarrow (\neg r \vee p) \\
\equiv&((\neg p \vee q) \wedge (\neg r \vee p)) \vee (\neg(\neg p \vee q) \wedge \neg(\neg r \vee p)) \\
\equiv&\cdots
\end{aligned}
$$

Via truth table

p	q	r	$(p \to q)$	$(r \to p)$	$(p \to q) \leftrightarrow (r \to p)$
T	T	T			
T	T	F			
T	F	T			
T	F	F			
F	T	T			
F	T	F			
F	F	T			
F	F	F			

Via truth table

p	q	r	$(p \to q)$	$(r \to p)$	$(p \to q) \leftrightarrow (r \to p)$
T	T	T	T		
T	T	F	T		
T	F	T	F		
T	F	F	F		
F	T	T	T		
F	T	F	T		
F	F	T	T		
F	F	F	T		

Via truth table

p	q	r	$(p \to q)$	$(r \to p)$	$(p \to q) \leftrightarrow (r \to p)$
T	T	T	T	T	
T	T	F	T	T	
T	F	T	F	T	
T	F	F	F	T	
F	T	T	T	F	
F	T	F	T	T	
F	F	T	T	F	
F	F	F	T	T	

Via truth table

p	q	r	$(p \to q)$	$(r \to p)$	$(p \to q) \leftrightarrow (r \to p)$
T	T	T	T	T	T
T	T	F	T	T	T
T	F	T	F	T	F
T	F	F	F	T	F
F	T	T	T	F	F
F	T	F	T	T	T
F	F	T	T	F	F
F	F	F	T	T	T

Via truth table

p	q	r	$(p \rightarrow q) \leftrightarrow (r \rightarrow p)$
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	F
F	T	T	F
F	T	F	T
F	F	T	F
F	F	F	T

LAND IN THESE ROWS!

Via truth table

p	q	r	$(p \to q) \leftrightarrow (r \to p)$
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	F
F	T	T	F
F	T	F	T
F	F	T	F
F	F	F	T

What compound proposition describes the first row?

A. $p \vee q \vee r$
B. $p \wedge q \wedge r$
C. $\neg p \vee \neg q \vee \neg r$
D. $\neg p \wedge \neg q \wedge \neg r$
E. None of the above

CENTR101: CA
PCYNH109: AB

Via truth table

p	q	r	$(p \to q) \leftrightarrow (r \to p)$	
T	T	T	T	$p \land q \land r$
T	T	F	T	$p \land q \land \neg r$
T	F	T	F	
T	F	F	F	
F	T	T	F	
F	T	F	T	$\neg p \land q \land \neg r$
F	F	T	F	
F	F	F	T	$\neg p \land \neg q \land \neg r$

Via truth table

p	q	r	$(p \to q) \leftrightarrow (r \to p)$
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	F
F	T	T	F
F	T	F	T
F	F	T	F
F	F	F	T

- $p \land q \land r$
- $p \land q \land \neg r$
- $\neg p \land q \land \neg r$
- $\neg p \land \neg q \land \neg r$

DNF: $(p \land q \land r) \lor (p \land q \land \neg r) \lor (\neg p \land q \land \neg r) \lor (\neg p \land \neg q \land \neg r)$

As a circuit (assume 3 and 4 input gates are available; otherwise cascade)

DNF: $(p \land q \land r) \lor (p \land q \land \neg r) \lor (\neg p \land q \land \neg r) \lor (\neg p \land \neg q \land \neg r)$

Via truth table

p	q	r	$(p \to q) \leftrightarrow (r \to p)$
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	F
F	T	T	F
F	T	F	T
F	F	T	F
F	F	F	T

AVOID THESE ROWS!

Via truth table

p	q	r	$(p \to q) \leftrightarrow (r \to p)$
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	F
F	T	T	F
F	T	F	T
F	F	T	F
F	F	F	T

What compound proposition describes avoiding the third row?

A. $p \land \neg q \land r$
B. $\neg(p \land \neg q \land r)$
C. $p \lor \neg q \lor r$
D. $\neg(p \lor \neg q \lor r)$
E. None of the above

Via truth table

p	q	r	$(p \to q) \leftrightarrow (r \to p)$	
T	T	T	T	
T	T	F	T	
T	F	T	F	$\neg(p \land \neg q \land r) \equiv \neg p \lor q \lor \neg r$
T	F	F	F	$\neg(p \land \neg q \land \neg r) \equiv \neg p \lor q \lor r$
F	T	T	F	$\neg(\neg p \land q \land r) \equiv p \lor \neg q \lor \neg r$
F	T	F	T	
F	F	T	F	$\neg(\neg p \land \neg q \land r) \equiv p \lor q \lor \neg r$
F	F	F	T	

Via truth table

p	q	r	$(p \to q) \leftrightarrow (r \to p)$	
T	T	T	T	
T	T	F	T	
T	F	T	F	$\neg(p \wedge \neg q \wedge r) \equiv \neg p \vee q \vee \neg r$
T	F	F	F	$\neg(p \wedge \neg q \wedge \neg r) \equiv \neg p \vee q \vee r$
F	T	T	F	$\neg(\neg p \wedge q \wedge r) \equiv p \vee \neg q \vee \neg r$
F	T	F	T	
F	F	T	F	$\neg(\neg p \wedge \neg q \wedge r) \equiv p \vee q \vee \neg r$
F	F	F	T	

CNF: $(\neg p \vee q \vee \neg r) \wedge (\neg p \vee q \vee r) \wedge (p \vee \neg q \vee \neg r) \wedge (p \vee q \vee \neg r)$

As a circuit (assume 3 and 4 input gates are available; otherwise cascade)

CNF: $(\neg p \vee q \vee \neg r) \wedge (\neg p \vee q \vee r) \wedge (p \vee \neg q \vee \neg r) \wedge (p \vee q \vee \neg r)$

Terminology

Rosen p. 25

Tautology: compound proposition that is always T

Contradiction: compound proposition that is always F

Contingency: compound proposition that is not tautology or contradiction

p	q	F	T
T	T	F	T
T	F	F	T
F	T	F	T
F	F	F	T

Predicates and Quantifiers

Rosen p. 37-44

Tautology: For every row in TT, always evaluates to T

Contradiction: For each row in TT, always evaluates to F

Contingency: There's some row in TT that makes it T, and some row that makes it F.

p	q	F	T
T	T	F	T
T	F	F	T
F	T	F	T
F	F	F	T

Predicates and Quantifiers

Rosen p. 37-44

Predicate: "Proposition with a hole"

P(x) is "x > 3"

Q(x) is "the word x contains the letter 'a'"

R(x) is "the compound proposition $p \wedge q$ evaluates to T in row x"

x has different possible values depending on predicate

Predicates and Quantifiers

Rosen p. 37-44

Predicate: "Proposition with a hole"

P(x) is "x > 3"

Q(x) is "the word x contains the letter 'a'"

R(x) is "the compound proposition $p \wedge q$ evaluates to T in row x"

Which of the following evaluates to T?

A. P(1)
B. Q(computer)
C. R(TT)
D. More than one of the above.
E. None of the above

Predicates and Quantifiers

Consider the predicate P(x) is "$x^2 - 4 = 0$"

Is there some value of x which makes P(x) true?

A. Yes, all integer values of x.

B. Yes, there's exactly one positive integer value of x that make this predicate evaluate to T.

C. Yes, there are exactly two integer values of x that make this predicate evaluate to T.

D. No.

Universal quantifiers

"P(x) for all values x in the domain"

$$\forall x P(x)$$

$\forall x P(x)$ is T when P(x) is "$\log_2 x < x$" and the domain is integers greater than 1.

$\forall x P(x)$ is F when P(x) "$x^2 > x$" and the domain is all real numbers.

Counterexample

"P(x) for all values x in the domain"

$$\forall x P(x)$$

To **disprove** a universal statement: give a counterexample.

- element in the domain
- which makes the predicate F.

Existential quantifiers

"There exists an element in the domain such that P(x)"

$$\exists x P(x)$$

$\exists x P(x)$ is T when P(x) is "$x^2 > x$" and the domain is all real numbers.

$\exists x P(x)$ is F when P(x) is "$x^2 + 1 = 0$" and the domain is all real numbers.

Construction proof

"There exists an element in the domain such that P(x)"

$$\exists x P(x)$$

To **prove** an existential statement: give an example.

- element in the domain
- which makes the predicate T.

"De Morgan"-ish

Rosen p. 45

$$\neg \forall x P(x) \equiv \exists x \neg P(x)$$
$$\neg \exists x P(x) \equiv \forall x \neg P(x)$$

Logical equivalence of quantified statements: no matter which predicates are substituted and no matter which domains of discourse, the statements have the same truth value

Edge case: empty domain?

What if the domain of discourse is empty?

A. $\forall x P(x)$, $\exists x P(x)$ both evaluate to T.
B. $\forall x P(x)$ evaluates to T and $\exists x P(x)$ evaluates to F.
C. $\forall x P(x)$ evaluates to F and $\exists x P(x)$ evaluates to T.
D. $\forall x P(x)$, $\exists x P(x)$ both evaluate to F.
E. Quantifiers are not defined in this case.

Restricting the domain

Rosen p. 44

Over the domain of real numbers,

$$\forall x > 1 (x^2 > x)$$

means

$$\forall x (x > 1 \to x^2 > x)$$

"Every real number greater than 1 makes $x^2 > x$ T."

Restricting the domain

Rosen p. 44

Over the domain of real numbers,

$\forall x > 1(x^2 > x)$ means $\forall x(x > 1 \rightarrow x^2 > x)$

"Every real number greater than 1 makes $x^2 > x$ T."

"There is a real number greater than 1 that makes P(x) true"

A. Translates as $\exists x(x > 1 \rightarrow x^2 > x)$
B. Translates as $\exists x(x > 1 \vee x^2 > x)$
C. Translates as $\exists x(x > 1 \wedge x^2 > x)$
D. Translates as $\exists x(x > 1 \leftrightarrow x^2 > x)$

Translations

If 2 is an integer then so is 3.

If 2 is an integer then so is 0.4.

If 2 is not an integer, then neither is 0.5.

If 2 is not an integer, then neither is 33.

Translations

Rosen p. 54 #28

Something is not in the correct place.

All tools are in the correct place and are in excellent condition.

Everything is in the correct place and in excellent condition.

Nothing is in the correct place and is in excellent condition.

One of the tools is not in the correct place, but it is in excellent condition.