CSE 20
DISCRETE MATH

SPRING 2016

http://cseweb.ucsd.edu/classes/sp16/cse20-ac/

Today's learning goals

- Explain the steps in a proof by (strong) mathematical induction
- Use (strong) mathematical induction to prove
  - correctness of identities and inequalities
  - properties of algorithms
  - properties of geometric constructions
- Apply recursive definitions of sets to determine membership in the set
- Use structural induction to prove properties of recursively defined sets

Induction: a road map

- Thursday
  - What is a proof by induction?
  - Examples: inequalities, algorithms, constructions
- Today
  - Recursive definitions: functions, sets, sigma notation
  - Strong induction
  - More examples / proofs: identities, constructions

A new proof strategy

Rosen p. 311

To ~~show that some~~ statement P(k) is true about all nonnegative integers k,

Need both for induction to be applicable

1. Show that it's true about 0 — i.e. P(
2. Show P(0) → P(1) — Hence conclude P(1)
3. Show P(1) → P(2) — Hence conclude P(2)
4. Show P(2) → P(3) — Hence conclude P(3)
5. …..

Mathematical induction

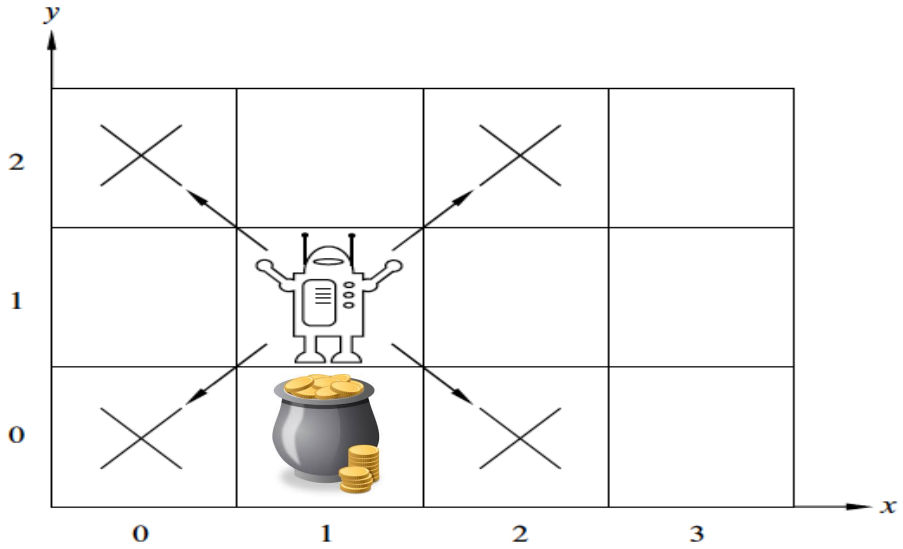
Rosen p. 311

To show that some statement P(k) is true about all nonnegative integers k,

1. Show that it's true about 0 i.e. P(0)
2. Show $\forall k(P(k) \rightarrow P(k+1))$ Hence conclude P(1),…

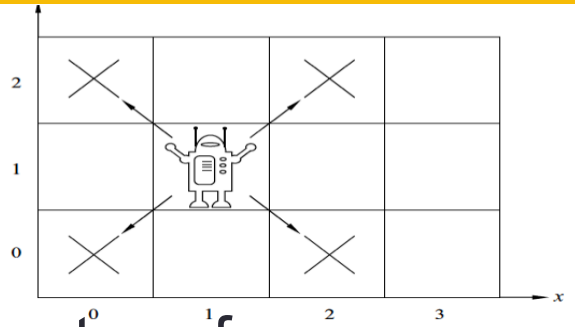
Robot

Start at origin, moves on infinite 2-dimensional integer grid.
At each step, move to diagonally adjacent grid point.



Can it ever reach (1,0)?

Robot



Lemma (Invariant): The sum of the coordinates of any state reachable by the robot is even.

Proof of Lemma: by mathematical induction on the number of steps.

Using proof: Sum of coordinates of (1,0) is 1 so not even!

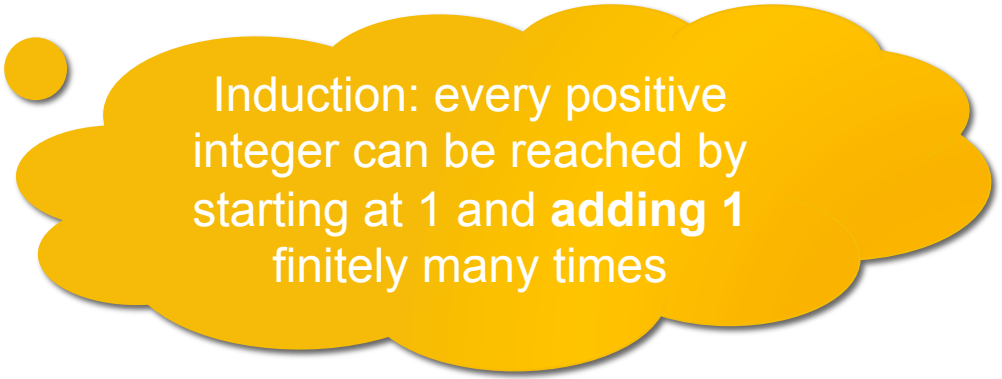
Recursive definitions

Rosen Sec 5.3

- Define a set S by

{ … }

{ x | P(x) }



Recursive definition:

Basis step – Specify initial collection of elements.

Recursive step – Provide rules for forming new elements in the set from those already known to be in the set.

Recursive definitions

Rosen Sec 5.3

Let S be the subset of the set of integers defined recursively by

Basis step: $1 \in S$

Recursive step: If $a \in S$, then $a + 2 \in S$.

What's an equivalent description of this set?

A. All positive multiples of 2.
B. All positive even integers.
C. All positive odd integers.
D. All integers.
E. None of the above.

Recursive definitions

Rosen Sec 5.3

The set of **binary strings** $\{0,1\}^*$ is defined recursively by

Basis step: $\lambda \in \{0,1\}^*$ where λ is the empty string.

Recursive step: If $w \in \{0,1\}^*$, then $w0 \in \{0,1\}^*$ and $w1 \in \{0,1\}^*$.

Which of the following are **not** binary strings?

A. λ
B. 1
C. 000
D. 010101
E. 10110011100011110000111110000011 1…

Structural induction

Rosen p. 354

To show that some statement P(k) is true about all elements of a recursively defined set S,

1. Show that it's true for each element specified in the **basis step** to be part of S.
2. Show that if it's true for each of the elements used to construct **new elements in recursive step**, then it holds for these new elements.

Structural Induction, example

Rosen Sec 5.3

Define the subset S of **binary strings** $\{0,1\}^*$ by

Basis step: $\lambda \in S$ where λ is the empty string.

Recursive step: If $w \in S$, then each of

$$10w \in S, 1w0 \in S, w10 \in S, 01w \in S, 0w1 \in S, w01 \in S$$

How many elements of S have been specified after the basis step, plus at most one application of the recursive step?

A. 1
B. 7
C. 6
D. 3
E. None of the above

Structural Induction, example *Rosen Sec 5.3*

Define the subset S of **binary strings** $\{0,1\}^*$ by

Basis step: $\lambda \in S$ where λ is the empty string.

Recursive step: If $w \in S$, then each of

$$10w \in S, 1w0 \in S, w10 \in S, 01w \in S, 0w1 \in S, w01 \in S$$

Claim: Every element in S has an equal number of 0s and 1s.

Structural Induction, example *Rosen Sec 5.3*

Define the subset S of **binary strings** $\{0,1\}^*$ by

Basis step: $\lambda \in S$ where λ is the empty string.

Recursive step: If $w \in S$, then each of

$$10w \in S, 1w0 \in S, w10 \in S, 01w \in S, 0w1 \in S, w01 \in S$$

Claim: Every element in S has an equal number of 0s and 1s.

Proof: *Basis step – WTS that empty string has equal # of 0s and 1s*

Recursive step – Let w be an arbitrary element of S.

*Assume, as the **IH** that w has equal # of 0s and 1s.*

WTS that 10w, 1w0, w10, 01w, 0w1, w01 each have equal # of 0s, 1s.

Recursive definitions, part 2

Rosen Sec 5.3

For functions f: **N** → X

- define by a (closed-form) formula or …

Sequences are functions too!

Recursive definitions, part 2

Rosen Sec 5.3

For functions f: **N** → X

- define by a (closed-form) formula or …

Sequences are functions too!

Basis step: *Specify the value of the function at 0*

Recursive step: *Give a rule for finding its value at an integer from its values at smaller integers*

Which of the following functions / sequences have recursive definitions?

A. n!

B. 2^n

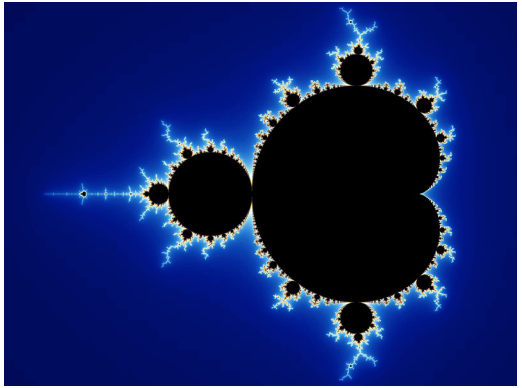
C. 2, -8, 32, -128, 512, …

D. $\sum_{i=1}^{n}(i^2+i)$

E. All of the above.

Recursive definitions, part 2

Rosen Sec 5.3



n! $0! = 1$, $n! = n\,(n-1)!$ for $n > 0$.

2^n $2^0 = 1$, $2^n = 2\,(2^{n-1})$ for $n > 0$.

2, -8, 32, -128, 512, …

$a_0 = 2$, $a_n = -4\,a_{n-1}$ for $n > 0$.

$$\sum_{i=1}^{n} g(n)$$

$$\sum_{i=1}^{0} g(i) = 0 \quad , \quad \sum_{i=1}^{1} g(i) = g(1) \quad \text{, and} \quad \sum_{i=1}^{n} g(i) = g(n) + \sum_{i=1}^{n-1} g(i) \quad \text{for n>1.}$$

An identity

Theorem: The sum of the first n positive integers is

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

An identity

Theorem: The sum of the first n positive integers is

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Proof: by Mathematical Induction.

1. Basis step WTS $\sum_{i=1}^{1} i = \frac{1(1+1)}{2}$
2. Inductive hypothesis Let k be a positive integer. Assume
$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$$
3. Induction step WTS $\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$

Nim

Two players take turns removing **any positive # of jellybeans** at a time from one of two piles in front of them. **The piles start out with equal #s.** The player who removes the **last jellybean wins** the game.

A. The first player has a strategy to always win.
B. The second player has a strategy to always win.
C. One of the players has a strategy to always win, but which player depends on how many jellybeans there are.
D. Who wins is random.
E. None of the above.

Nim

Two players take turns removing **any positive # of jellybeans**
at a time from one of two piles in front of them. **The piles start out with equal #s.**
The player who removes the **last jellybean wins** the game.

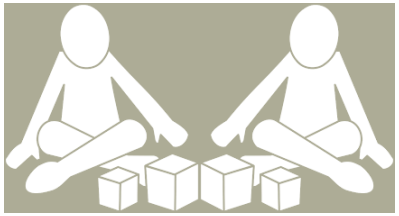
n=1 *Who wins?*

Nim

Two players take turns removing **any positive # of jellybeans**
at a time from one of two piles in front of them. **The piles start out with equal #s.**
The player who removes the **last jellybean wins** the game.

n=2 *Who wins?*

Nim



Two players take turns removing
any positive # of jellybeans
at a time from one of two piles in front of
them. **The piles start out with equal #s.**
The player who removes the **last jellybean wins** the game.

A. The first player has a strategy to always win.
B. The second player has a strategy to always win.
C. One of the players has a strategy to always win, but which player depends on how many jellybeans there are.
D. Who wins is random.
E. None of the above.

Nim

Two players take turns removing **any positive # of jellybeans**
at a time from one of two piles in front of them. **The piles start out with equal #s.**
The player who removes the **last jellybean wins** the game.

Idea: 2nd player takes the same amount 1st player took but from opposite pile.

*…Game reduces to **same setup** but with fewer jellybeans.*

Strong induction

Rosen p. 334

To show that some statement P(n) is true about all positive integers n,

1. Verify that P(1) is true.
2. Let k be an arbitrary positive integer. Show that

$$[P(1) \wedge P(2) \wedge \cdots \wedge P(k)] \rightarrow P(k+1)$$

is true.

Strong induction hypothesis

Nim

Two players take turns removing **any positive # of jellybeans** at a time from one of two piles in front of them. **The piles start out with equal #s.** The player who removes the **last jellybean wins** the game.

Theorem: the second player can always guarantee a wing.

Proof: By Strong Mathematical Induction, on # jellybeans in each pile.

1. Basis step WTS if piles each have 1, then 2nd player can win.
2. Strong Induction hypothesis Let k be a positive integer. Assume that 2nd player can win whenever there are j jellybeans in each pile, for each j between 1 and k (inclusive).
3. Induction step WTS 2nd player has winning strategy when start with k+1 jellybeans in each pile.

Fibonacci numbers

Rosen p. 158, 347

$$f_0 = 1,\ f_1 = 1,\ f_n = f_{n-1} + f_{n-2}$$

Theorem: For each integer n>=2, $f_n >= 2^{n-2}$.

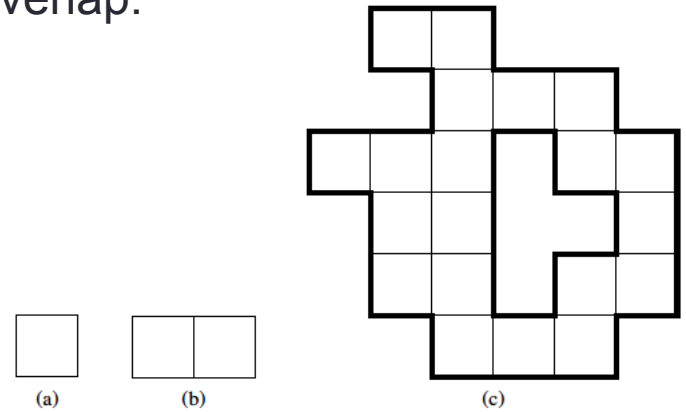
Proof: By Strong Mathematical Induction, on n>=2.

1. Basis step WTS $f_2 >= 2^0$.
2. Strong Induction hypothesis Let k be a positive integer. Assume inequality is true for each **earlier** Fibonacci term.
3. Induction step WTS statement is true about f_k.

Shapes

Place squares so that each new square shares an edge with at least one previously-placed square and no squares overlap.

Each side of each square has length 1.



Use mathematical induction on the number of squares to prove that the perimeter of the shape (length of the boundary) is always even, no matter how many squares are placed or how they are arranged.