

Background material

Relations

- A relation over a set S is a set $R \subseteq S \times S$
 - We write $a R b$ for $(a,b) \in R$
- A relation R is:
 - reflexive iff
$$\forall a \in S . a R a$$
 - transitive iff
$$\forall a \in S, b \in S, c \in S . a R b \wedge b R c \Rightarrow a R c$$
 - symmetric iff
$$\forall a, b \in S . a R b \Rightarrow b R a$$
 - anti-symmetric iff
$$\forall a, b, \in S . a R b \Rightarrow \neg(b R a)$$

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$$\forall a, b, \in S . a R b \wedge b R a \Rightarrow a = b$$

Partial orders

- An equivalence class is a relation that is:
- A partial order is a relation that is:

Partial orders

- An equivalence class is a relation that is:
 - reflexive, transitive, symmetric
- A partial order is a relation that is:
 - reflexive, transitive, anti-symmetric
- A partially ordered set (a poset) is a pair (S, \leq) of a set S and a partial order \leq over the set
- Examples of posets: $(2^S, \subseteq)$, (\mathbb{Z}, \leq) , $(\mathbb{Z}, \text{divides})$

Lub and glb

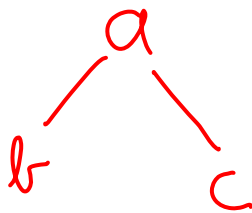
- Given a poset (S, \leq) , and two elements $a \in S$ and $b \in S$, then the:
 - least upper bound (lub) is an element c such that $a \leq c$, $b \leq c$, and $\forall d \in S . (a \leq d \wedge b \leq d) \Rightarrow c \leq d$
 - greatest lower bound (glb) is an element c such that $c \leq a$, $c \leq b$, and $\forall d \in S . (d \leq a \wedge d \leq b) \Rightarrow d \leq c$

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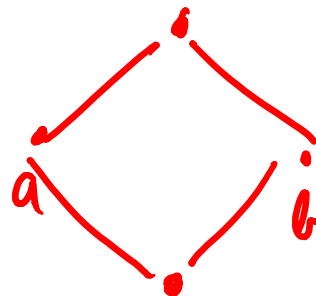
glb of b & c ?

Lattices

- A lattice is a tuple $(S, \sqsubseteq, \perp, \top, \sqcup, \sqcap)$ such that:
 - (S, \sqsubseteq) is a poset
 - $\forall a \in S . \perp \sqsubseteq a$
 - $\forall a \in S . a \sqsubseteq \top$
 - Every two elements from S have a lub and a glb
 - \sqcup is the least upper bound operator, called a join
 - \sqcap is the greatest lower bound operator, called a meet

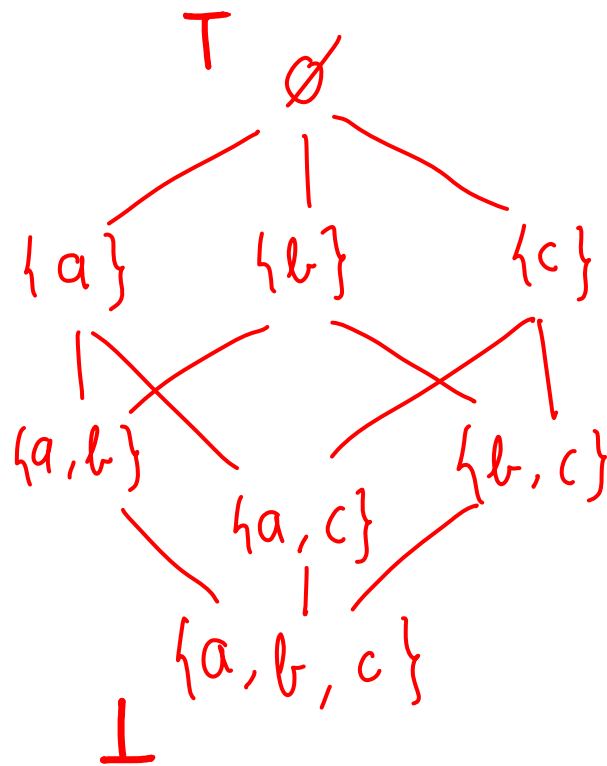
$$a \sqcup b$$

$$a \sqcap b$$



Examples of lattices

- Powerset lattice



$$S \triangleq \mathcal{P}(\{a, b, c\})$$

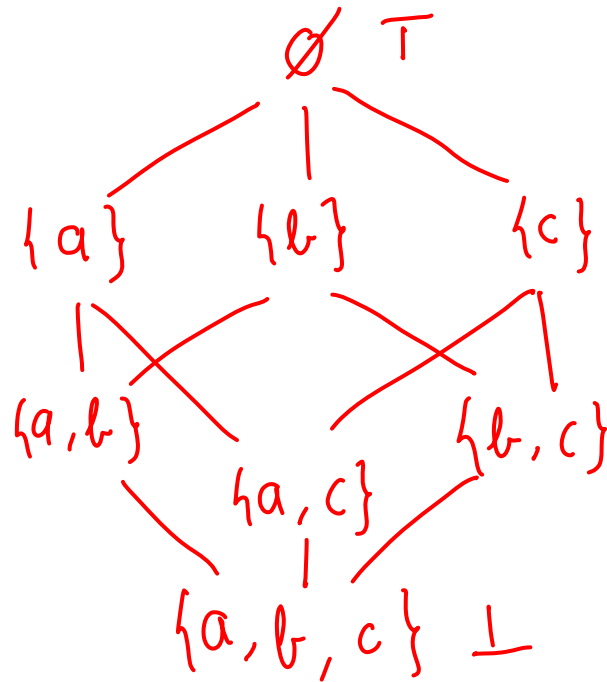
$$\sqsubseteq \triangleq \supseteq$$

$$\sqcup \triangleq \cap$$

$$\sqcap \triangleq \cup$$

Examples of lattices

- Powerset lattice



$$S \triangleq 2^{\{a, b, c\}}$$

$$\sqsubseteq \triangle \supseteq$$

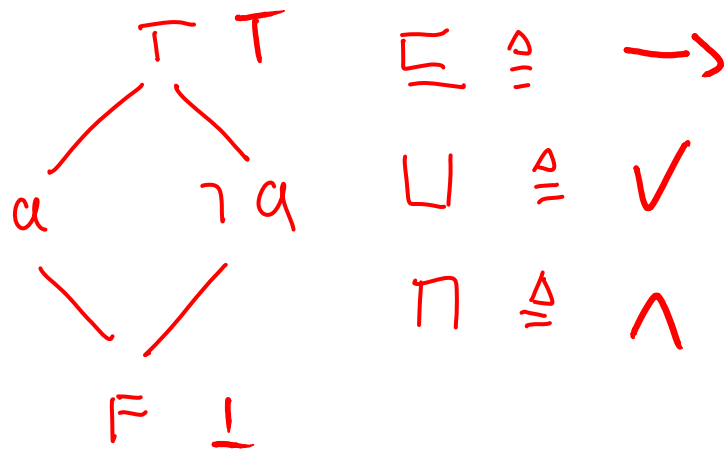
$$\sqcup \triangle \sqcap$$

$$\sqcap \triangle \sqcup$$

Examples of lattices

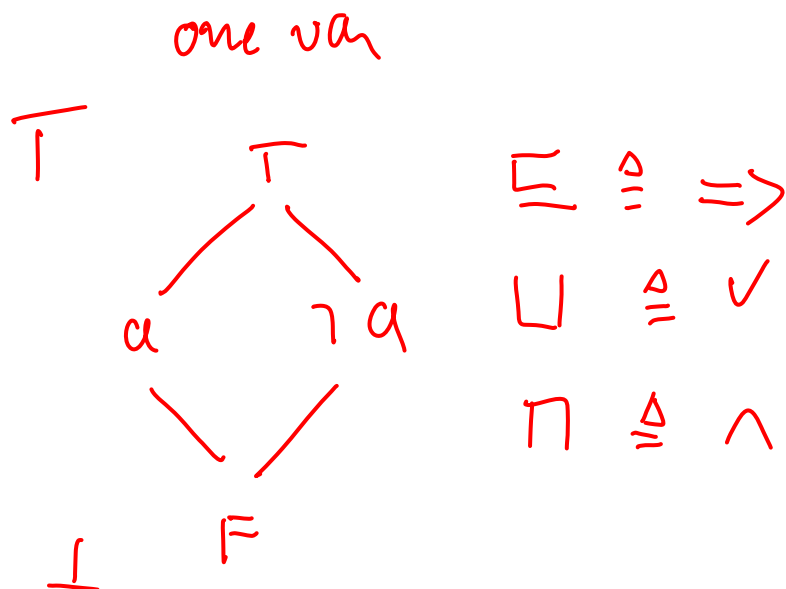
- Booleans expressions

one var



Examples of lattices

- Booleans expressions



Examples of lattices

- Booleans expressions

two way

T

⋮

F

Examples of lattices

- Booleans expressions

two vars

T

∴ 16 expressions ...

F

$2^{(2^2)}$

a	b	
0	0	□
0	1	□
1	0	□
1	1	□