

CSE 20

DISCRETE MATH

SPRING 2016

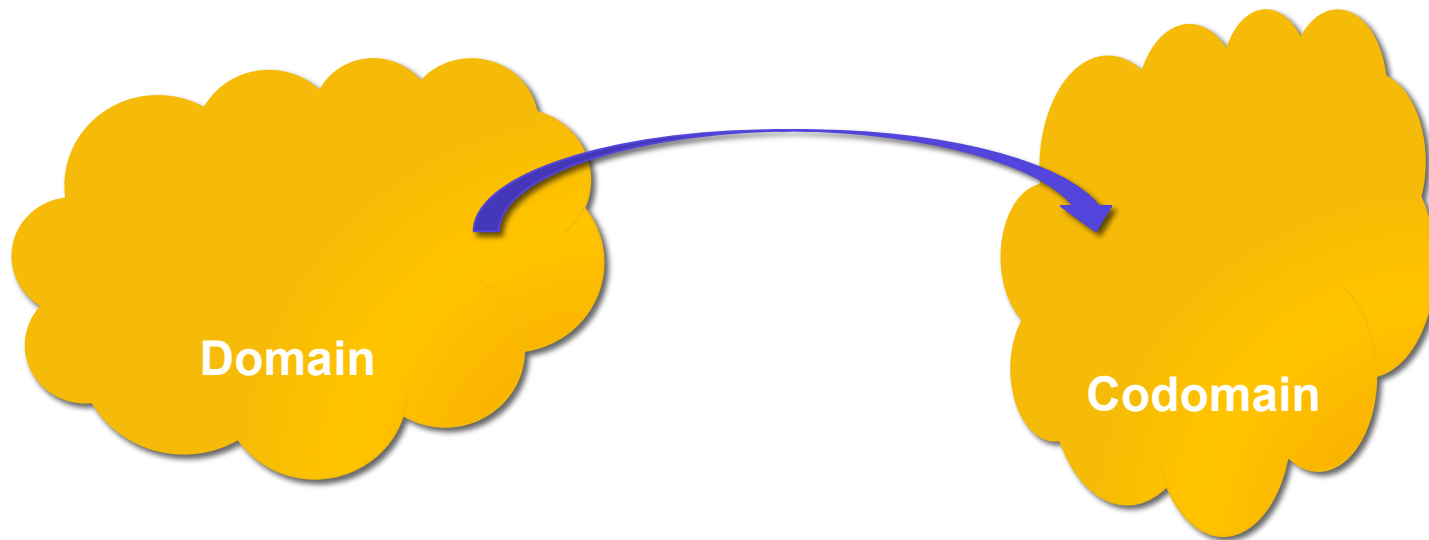
<http://cseweb.ucsd.edu/classes/sp16/cse20-ac/>

Today's learning goals

- Represent functions in multiple ways
- Define and prove properties of: domain of a function, image of a function, composition of functions
- Determine and prove whether a function is one-to-one, onto, bijective
- Apply the definition and properties of floor function, ceiling function, factorial function
- Define and compute the cardinality of a set: Finite sets, countable sets, uncountable sets
- Use functions to compare the sizes of sets
- Use functions to define sequences: arithmetic progressions, geometric progressions
- Use and prove properties of recursively defined functions and recurrence relations (using induction)
- Use and interpret Sigma notation

Functions

Rosen p. 138



Function

Mapping

Transformation

$$\forall a(a \in D \rightarrow \exists! b(b \in C \wedge f(a) = b))$$

To specify a function

(1) Domain

(2) Codomain

(3) Assignment

Operations on functions

Rosen p. 141,147

If $f: A \rightarrow \mathbf{R}$, $g: A \rightarrow \mathbf{R}$

$f+g: A \rightarrow \mathbf{R}$

$fg: A \rightarrow \mathbf{R}$

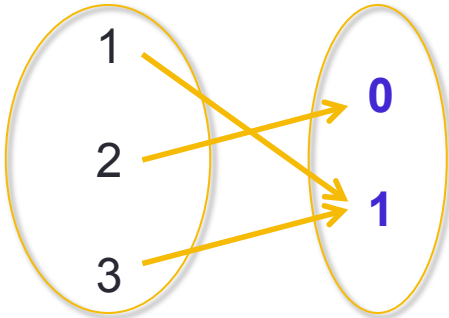
If $f: B \rightarrow C$, $g: A \rightarrow B$

$f \circ g: A \rightarrow C$

Properties of functions

possible

- A function f is means **at least one input for every output**



How can we formalize this?

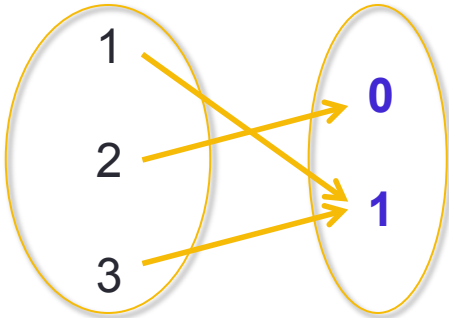
- A. $\forall a(a \in D \rightarrow \exists!b (b \in C \wedge f(a) = b))$
- B. $\forall b(b \in C \rightarrow \exists!a (a \in D \wedge f(a) = b))$
- C. $\forall a\forall b((a \in D \wedge b \in C) \rightarrow f(a) = b)$
- D. $\forall b(b \in C \rightarrow \exists a (a \in D \wedge f(a) = b))$
- E. None of the above

Properties of functions

Rosen p. 143

possible

- A function f is **onto** means **at least one input for every output** (surjective)

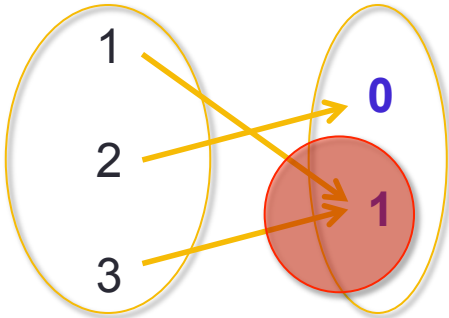


$$\forall b(b \in C \rightarrow \exists a(a \in D \wedge f(a) = b))$$

Properties of functions

Rosen p. 141

- A function f is **one-to-one** means **no duplicate images** (injective)



How can we formalize this?

A. $\forall a \forall b ((a \in D \wedge b \in D) \rightarrow f(a) \neq f(b))$

B. $\forall a \forall b ((a \in D \wedge b \in D) \rightarrow (f(a) = f(b) \rightarrow a = b))$

C. $\forall a \forall b ((a \in C \wedge b \in C) \rightarrow a \neq b)$

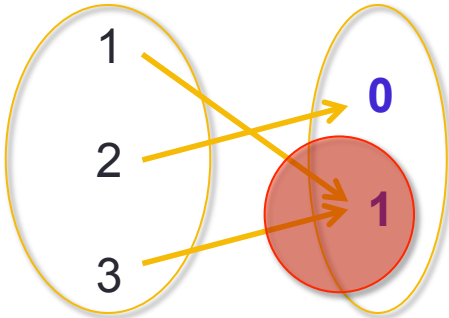
D. $\forall a \forall b ((a \in C \wedge b \in C) \rightarrow f(a) \neq f(b))$

E. None of the above

Properties of functions

Rosen p. 141

- A function f is **one-to-one** means **no duplicate images** (injective)

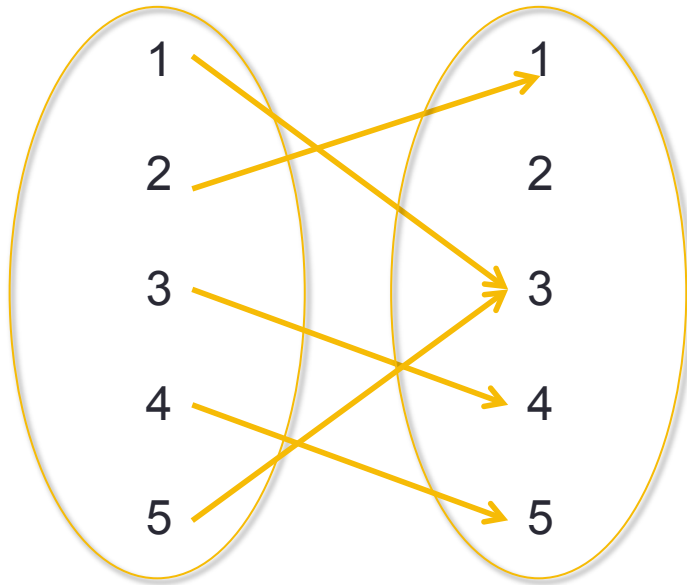


$$\forall a \forall b ((a \in D \wedge b \in D) \rightarrow (f(a) = f(b) \rightarrow a = b))$$

$$\forall a \forall b ((a \in D \wedge b \in D) \rightarrow (a \neq b \rightarrow f(a) \neq f(b)))$$

Onto? One-to-one?

Consider the function over domain and codomain $\{1,2,3,4,5\}$ defined by



This function is

- A. Well defined, onto, and one-to-one.
- B. Well defined, but neither onto nor one-to-one.
- C. Well defined, onto, but not one-to-one.
- D. Not well-defined, not onto, not one-to-one.
- E. None of the above.

Onto? One-to-one?

Consider the function over domain and codomain $\mathbf{R}^{\geq 0}$ defined by

$$f(x) = x^2$$

This function is

- A. Well defined, onto, and one-to-one.
- B. Well defined, but neither onto nor one-to-one.
- C. Well defined, onto, but not one-to-one.
- D. Not well-defined, not onto, not one-to-one.
- E. None of the above.

Proving a function is ...

Define $f: \{0,1\}^* \rightarrow \mathbf{N}$ by $f(w) = |w|$, or formally, f is defined recursively by

$$\begin{cases} f(\lambda) = 0 \\ f(w0) = f(w) + 1 \\ f(w1) = f(w) + 1 \end{cases}$$

Fact: This function is onto.

Proving a function is ...

Define $f: \{0,1\}^* \rightarrow \mathbf{N}$ by $f(w) = |w|$, or formally, f is defined recursively by

$$\begin{cases} f(\lambda) = 0 \\ f(w0) = f(w) + 1 \\ f(w1) = f(w) + 1 \end{cases}$$

Fact: This function is not one-to-one.

Proving a function is ...

Let $A = \{1,2,3\}$ and $B = \{2,4,6\}$.

Define a function from the power set of A to the power set of B by:

$$f : \mathcal{P}(A) \rightarrow \mathcal{P}(B)$$

$$f(X) = X \cap B$$

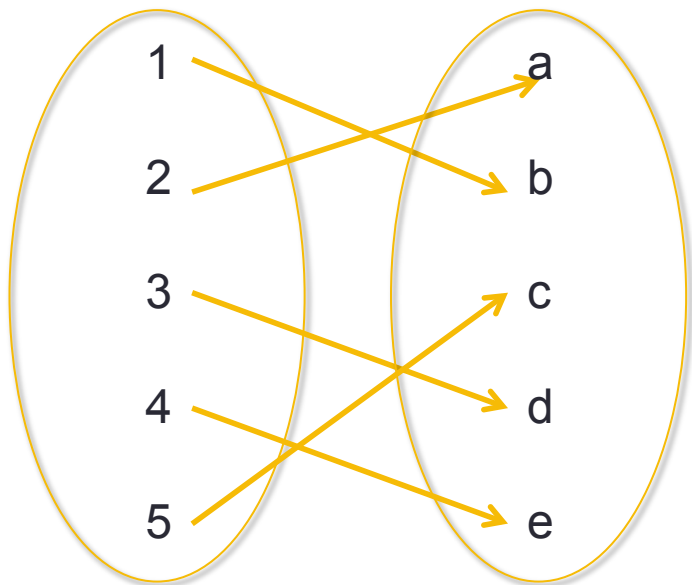
Well-defined?

Onto?

One-to-one?

One-to-one + onto

Rosen p. 144



one-to-one correspondence

bijection

invertible

The **inverse** of a function $f: A \rightarrow B$ is the function $g: B \rightarrow A$ such that

$$\forall b(b \in B \rightarrow (g(b) = a \leftrightarrow f(a) = b))$$

One-to-one + onto

Rosen p. 144

What's the inverse function of

$$f(x) = x^3 + 1 ?$$

A. $g(x) = x^{1/3} - 1$

B. $g(x) = (x - 1)^{1/3}$

C. $g(x) = \frac{1}{x^3 + 1}$

D. More than one of the above.

E. None of the above.

one-to-one correspondence

bijection

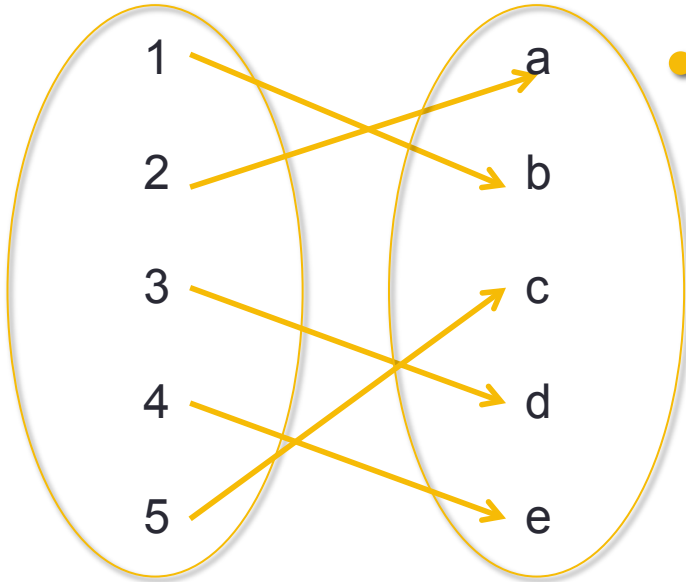
invertible

The **inverse** of a function $f: A \rightarrow B$ is the function $g: B \rightarrow A$ such that

$$\forall b(b \in B \rightarrow (g(b) = a \leftrightarrow f(a) = b))$$

One-to-one + onto

Rosen p. 144



Fact: for finite sets A and B, there is a bijection between them if and only if $|A| = |B|$.

Beyond finite sets

Rosen Section 2.5

For all sets, we say

$|A| = |B|$ if and only if there is a bijection between them.

- A. $|\mathbf{Z}| = |\mathbf{N}|$
- B. $|\mathbf{N}| = |\mathbf{Q}|$
- C. $|\mathbf{Z}| = |\{0,1\}^*|$
- D. All of the above.
- E. None of the above.

Beyond finite sets

For all sets, we say

$|A| = |B|$ if and only if there is a bijection between them.

0, 1, 2, 3, 4, 5, 6,

0, -1, 1, -2, 2, -3, 3, ...

0, 1, -1, $\frac{1}{2}$, $-\frac{1}{2}$, ...

λ , 0, 1, 00, 01, 10, 11, ...

Beyond finite sets

For all sets, we say

$|A| = |B|$ if and only if there is a bijection between them.

0, 1, 2, 3, 4, 5, 6,

0, -1, 1, -2, 2, -3, 3, ...

0, 1, -1, $\frac{1}{2}$, $-\frac{1}{2}$, ...

λ , 0, 1, 00, 01, 10, 11, ...

Which of the following is **not** true?

- A. There are sets A, B with $A \subsetneq B$ and $|A| = |B|$
- B. There are **disjoint** sets A, B with $|A| = |B|$
- C. There are one-to-one function that are not onto.
- D. There are onto functions that are not one-to-one.
- E. All infinite sets are of the same size.

Cardinality

Rosen Defn 3 p. 171

- Finite sets
- Countably infinite sets
- Uncountable sets

$|A| = n$ for some nonnegative int n

$|A| = |\mathbf{Z}^+|$ (informally, can be listed out)

Infinite but not in bijection with \mathbf{N}