CSE 20
DISCRETE MATH

Spring 2016

http://cseweb.ucsd.edu/classes/sp16/cse20-ac/

Today's learning goals

- Define the decimal, binary, hexadecimal, and octal expansions of a positive integer.
- Describe and use algorithms for integer operations based on their expansions
- Define and use the DIV and MOD operators.

About you

CENTR101: CA PCYNH109: AB

To change your remote frequency

1. Press and hold power button until flashing
2. Enter two-letter code
3. Checkmark / green light indicates success

How many people in this class have you met so far?

A. None.
B. Less than 5.
C. 5-10.
D. 10-15.
E. More than 15.

Algorithms!

From last time

An **algorithm** is a finite sequence of precise instructions for performing a computation or for solving a problem.

… arithmetic

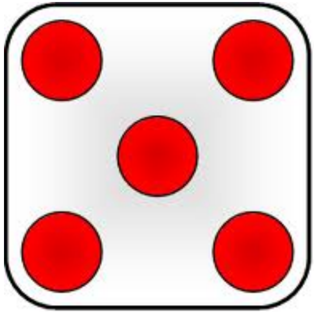
… optimization

Representation

$\log_2(32)$

101

Five

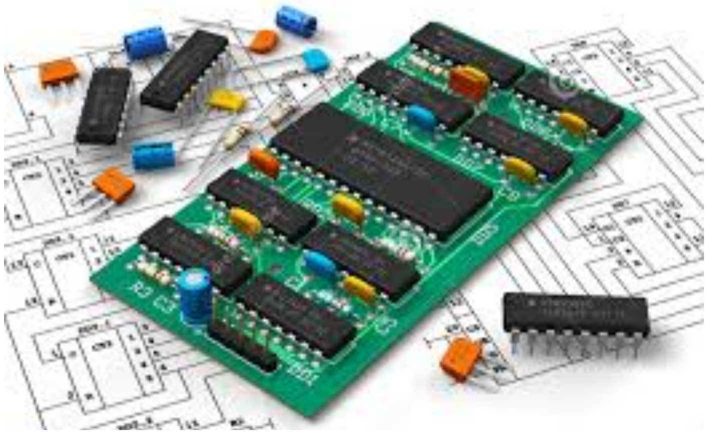


5

Integer representations

Different contexts call for different representations.

Base 10



Base 2

Bases

Are base 2 and base 10 the only possible ways to express positive integers?

A. Yes.
B. No: there's also base 8 (octal) and base 16 (hexadecimal), but that's it.
C. No: there's a few other special possible bases.
D. No: any integer greater than 1 can be a base.

Base expansion

For what positive integer k can we choose
$a_0, a_1, a_2, a_3, \ldots, a_k$ so that $a_k \neq 0$ and each a_i is 0 or 1 with

$$17 = a_k 2^k + a_{k-1} 2^{k-1} + \ldots + a_1 2 + a_0$$

A. k = 17
B. k = 2
C. k = 4
D. k = 5
E. Multiple values of k would work.

Base expansion

For what positive integer k can we choose

$a_0, a_1, a_2, a_3, \ldots, a_k$ so that $a_k \neq 0$ and each a_i is 0 or 1 with

$$17 = a_k 2^k + a_{k-1} 2^{k-1} + \ \ldots + a_1 2 + a_0$$

Writing 17 as a sum of powers of 2!

Base expansion

Find positive integer k and coefficients $a_0, a_1, a_2, a_3, \ldots, a_k$ so that

$$17 = a_k 2^k + a_{k-1} 2^{k-1} + \ldots + a_1 2 + a_0$$

$$= 16 + 1$$

$$= 1*2^4 + 1$$

So k=4 and $a_4=1$, $a_3=0$, $a_2=0$, $a_1=0$, $a_0=1$.

Base expansion

Rosen p. 246

For base b (integer greater than 1) and positive integer n there is unique choice of

- k, a nonnegative integer
- $a_0, a_1, a_2, a_3, \ldots, a_k$ integers between 0 and b-1, where
- $a_k \neq 0$ and

$$n = a_k b^k + a_{k-1} b^{k-1} + \ldots + a_1 b + a_0$$

Base expansion

Rosen p. 246

Notation: for positive integer n

Write

$$(a_k\, a_{k-1} \ldots a_1\, a_0)_b$$

when

$$n = a_k b^k + a_{k-1} b^{k-1} + \ldots + a_1 b + a_0$$

Base b expansion of n

Common bases

Rosen p. 246

- **Decimal expansion** base 10 $(\mathbf{17})_{10}$
- **Binary expansion** base 2 $(\mathbf{10001})_{2}$
- **Octal expansion** base 8 $(\quad)_{8}$
- **Hexadecimal expansion** base 16 $(\quad)_{16}$

Common bases

Rosen p. 246

- **Decimal expansion** base 10 $(17)_{10}$
- **Binary expansion** base 2 $(10001)_2$
- **Octal expansion** base 8 $(\ 21\ )_8$
- **Hexadecimal expansion** base 16 $(\ 11\ )_{16}$

Common bases

Rosen p. 246

- **Decimal expansion** base 10 $(17)_{10}$
- **Binary expansion** base 2 $(10001)_2$
- **Octal expansion** base 8 $(\ 21\ )_8$
- **Hexadecimal expansion** base 16 $(\ 11\ )_{16}$

What's different?

Base expansion

In what base could this expansion be

$$(1401)_?$$

A. Binary (base 2)
B. Octal (base 8)
C. Decimal (base 10)
D. Hexadecimal (base 16)
E. More than one of the above

Base expansion

In what base could this expansion be

$$(1401)_?$$

A. Binary (base 2)

B. Octal (base 8) — Value: $1*8^3+4*8^2+1=768$

C. Decimal (base 10) — Value: $1*10^3+4*10^2+1=1401$

D. Hexadecimal (base 16) — Value: $1*16^3+4*16^2+1=5121$

E. More than one of the above

Algorithm: constructing base b expansion *Rosen p. 249*

Input n,b **Output** k, coefficients in expansion

- English description.

- Pseudocode.

Algorithm: constructing base b expansion *Rosen p. 249*

Input n,b **Output** k, coefficients in expansion

- English description.

  Find k

  Work down to find a_k, then a_{k-1}, etc.

Algorithm: constructing base b expansion *Rosen p. 249*

Input n,b **Output** k, coefficients in expansion

- English description.

  Find k

  Work down to find a_k, then a_{k-1}, etc.

What's the length of the base b expansion of n?

Start with related question:
What's the biggest number that can be represented with k digits?

Bounds

Base 2

0	**0**
1	**1**
10	**2**
11	**3**
100	**4**
101	**5**
110	**6**
111	**7**
1000	**8**
1001	**9**
1010	**10**

Base 3

0	**0**	102	**11**	211	**22**
1	**1**	110	**12**	212	**23**
2	**2**	111	**13**	220	**24**
10	**3**	112	**14**	221	**25**
11	**4**	120	**15**	222	**26**
12	**5**	121	**16**	1000	**27**
20	**6**	122	**17**	1001	**28**
21	**7**	200	**18**	1002	**29**
22	**8**	201	**19**	1010	**30**
100	**9**	202	**20**	1011	**31**
101	**10**	210	**21**	1012	**32**

Bounds

Base 2

0	**0**
1	**1**
10	**2**
11	**3**
100	**4**
101	**5**
110	**6**
111	**7**
1000	**8**
1001	**9**
1010	**10**

What's the biggest number whose base b expansion has k digits, i.e.

$$(a_{k-1} \ldots a_1 \, a_0)_b?$$

A. b^k
B. b^{k-1}
C. b^k-1
D. $b^{k-1}-1$
E. None of the above.

Bounds

Base 2

0	**0**
1	**1**
10	**2**
11	**3**
100	**4**
101	**5**
110	**6**
111	**7**
1000	**8**
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What's the biggest number whose base b expansion has k digits, i.e.

$$(a_{k-1} \ldots a_1 a_0)_b?$$

A. b^k
B. b^{k-1}
C. b^k-1
D. $b^{k-1}-1$
E. None of the above.

Algorithm: constructing base b expansion *Rosen p. 249*

Input n,b **Output** k, coefficients in expansion

- English description.

  Find k by computing successive powers of b until find smallest k such that

  $$\mathbf{b^{k-1} \leq n < b^k}$$

  For each value of i from 1 to k

  Set a_{k-i} to be the largest number between 0 and b-1 for which $a_{k-i}\, b^{k-i} < n$.

  Update current value remaining $n := n - a_{k-i}\, b^{k-i}$

Algorithm: constructing base b expansion *Rosen p. 249*

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  Update current value remaining n := n - $a_{k-i}\, b^{k-i}$

What's the the base 3 expansion of 17?

A. $(10001)_3$
B. $(210)_3$
C. $(111)_3$
D. $(222)_3$
E. None of the above.

Algorithm: constructing base b expansion *Rosen p. 249*

Input n,b **Output** k, coefficients in expansion

- English description.

  Find k by computing successive powers of b until find smallest k such that

  $$b^{k-1} \leq n < b^k$$

  For each value of i from 1 to k

  Set a_{k-i} to be the largest number between 0 and b-1 for which $a_{k-i} b^{k-i} < n$.

  Update current value remaining $n := n - a_{k-i} b^{k-i}$

Definite? Finite? Correct?

Challenge: translate to pseudocode!

Algorithm: constructing base b expansion *Rosen p. 249*

Input n,b **Output** k, coefficients in expansion

- English description.

  Idea: Find smallest digit first, then next smallest, etc.

  …. **but how?**

- Pseudocode.

Bases and Divisibility

Rosen p. 237-239

When k>0

$$n = a_k b^k + a_{k-1} b^{k-1} + \ldots + a_1 b + a_0$$
$$= b\,(a_k b^{k-1} + a_{k-1} b^{k-2} + \ldots + a_1) + a_0$$

Quotient when divide n by b

Remainder when divide n by b

Bases and Divisibility

Rosen p. 237-239

When k>0

$$n = a_k b^k + a_{k-1} b^{k-1} + \dots + a_1 b + a_0$$
$$= b\,(a_k b^{k-1} + a_{k-1} b^{k-2} + \dots + a_1) + a_0$$

Quotient when divide n by b

Remainder when divide n by b

Theorem: For a an integer and d a positive integer, there are unique integers q and r with $0 \le r < d$ and $a = qd + r$.

Notation: $q = a$ **div** d $r = a$ **mod** d

Algorithm: constructing base b expansion *Rosen p. 249*

Input n,b **Output** k, coefficients in expansion

- English description.

  Compute n **mod** b to obtain a_0.

  Update value n := n **div** b of integer whose expansion we need.

  Repeat.

Algorithm: constructing base b expansion *Rosen p. 249*

Update value n := n **div** b of integer whose expansion we need.

How is the base b expansion of n **div** b related to the base b expansion of n?

A. They're the same.
B. They're unrelated.
C. The expansion of n has an extra symbol at the end.
D. The expansion of n has an extra symbol at the front.
E. None of the above

Relationship between expansions

$$n = a_k b^k + a_{k-1} b^{k-1} + \ \ldots + a_1 b + a_0$$
$$= b\,(a_k b^{k-1} + a_{k-1} b^{k-2} + \ \ldots + a_1) + a_0$$

So base b expansion of n is

$$(a_k\ a_{k-1}\ \ldots\ a_1\ a_0)_b$$

and n div b is $a_k b^{k-1} + a_{k-1} b^{k-2} + \ \ldots + a_1$, which has the expansion

$$(a_k\ \ldots\ a_1)_b$$

So: **base b expansion of n is the result of taking the base b expansion of n div b, shifting one column to the left and *adding a coefficient* appended to the end.**

Algorithm: constructing base b expansion *Rosen p. 249*

Input n,b **Output** k, coefficients in expansion

- Pseudocode.

procedure *base b expansion*$(n, b:$ pos ints with $b > 1)$

$q := n$

$k := 0$

while $q \neq 0$

  $a_k := q \textbf{ mod } b$

  $q := q \textbf{ div } b$

  $k := k + 1$

return $(a_{k-1}, \ldots, a_1, a_0)$

Algorithm: constructing base b expansion *Rosen p. 249*

procedure *base b expansion*$(n, b:$ pos ints with $b > 1)$

$q := n$
$k := 0$
while $q \neq 0$
    $a_k := q$ **mod** b
    $q := q$ **div** b
    $k := k + 1$
return $(a_{k-1}, \ldots, a_1, a_0)$

n	b	q	k	a_k
17	**3**	17	0	17 **mod** 3 = 2
		17 **div** 3 = 5	1	5 **mod** 3 = 2
		5 **div** 3 = 1	2	1 **mod** 3 = 1
		1 **div** 3 = 0	3	return!

Definite? Finite? Correct?

Arithmetic + Representations *Rosen p. 251*

What is the sum of $(A)_{16}$ and $(12)_{16}$?

A. $(12A)_{16}$
B. $(1A)_{16}$
C. $(22)_{16}$
D. $(16)_{16}$
E. $(1C)_{16}$

Hexadecimal digits

0	8
1	9
2	A
3	B
4	C
5	D
6	E
7	F

Arithmetic + Representations *Rosen p. 251*

What is the product of $(A)_{16}$ and $(12)_{16}$?

A. $(B4)_{16}$
B. $(78)_{16}$
C. $(12A)_{16}$
D. $(120)_{16}$
E. $(A12)_{16}$

Hexadecimal digits

0	8
1	9
2	A
3	B
4	C
5	D
6	E
7	F

Arithmetic + Representations *Rosen p. 251*

procedure $mystery1(a, b: \text{ pos ints with binary expansions } (a_{n-1}, \ldots, a_0), (b_{n-1}, \ldots, b_0))$

$c := 0$

for $j := 0$ **to** $n - 1$

  $d := (a_j + b_j + c)$ **div** 2

  $s_j := a_j + b_j + c - 2d$

  $c := d$

$s_n := c$

return $(s_n, \ldots, a_1, a_0)$

Trace this procedure on inputs

$$a = 5 = (101)_2,\ b = 4 = (100)_2$$

What is the return value?

Arithmetic + Representations *Rosen p. 251*

procedure $mystery1(a, b:$ pos ints with binary expansions $(a_{n-1}, \ldots, a_0), (b_{n-1}, \ldots, b_0))$

$c := 0$
for $j := 0$ **to** $n - 1$
    $d := (a_j + b_j + c)$ **div** 2
    $s_j := a_j + b_j + c - 2d$
    $c := d$
$s_n := c$
return $(s_n, \ldots, a_1, a_0)$

What's a higher-level description of the function computed by this procedure?

A. It adds up the bits of the binary expansions and divides each by 2.
B. It returns the binary expansion of a number that is the sum of the two input integers, divided by 2.
C. It returns the binary expansion of the sum of the two input integers.
D. None of the above.

Arithmetic + Representations *Rosen p. 251*

procedure $mystery1(a, b:$ pos ints with binary expansions $(a_{n-1}, \ldots, a_0), (b_{n-1}, \ldots, b_0))$

$c := 0$
for $j := 0$ **to** $n - 1$
    $d := (a_j + b_j + c)$ **div** 2
    $s_j := a_j + b_j + c - 2d$
    $c := d$
$s_n := c$
return $(s_n, \ldots, a_1, a_0)$

On Tuesday, we'll see how the binary case works well with logic circuits and Boolean algebra…

Reminders

- Homework 1 due tomorrow
  - Set up course tools
  - Pseudocode and algorithms
- Office hours