

# Formalization of DFA using lattices

---

# Recall worklist algorithm

---

```
let m: map from edge to computed value at edge
let worklist: work list of nodes

for each edge e in CFG do
    m(e) :=  $\emptyset$ 

for each node n do
    worklist.add(n)

while (worklist.empty.not) do
    let n := worklist.remove_any;
    let info_in := m(n.incoming_edges);
    let info_out := F(n, info_in);
    for i := 0 .. info_out.length do
        let new_info := m(n.outgoing_edges[i]) U
            info_out[i];
        if (m(n.outgoing_edges[i])  $\neq$  new_info)
            m(n.outgoing_edges[i]) := new_info;
        worklist.add(n.outgoing_edges[i].dst);
```

# Using lattices

---

- We formalize our domain with a powerset lattice
- What should be top and what should be bottom?

# Using lattices

---

- We formalize our domain with a powerset lattice
- What should be top and what should be bottom?
- Does it matter?
  - It matters because, as we've seen, there is a notion of approximation, and this notion shows up in the lattice

# Using lattices

---

- Unfortunately:
  - dataflow analysis community has picked one direction
  - abstract interpretation community has picked the other
- We will work with the abstract interpretation direction
- Bottom is the most precise (optimistic) answer, Top the most imprecise (conservative)

# Direction of lattice

---

- Always safe to go up in the lattice
- Can always set the result to  $\top$
- Hard to go down in the lattice
- So ... Bottom will be the empty set in reaching defs

# Worklist algorithm using lattices

---

```
let m: map from edge to computed value at edge
let worklist: work list of nodes

for each edge e in CFG do
    m(e) :=  $\perp$ 

for each node n do
    worklist.add(n)

while (worklist.empty.not) do
    let n := worklist.remove_any;
    let info_in := m(n.incoming_edges);
    let info_out := F(n, info_in);
    for i := 0 .. info_out.length do
        let new_info := m(n.outgoing_edges[i])  $\sqcup$ 
            info_out[i];
        if (m(n.outgoing_edges[i])  $\neq$  new_info)
            m(n.outgoing_edges[i]) := new_info;
        worklist.add(n.outgoing_edges[i].dst);
```

# Termination of this algorithm?

---

- For reaching definitions, it terminates...
- Why?
  - lattice is finite
- Can we loosen this requirement?
  - Yes, we only require the lattice to have a finite height
- Height of a lattice: length of the longest ascending or descending chain
- Height of lattice  $(2^S, \subseteq) = |S|$



# Termination of this algorithm?

---

- For reaching definitions, it terminates...
- Why?
  - lattice is finite
- Can we loosen this requirement?
  - Yes, we only require the lattice to have a finite height
- Height of a lattice: length of the longest ascending or descending chain
- Height of lattice  $(2^S, \subseteq) = |S|$

# Termination

---

- Still, it's annoying to have to perform a join in the worklist algorithm

```
while (worklist.empty.not) do
  let n := worklist.remove_any;
  let info_in := m(n.incoming_edges);
  let info_out := F(n, info_in);
  for i := 0 .. info_out.length do
    let new_info := m(n.outgoing_edges[i])  $\sqcup$ 
      info_out[i];
    if (m(n.outgoing_edges[i])  $\neq$  new_info)
      m(n.outgoing_edges[i]) := new_info;
      worklist.add(n.outgoing_edges[i].dst);
```

- It would be nice to get rid of it, if there is a property of the flow functions that would allow us to do so

# Even more formal

---

- To reason more formally about termination and precision, we re-express our worklist algorithm mathematically
- We will use fixed points to formalize our algorithm

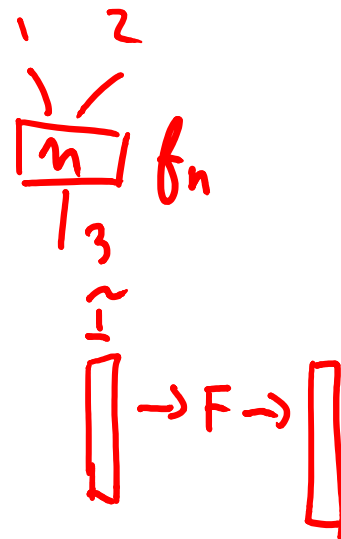
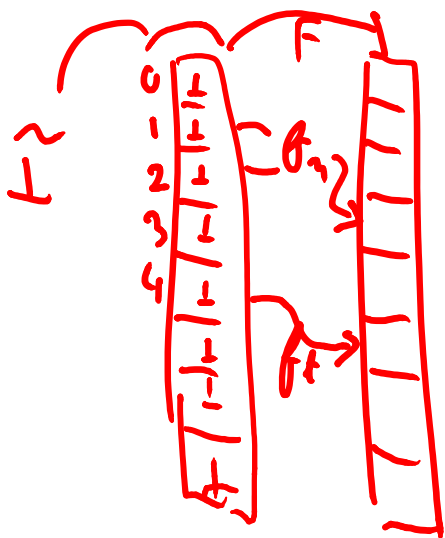
# Fixed points

$$m = F(m)$$

- Recall, we are computing  $m$ , a map from edges to dataflow information



- Define a global flow function  $F$  as follows:  $F$  takes a map  $m$  as a parameter and returns a new map  $m'$ , in which individual local flow functions have been applied



# Fixed points

---

- We want to find a fixed point of  $F$ , that is to say a map  $m$  such that  $m = F(m)$
- Approach to doing this?
- Define  $\tilde{\perp}$ , which is  $\perp$  lifted to be a map:  
$$\tilde{\perp} = \lambda e. \perp$$
- Compute  $F(\tilde{\perp})$ , then  $F(F(\tilde{\perp}))$ , then  $F(F(F(\tilde{\perp})))$ , ... until the result doesn't change anymore

# Fixed points

- Formally:

$$\text{Soln} = \bigsqcup_{i=0}^{\infty} F^i(\hat{\perp}) \quad \sqcup \left( F(\perp) \quad F^2(\perp) \quad F^3(\perp) \dots \right)$$

- We would like the sequence  $F^i(\hat{\perp})$  for  $i = 0, 1, 2 \dots$  to be increasing, so we can get rid of the outer join

- Require that  $F$  be monotonic:

$$- \forall a, b. a \sqsubseteq b \Rightarrow F(a) \sqsubseteq F(b)$$

$$f_{x := *p}(i_{m_1}) = \dots$$

$$i_{m_1} \sqsubseteq i_{m_2}$$

$$\Rightarrow f_{x := *p}(i_{m_1}) \sqsubseteq f_{x := *p}(i_{m_2})$$

# Fixed points

---

$$a \in b \\ \Rightarrow F(a) \subseteq F(b)$$

$$\begin{array}{l} \tilde{I} \subseteq F(\tilde{I}) \\ \quad \swarrow \\ F(\tilde{I}) \subseteq FF(\tilde{I}) \\ \quad \swarrow \\ F^2(\tilde{I}) \subseteq F^3(\tilde{I}) \end{array}$$

# Fixed points

---

$$\begin{aligned} \tilde{I} &\subseteq F(\tilde{I}) \\ F(\tilde{I}) &\subseteq F(F(\tilde{I})) \\ &\vdots \\ F^k(\tilde{I}) &\subseteq F^{k+1}(\tilde{I}) \\ F^{k+1}(\tilde{I}) &\subseteq F^{k+2}(\tilde{I}) \end{aligned}$$



# Back to termination

---

- So if  $F$  is monotonic, we have what we want: finite height  $\Rightarrow$  termination, without the outer join
- Also, if the local flow functions are monotonic, then global flow function  $F$  is monotonic

# Another benefit of monotonicity

---

- Suppose Marsians came to earth, and miraculously give you a fixed point of  $F$ , call it  $fp$ .
- Then:

$$\tilde{I} \sqsubseteq fp$$

$$F(\tilde{I}) \sqsubseteq fp$$

$$FF(\tilde{I}) \sqsubseteq fp$$

$$ofp \sqsubseteq fp$$

# Another benefit of monotonicity

---

- Suppose Marsians came to earth, and miraculously give you a fixed point of  $F$ , call it  $fp$ .
- Then:

$$\begin{aligned} \tilde{I} &\sqsubseteq fp \\ F(\tilde{I}) &\sqsubseteq F(fp) \\ F(\tilde{I}) &\sqsubseteq fp \\ F^2(\tilde{I}) &\sqsubseteq fp \\ &\vdots \\ 0fp &\sqsubseteq fp \end{aligned}$$

# Another benefit of monotonicity

---

- We are computing the least fixed point...

# Recap

---

- Let's do a recap of what we've seen so far
- Started with worklist algorithm for reaching definitions

# Worklist algorithm for reaching defns

---

```
let m: map from edge to computed value at edge
let worklist: work list of nodes

for each edge e in CFG do
    m(e) :=  $\emptyset$ 

for each node n do
    worklist.add(n)

while (worklist.empty.not) do
    let n := worklist.remove_any;
    let info_in := m(n.incoming_edges);
    let info_out := F(n, info_in);
    for i := 0 .. info_out.length do
        let new_info := m(n.outgoing_edges[i]) U
            info_out[i];
        if (m(n.outgoing_edges[i])  $\neq$  new_info)
            m(n.outgoing_edges[i]) := new_info;
        worklist.add(n.outgoing_edges[i].dst);
```

# Generalized algorithm using lattices

---

```
let m: map from edge to computed value at edge
let worklist: work list of nodes

for each edge e in CFG do
    m(e) :=  $\perp$ 

for each node n do
    worklist.add(n)

while (worklist.empty.not) do
    let n := worklist.remove_any;
    let info_in := m(n.incoming_edges);
    let info_out := F(n, info_in);
    for i := 0 .. info_out.length do
        let new_info := m(n.outgoing_edges[i])  $\sqcup$ 
            info_out[i];
        if (m(n.outgoing_edges[i])  $\neq$  new_info)
            m(n.outgoing_edges[i]) := new_info;
        worklist.add(n.outgoing_edges[i].dst);
```

# Next step: removed outer join

---

- Wanted to remove the outer join, while still providing termination guarantee
- To do this, we re-expressed our algorithm more formally
- We first defined a “global” flow function  $F$ , and then expressed our algorithm as a fixed point computation



# Guarantees

---

- If  $F$  is monotonic, don't need outer join
- If  $F$  is monotonic and height of lattice is finite: iterative algorithm terminates
- If  $F$  is monotonic, the fixed point we find is the least fixed point.

# What about if we start at top?

---

- What if we start with  $\tilde{T}: F(\tilde{T}), F(F(\tilde{T})), F(F(F(\tilde{T})))$

$\tilde{T}$

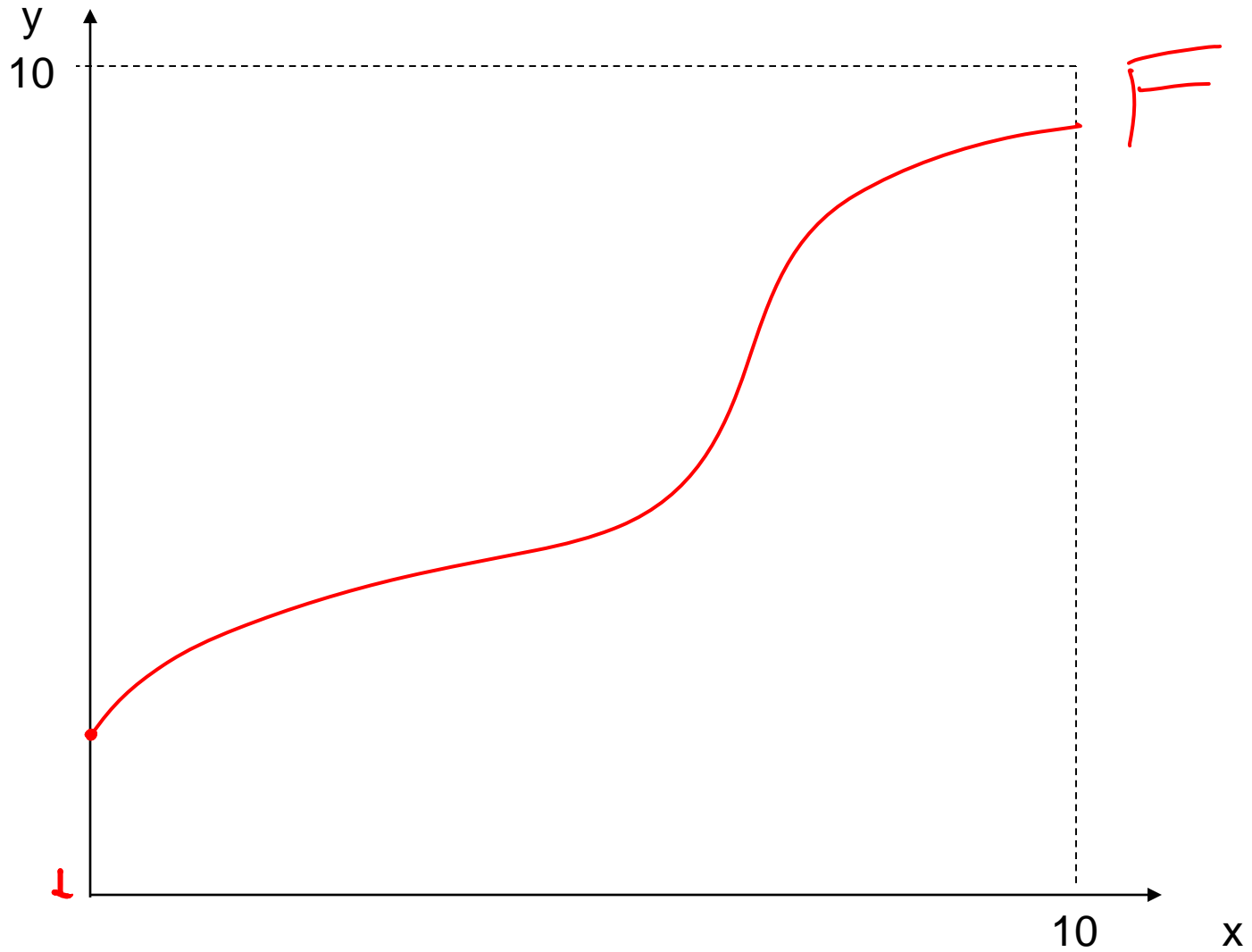
$\tilde{T}$

# What about if we start at top?

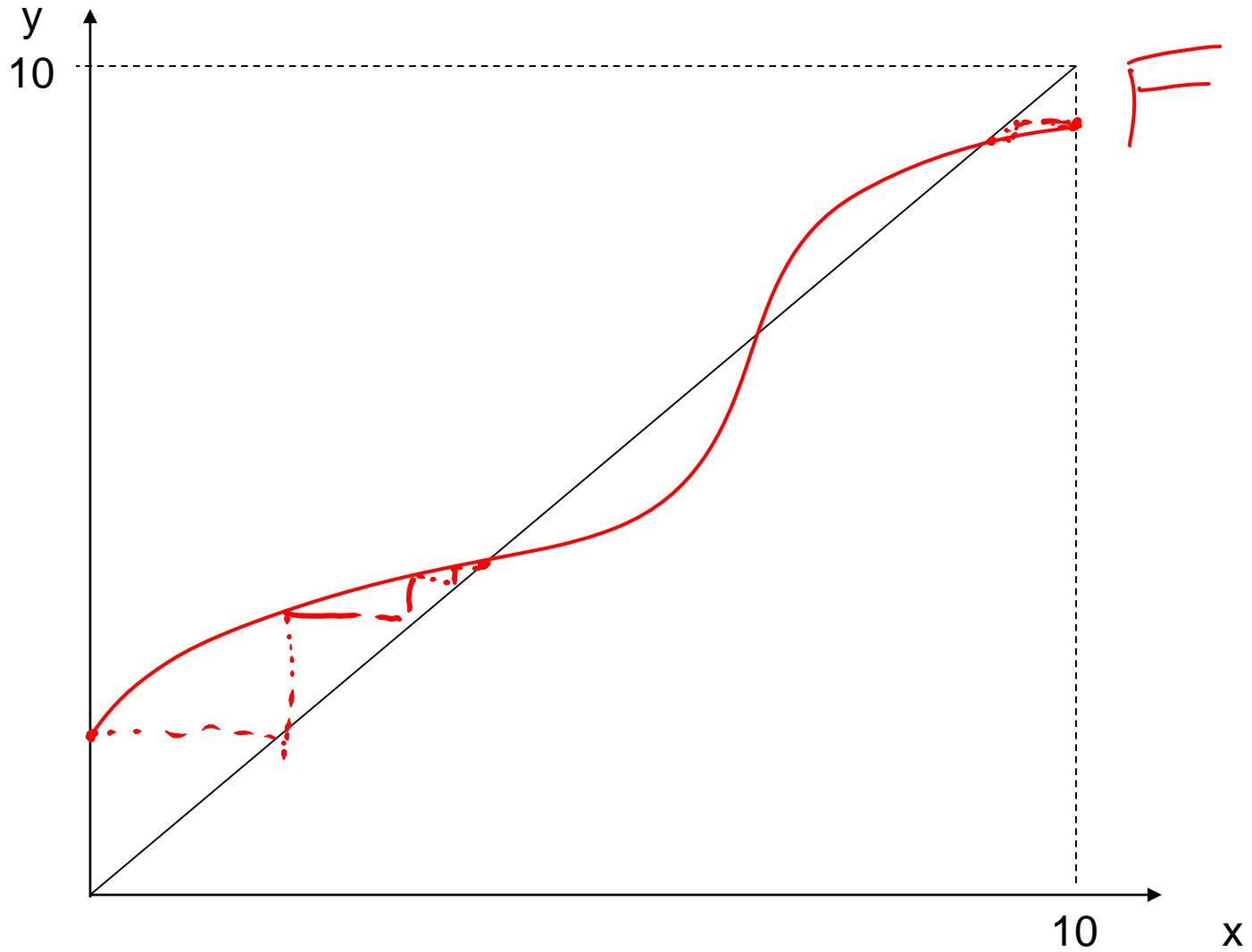
---

- What if we start with  $\tilde{T}: F(\tilde{T}), F(F(\tilde{T})), F(F(F(\tilde{T})))$
- We get the greatest fixed point
- Why do we prefer the least fixed point?
  - More precise

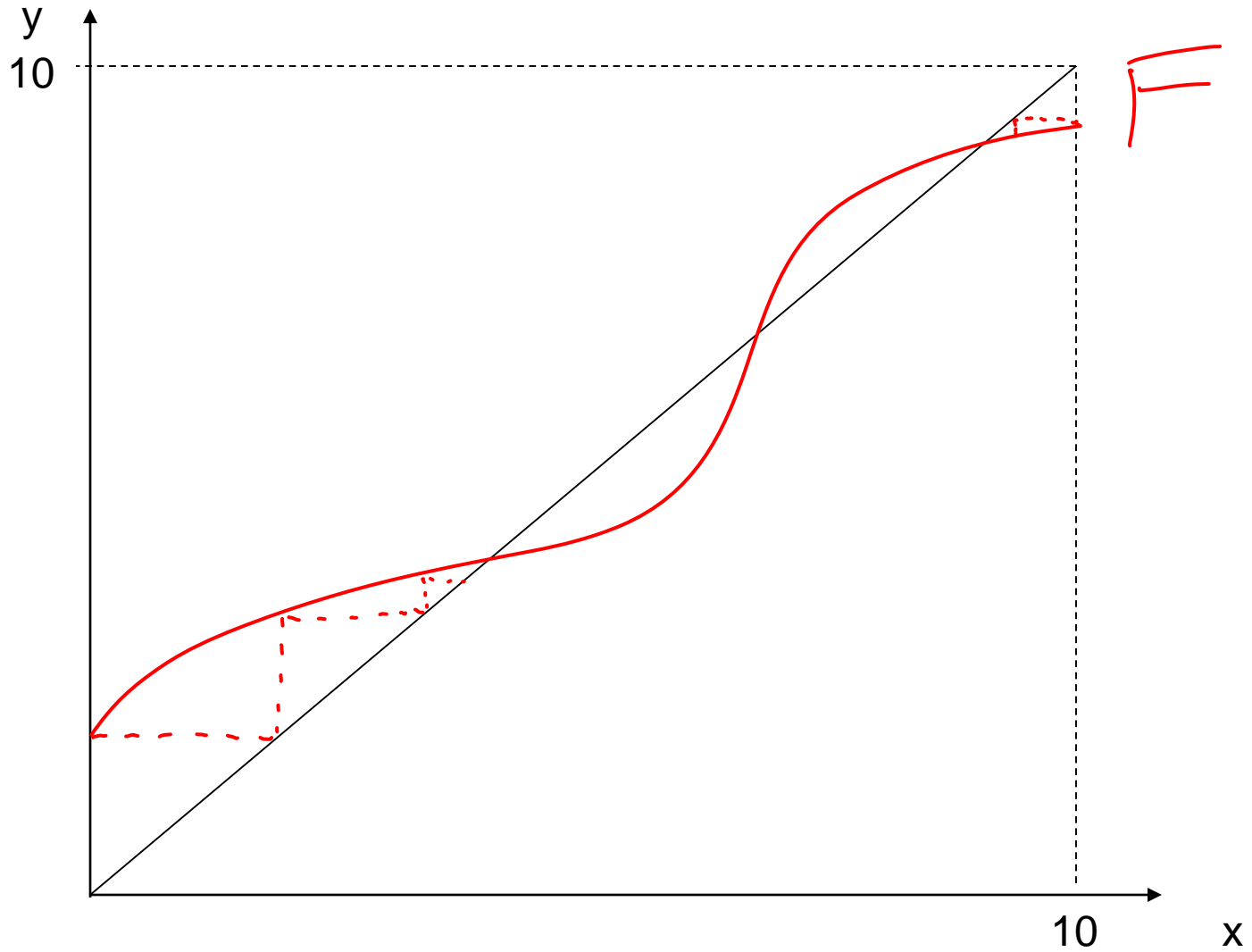
# Graphically



# Graphically



# Graphically



# Graphically, another way

---

