CSE 20
DISCRETE MATH

SPRING 2016

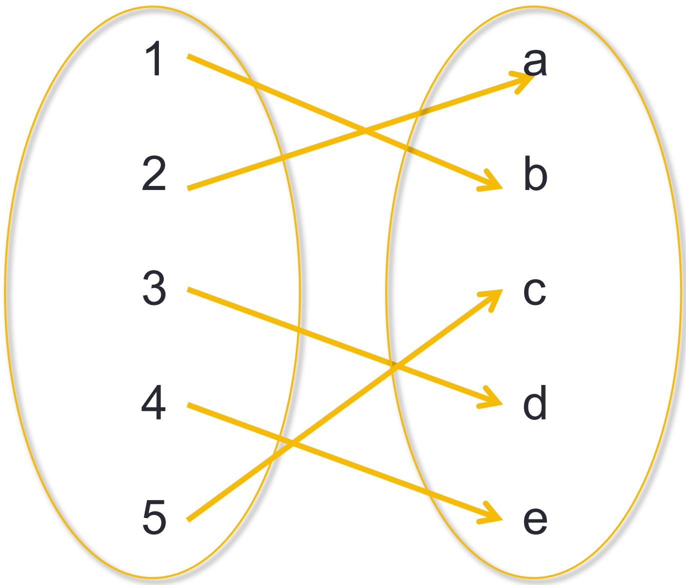
http://cseweb.ucsd.edu/classes/sp16/cse20-ac/

Today's learning goals

- Define and compute the cardinality of a set: Finite sets, countable sets, uncountable sets
- Use functions to compare the sizes of sets
- Determine and prove whether a given binary relation is
  - symmetric
  - antisymmetric
  - reflexive
  - transitive

One-to-one + onto

Rosen p. 144



one-to-one correspondence

bijection

invertible

The **inverse** of a function f: A→B is the function g: B→A such that

$$\forall b(b \in B \rightarrow (g(b) = a \leftrightarrow f(a) = b))$$

Beyond finite sets

Rosen Section 2.5

For all sets, we say

|A| = |B| if and only if there is a bijection between them.

Useful fact (Cantor-Schroder-Bernstein Theorem):
|A| = |B| iff there are two one-to-one functions f:A→B and g:B→A

Rosen Theorem 2, p 174

Cardinality

Rosen Defn 3 p. 171

- Finite sets — $|A| = n$ for some nonnegative int n
- Countably infinite sets — $|A| = |\mathbf{Z}^+|$ (informally, can be listed out)
- Uncountable sets — Infinite but not in bijection with $\mathbf{Z}^+$

Cardinality

Rosen Defn 3 p. 171

- Finite sets

|A| = n for some nonnegative int n

|{1,2,3}| = 3 = |{cat, dog, mouse}|

What is a bijection between {1,2,3} and {cat,dog,mouse}?

Cardinality

Rosen Defn 3 p. 171

- Finite sets

|A| = n for some nonnegative int n

Which of the following sets is **not** finite?

A. $\emptyset$

B. $[0, 1]$

C. $\{x \in \mathbb{Z} | x^2 = 1\}$

D. $\mathcal{P}(\{1, 2, 3\})$

E. None of the above (they're all finite)

Cardinality

Rosen p. 172

- Countable sets   A is finite or |A| = |**Z**$^+$| (informally, can be listed out)

Examples: $\emptyset$ $\{x \in \mathbb{Z} | x^2 = 1\}$ $\mathcal{P}(\{1,2,3\})$ $\mathbb{Z}^+$

and also …

- the set of **odd positive** integers   Example 1
- the set of **all integers**   Example 3
- the set of **positive rationals**   Example 4

Useful facts for proofs

- If A, B are both countable then A U B is also countable

  Theorem 1, p. 174

- A subset of a countable set is also countable

  Rosen 2.5 Ex. 16

- If A is a subset of B and A is uncountable, then B is uncountable.

  Rosen 2.5 Ex. 15

Useful facts for proofs

Claim: The set of all rational numbers, **Q**, is countable.

Useful facts for proofs

Claim: The set of all rational numbers, **Q**, is countable.

Proof:

Use that $\mathbf{Q}^+$ is countable to prove that $\mathbf{Q}^-$ is also countable.

Then, use that the union of two countable sets is countable…

Cardinality

Rosen p. 172

- Countable sets A is finite or |A| = $|\mathbf{Z}^+|$ (informally, can be listed out)

Examples: $\emptyset$ $\{x \in \mathbb{Z} | x^2 = 1\}$ $\mathcal{P}(\{1,2,3\})$ $\mathbb{Z}^+$

and also …

- the set of **odd positive** integers Example 1
- the set of **all integers** Example 3
- the set of **positive rationals** Example 4
- the set of **negative rationals**
- the set of **rationals**

Cardinality

Rosen p. 172

- Uncountable sets

Infinite but not in bijection with $\mathbf{Z}^+$

Which of the following is **not** true?

A. There are sets A, B with $A \subsetneq B$ and $|A| = |B|$

B. There are **disjoint** sets A, B with $|A| = |B|$

C. There are one-to-one function that are not onto.

D. There are onto functions that are not one-to-one.

E. All infinite sets are of the same size

$|Z^+| \neq |R|$

Rosen example 5, page 173-174

Cantor's diagonalization argument

Theorem: For every set A, $|A| \neq |\mathcal{P}(A)|$

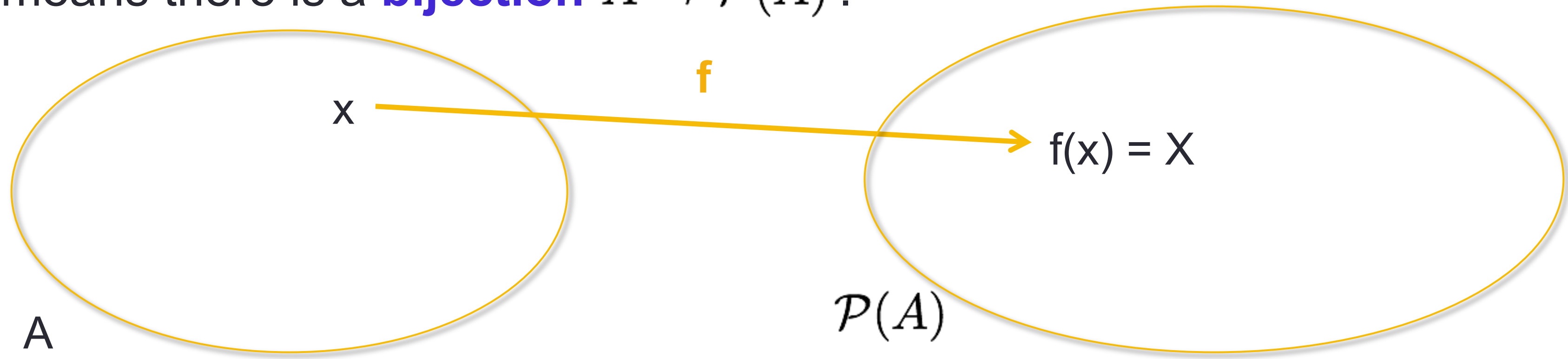
$|\mathbb{Z}^+| \neq |\mathbb{R}|$

Cantor's diagonalization argument

Theorem: For every set A, $|A| \neq |\mathcal{P}(A)|$

Proof: (Proof by contradiction)

Assume towards a contradiction that $|A| = |\mathcal{P}(A)|$. By definition, that means there is a **bijection** $A \to \mathcal{P}(A)$.

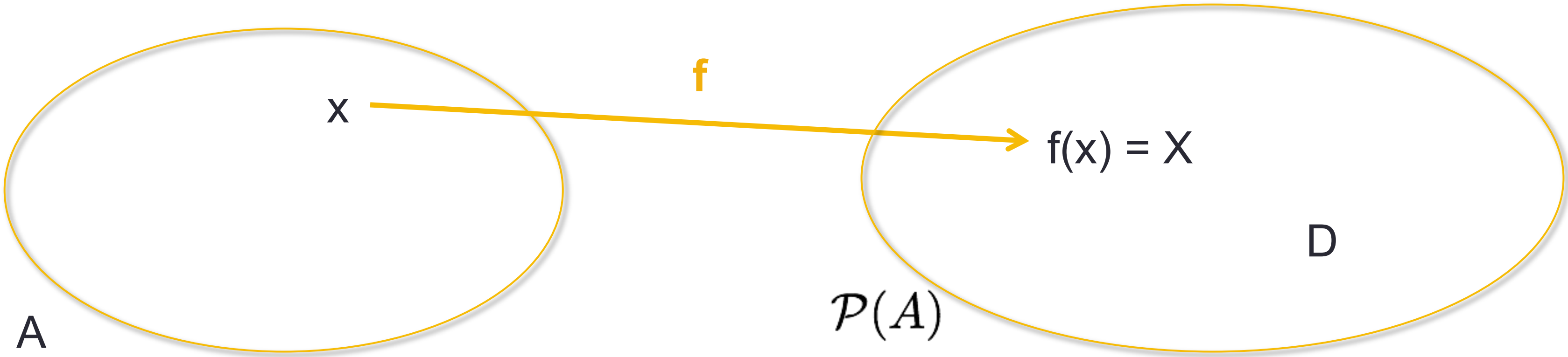


$|Z^+| \neq |R|$

Cantor's diagonalization argument

Consider the subset D of A defined by, for each a in A:

$$a \in D \quad \text{iff} \quad a \notin f(a)$$



$|Z^+| \neq |R|$

Cantor's diagonalization argument

Consider the subset D of A defined by, for each a in A:

$$a \in D \qquad \text{iff} \qquad a \notin f(a)$$

Define d to be the pre-image of D in A under f   f(d) = D

Is d in D?

- If yes, then by definition of D, $d \notin f(d) = D$ **a contradiction!**
- Else, by definition of D, $\neg(d \notin f(d))$ so $d \in f(D) = D$ **a contradiction!**

Cardinality

Rosen p. 172

- Uncountable sets — Infinite but not in bijection with $\mathbf{Z}^+$

Examples: the power set of any countably infinite set

and also …

- the set of **real** numbers — Example 5
- (0,1) — Example 6 (++)
- (0,1] — Example 6 (++)

Exercises 33, 34

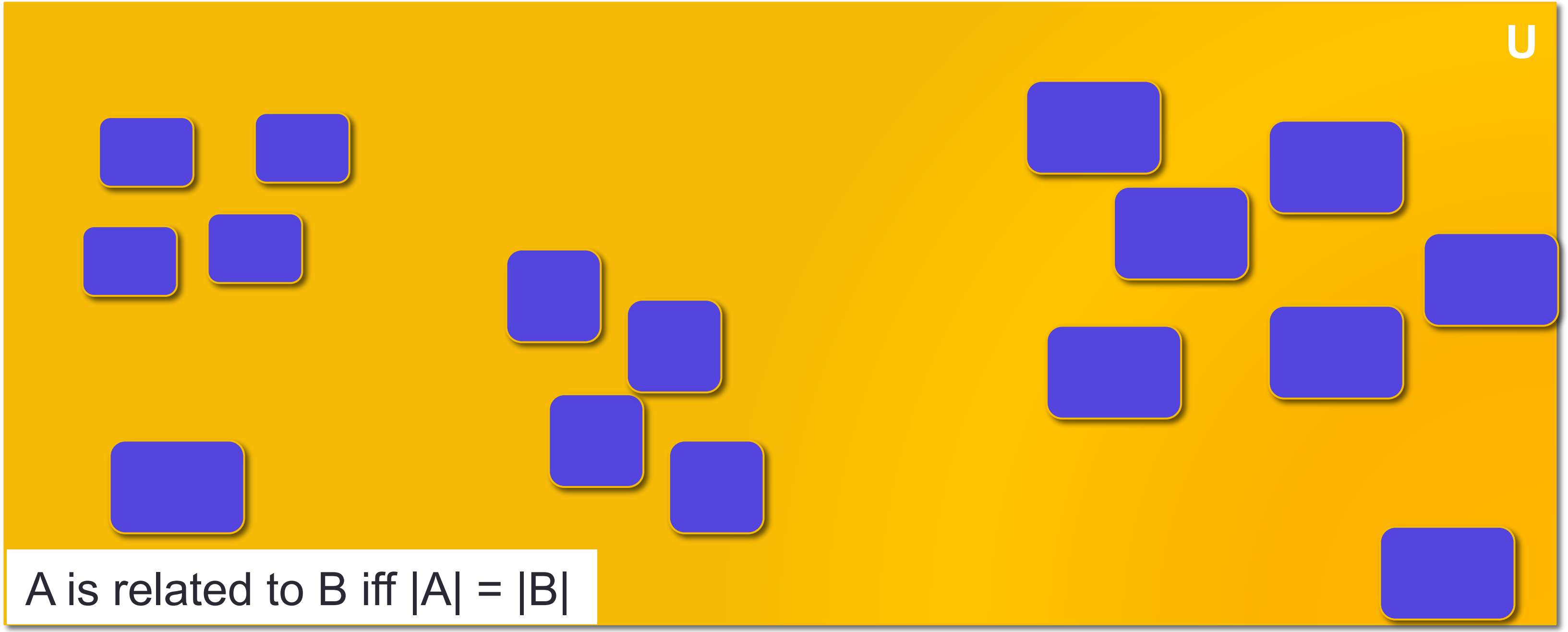
Cardinality and subsets

Suppose A and B are sets and $A \subseteq B$.

A. If A is finite then B is finite.
B. If A is countable then B is uncountable.
C. If B is infinite then A is finite.
D. If B is uncountable then A is uncountable.
E. None of the above.

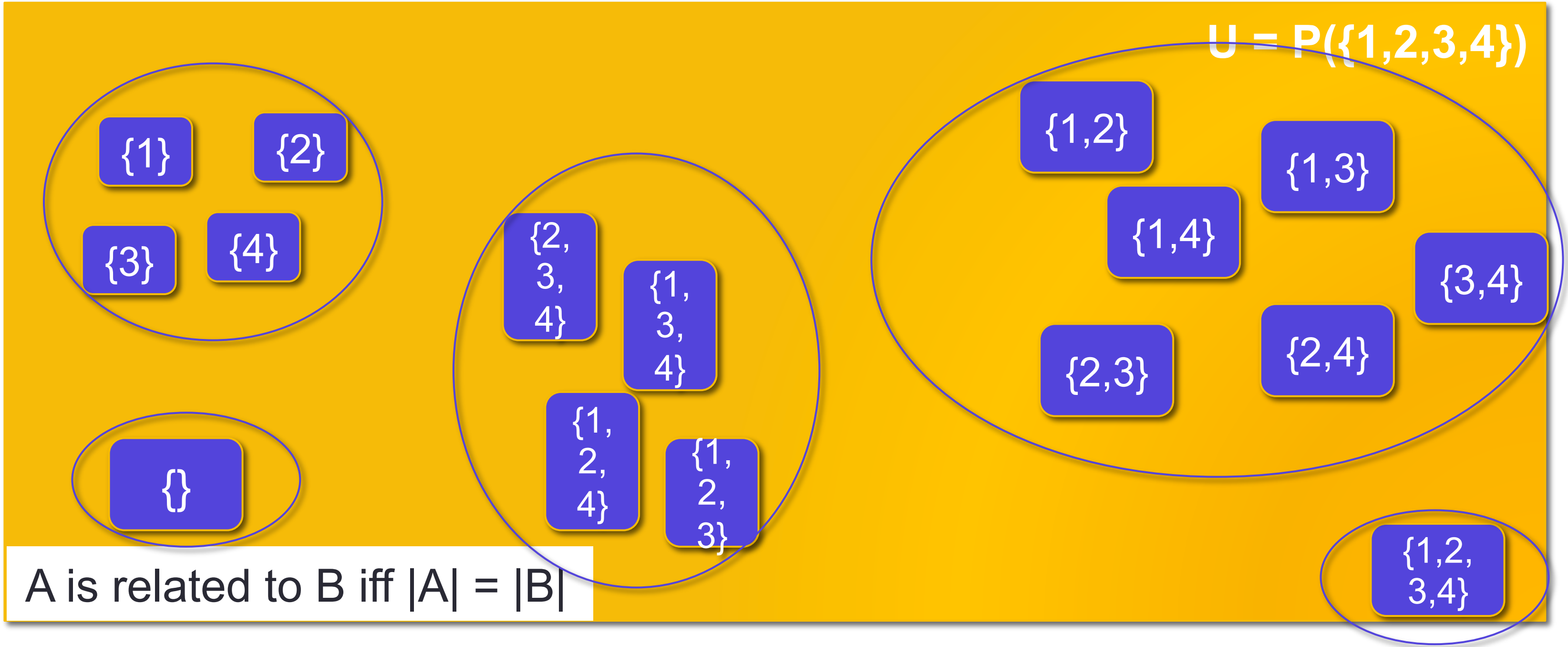
Size as a relation

- Cardinality lets us compare and group sets.



Size as a relation

- Cardinality lets us compare and group sets.



Relations, more generally

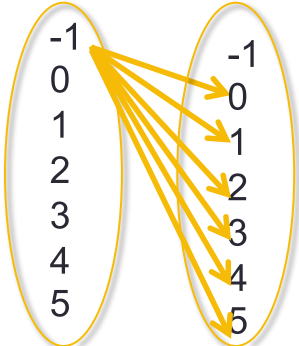
Rosen Chapter 9

- Let A, B be sets. **Binary relation from A to B** is (any) subset of A x B.

Examples

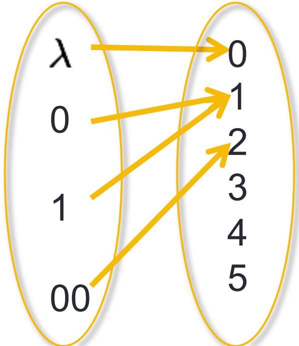
A = B = **Z**

R={(x,y) : x < y}



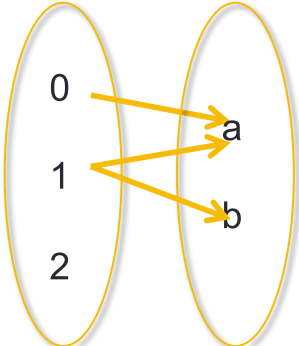
A = $\{0,1\}^*$ B=**N**

R={(w, n) : |w|=n}



A = {0,1,2} B={a,b}

R={(0,a), (1,a), (1,b)}



Relation on a set A

Rosen pp 576-578

R is subset of A x A. It is called

reflexive iff $\forall a(\ (a,a) \in R\ )$

symmetric iff $\forall a \forall b(\ (a,b) \in R \rightarrow (b,a) \in R\ )$

antisymmetric iff $\forall a \forall b(\ [(a,b) \in R \wedge (b,a) \in R] \rightarrow a = b\ )$

transitive iff $\forall a \forall b \forall c(\ [(a,b) \in R \wedge (b,c) \in R] \rightarrow (a,c) \in R\ )$

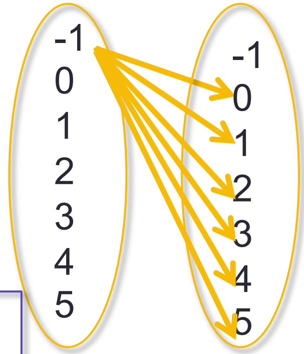
Relations, more generally

Rosen Chapter 9

Examples

A = B = **Z**

R={(x,y) : x < y}



Which of the following properties hold for R?

A. Reflexive, i.e. $\forall a(\ (a,a) \in R\ )$

B. Symmetric, i.e. $\forall a \forall b(\ (a,b) \in R \rightarrow (b,a) \in R\ )$

C. Antisymmetric, i.e. $\forall a \forall b(\ [(a,b) \in R \wedge (b,a) \in R] \rightarrow a = b\ )$

D. Transitive, i.e. $\forall a \forall b \forall c(\ [(a,b) \in R \wedge (b,c) \in R] \rightarrow (a,c) \in R\ )$

E. None of the above.