Lecture 12-13 Hilbert-Huang Transform

Background:

- An examination of Fourier Analysis
- Existing non-stationary data handling method
- Instantaneous frequency
- Intrinsic mode functions(IMF)
- Empirical mode decomposition(EMD)
- Mathematical considerations

HHT Transform Sources:

Original Paper: Huang, et al. "The empirical mode decomposition and the Hilbert spectrum for nonlinear and non-stationary time series analysis." Proc. R. Soc. Lond. A (1998) 454, 903–995

NASA: http://techtransfer.gsfc.nasa.gov/HHT/

Book: HILBERT-HUANG TRANSFORM AND ITS APPLICATIONS
Ed. by Norden E Huang and Samuel S P Shen

EMD code: http://perso.ens-lyon.fr/patrick.flandrin/emd.html

Other goodies:
http://www.mathworks.com/matlabcentral/fileexchange/21409

HHT-based Identification codes:
http://hitech.technion.ac.il/feldman/

Why not the Fourier analysis (FA)?

- **The FA performs well when the system is linear;**

  Measure at least two output: $y_1(t)$ and $y_2(t)$ corresponding to input $x_1(t)$ and $x_2(t)$.

  Now apply input: $x(t) = a\, x_1(t) + b\, x_2(t) + c\, x_3(t) + \ldots$
  if the output is given by $y(t) = a\, y_1(t) + b\, y_2(t) + c\, y_3(t) + \ldots$
  then the system is deemed to be linear,

- **And when data are periodic or stationary;**

  $X(t)$, is stationary in the wide sense, if, for all t,

$$E(|X(t)^2|) < \infty,$$

$$E(X(t)) = m,$$

$$C(X(t_1), X(t_2)) = C(X(t_1 + \tau), X(t_2 + \tau)) = C(t_1 - t_2)$$

And when is the FA not so best?

- when data are nonstationary;
- the FA basis functions are global, hence they cannot treat local nonlinearity without significant dispersions (spreading);
- The above is especially true when the wave forms deviate significantly from sinusoidal form;
- For delta function-like waves, an excessive number of harmonic terms are required, let alone the Gibbs phenomena.

Nonstationary data processing methods

- **Spectrogram**
- **Wavelets analysis**
- **Wigner-Ville distribution**
- **Evolutionary spectrum**
- **Empirical orthogonal function expansion (EOF)**
- **Smoothed moving average**
- **Trend least-squares estimation**

Instantaneous Frequency

Definition of Hilbert Transform:

$$Y(t) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{X(t')}{t - t'} \, \mathrm{d}t'$$

Complexification:

$$Z(t) = X(t) + \mathrm{i}Y(t) = a(t)\mathrm{e}^{\mathrm{i}\theta(t)}$$

$$a(t) = [X^2(t) + Y^2(t)]^{1/2}, \quad \theta(t) = \arctan\left(\frac{Y(t)}{X(t)}\right)$$

Frequency: $$\omega = \frac{\mathrm{d}\theta(t)}{\mathrm{d}t}$$

Instantaneous frequency - cont'd

The instantaneous frequency defined is a scalar; which means that ω is a monocomponent. In reality, the signal may not represent a monocomponent. Therefore, one should interpret it as a localized frequency within a narrow band. As the concept of bandwidth plays a crucial role, we borrow its definition from the signal processing:

The number of zero crossing per unit time is given by

$$N_0 = \frac{1}{\pi}\left(\frac{m_2}{m_0}\right)^{1/2}$$

while the expected number of extrema per unit time is given by

$$N_1 = \frac{1}{\pi}\left(\frac{m_4}{m_2}\right)^{1/2}$$

where m_i is the i-th moment of the spectrum.

Instantaneous frequency - cont'd

Hence, a standard bandwidth measure can be given by

$$\nu^2 = \pi^2 (N^2_1 - N^2_0)$$

Note that if $\nu = 0$, the expected numbers of extrema and zero crossings are equal. It is this observation we will exploit in the empirical mode decomposition later on.

However, the instantaneous frequency defined previously still yields a global measure. Hence, when one decomposes the signal into multi-components, a key criterion is to ensure the associated frequency is locally valid. This is discussed in the next,

Intrinsic Mode Decomposition.

Intrinsic Mode Decomposition (IMF)
(implies oscillations embedded in the data)

Suppose a function is symmetric with respect to the local zero mean, and have the same numbers of extrema and zero crossing. Then a physically meaningful local instantaneous frequency can be discerned from the function.

Exploiting this concept, an intrinsic mode function satisfies the following two conditions:

1. **In the whole data set, the number of extrema and the number of zero crossings must either be equal or differ at most by one; (adaptation of narrow band concept)**
2. **At any point, the mean value of the envelope defined by the local maxima and the envelope defined by the local minima is zero (new - adoption of local properties).**

Intrinsic Mode Decomposition - Cont'd

Modification of local mean: the local mean of the two envelopes defined by the local maxima and local minima - this forces the local symmetry. However, it does engender an alias in the instantaneous frequency for nonlinearly deformed waves.

The IMF properties:

- **each IMF involves only one mode of oscillation;**
- **each IMF characterizes not only a narrow band but both amplitude and frequency modulations;**
- **an IMF can thus be nonstationary.**

Huang et al Statement on why IMF-based instantaneous frequency makes sense (Proc. R. Soc. Lond. A (1998), p.916):

Having defined IMF, we will show that the definition given in equation (3.4) gives the best instantaneous frequency. An IMF after the Hilbert transform can be expressed as in equation (3.2). If we perform a Fourier transform on $Z(t)$, we have

$$W(\omega) = \int_{-\infty}^{\infty} a(t)\mathrm{e}^{\mathrm{i}\theta(t)}\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t, = \int_{-\infty}^{\infty} a(t)\mathrm{e}^{\mathrm{i}(\theta(t)-\omega t)}\,\mathrm{d}t. \tag{4.1}$$

Then by the stationary phase method (see, for example, Copson 1967), the maximum contribution to $W(\omega)$ is given by the frequency satisfying the condition

$$\frac{\mathrm{d}}{\mathrm{d}t}(\theta(t) - \omega t) = 0; \tag{4.2}$$

therefore, equation (3.4) follows. Although mathematically, the application of the stationary phase method requires a large parameter for the exponential function, the adoption here can be justified if the frequency, ω, is high compared with the inversed local time scale of the amplitude variation. Therefore, this definition fits the best for gradually changing amplitude. Even with this condition, this is still a much better definition for instantaneous frequency than the zero-crossing frequency; it is also better than the integral definition suggested by Cohen (1995) as given in equation (3.10). Furthermore, it agrees with the definition of frequency for the classic wave theory (see, for example, Whitham 1975).

Empirical mode decomposition method (EMD) in a nutshell

EMD identifies the intrinsic oscillatory modes by their characteristic time scales in the data empirically, then decomposes the data into the corresponding IMFs via the sifting process.

Thus, it is an algorithm to assign an instantaneous frequency to each IMF in order to decompose an arbitrary set of data; this means, for complex data, we can allow more than one instantaneous frequency at a time locally. In doing so we obtain IMFs as most data do not consist of IMFs.

In other words, EMD decomposes an arbitrary data set, whether they are linear, nonlinear or nonstationary, into a set of IMFs.

Assumptions introduced in EMD

1. The signal has at least two extrema - one maxima and one minima;

2. The characteristic time scale is defined by the time lapse between the extrema;

3. If the data is totally devoid of extrema but contained only inflection points, then it can be differentiated once or more times to reveal the extrema. Final results then can be obtained by integration(s) of the components.

The sifting process

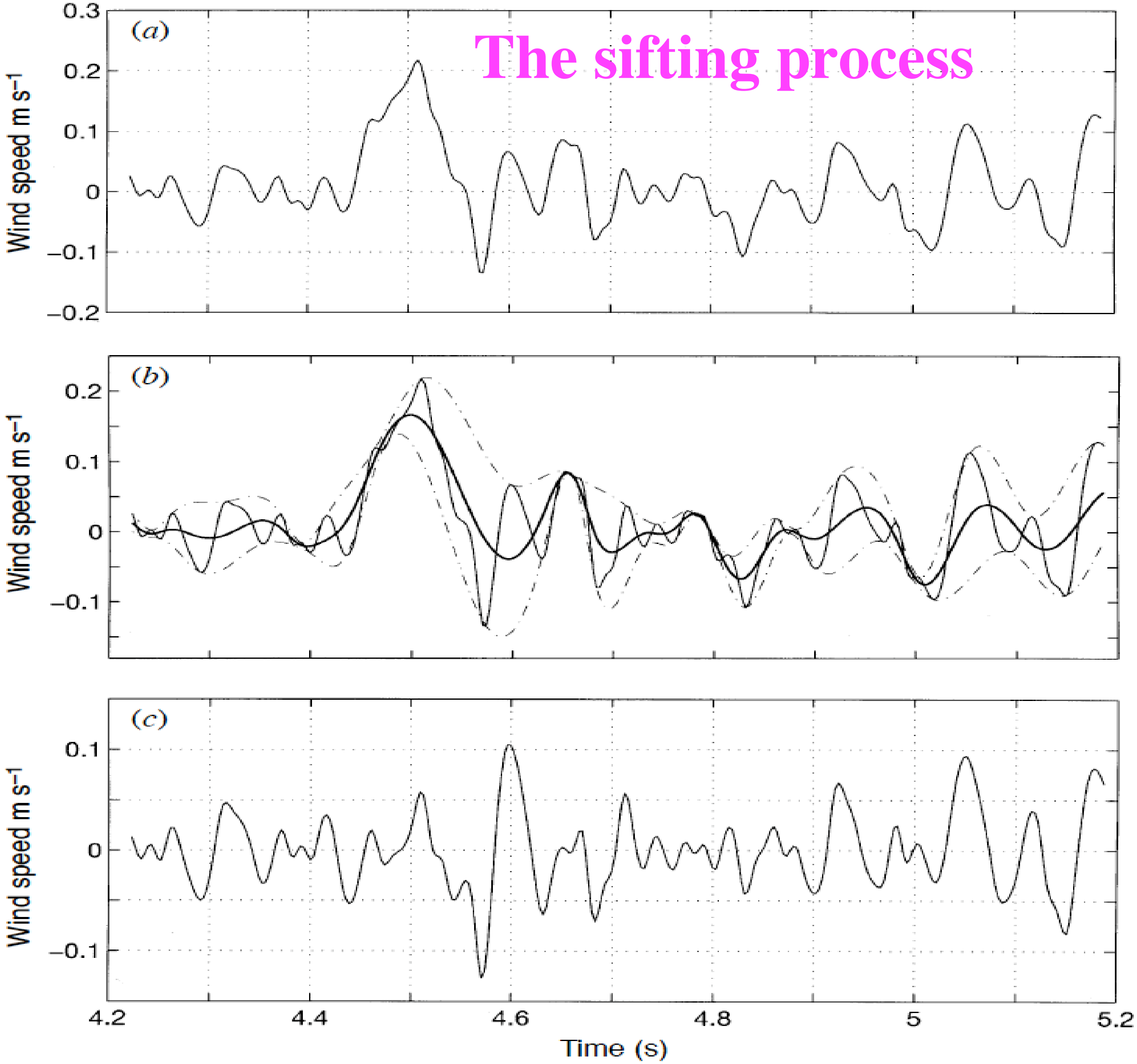


Figure 3. Illustration of the sifting processes: (*a*) the original data; (*b*) the data in thin solid line, with the upper and lower envelopes in dot-dashed lines and the mean in thick solid line; (*c*) the difference between the data and m_1. This is still not an IMF, for there are negative local maxima and positive minima suggesting riding waves.

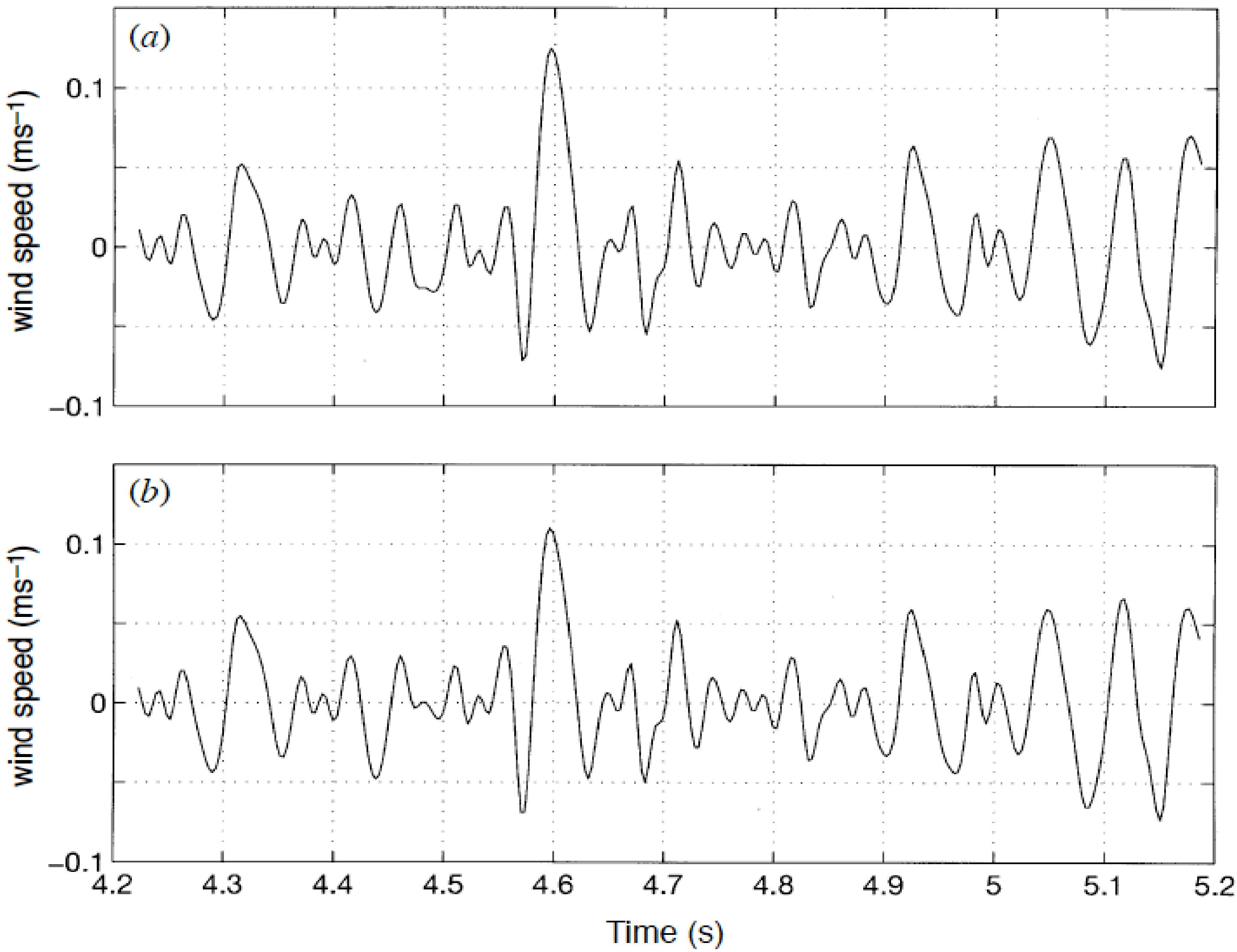


Figure 4. Illustration of the effects of repeated sifting process: (*a*) after one more sifting of the result in figure 3*c*, the result is still asymmetric and still not a IMF; (*b*) after three siftings, the result is much improved, but more sifting needed to eliminate the asymmetry. The final IMF is shown in figure 2 after nine siftings.

Signal from Final Shifting for c_1

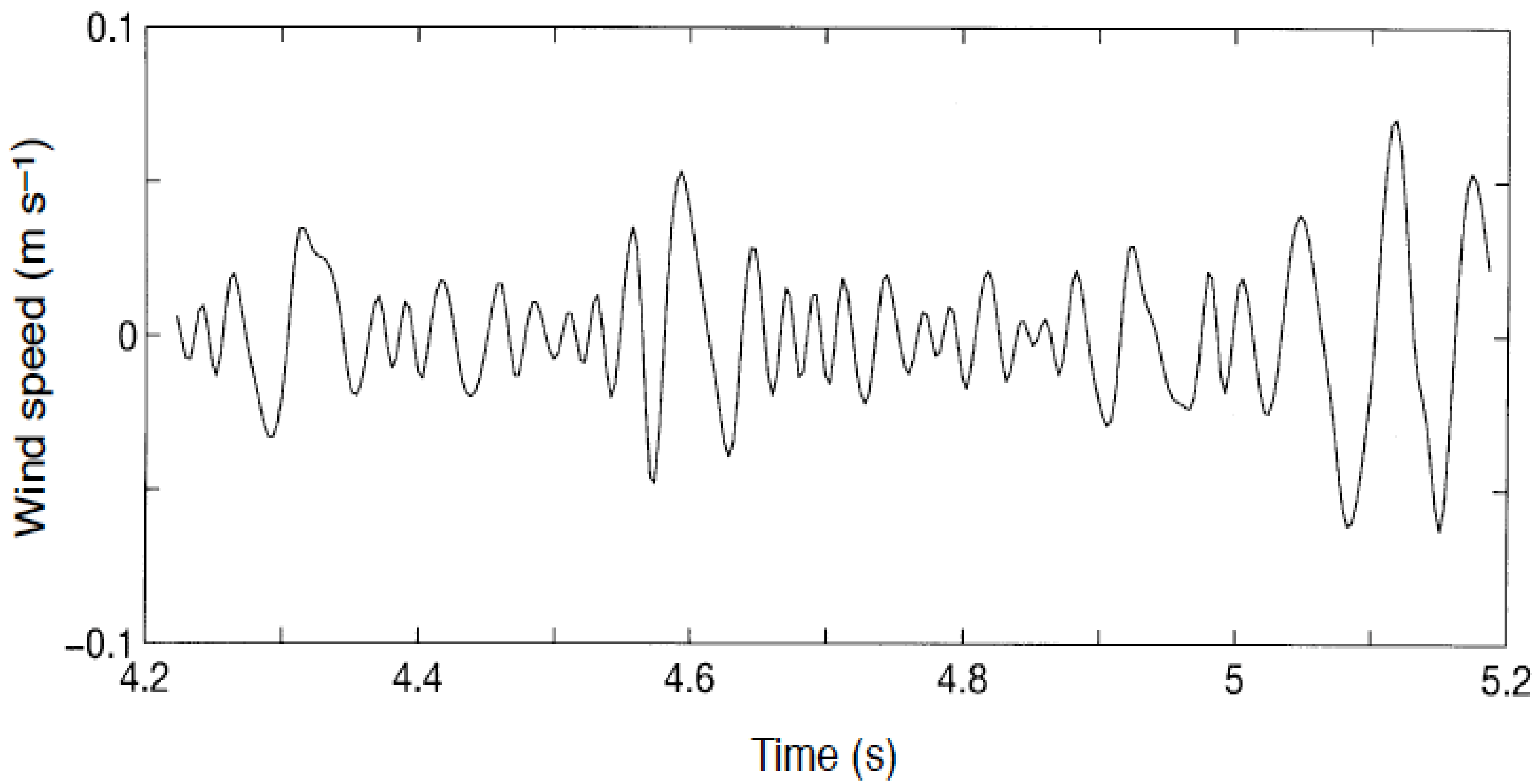


Figure 2. A typical intrinsic mode function with the same numbers of zero crossings and extrema, and symmetry of the upper and lower envelopes with respect to zero.

Algorithmic statement of the visual process demonstrated so far

1. For data X(t), we mark the local maxima and local minima, and interpolate the extrema points via, e.g., splines to obtain upper And lower envelopes.

2. Obtain the mean of the two envelopes, m_1 .

3. Obtain $h_1 = X(t) - m_1$ and inspect whether the the number of extrema and the number of zero crossings must either be equal or differ at most by one. Plus, inspect whether all the local maxima are positive and all the local minima are negative.

3. If not, repeat the sifting process and obtain $h_1 - m_{11} = h_{11}$. and repeat to obtain

$$h_{1(k-1)} - m_{1k} = h_{1k}$$

Algorithmic statement - cont'd

If h_{1k} constitutes an IMF, then designate it $c_1 = h_{1k}$.

Now we obtain the first residual r_1 via

$$r_1 = X(t) - c_1$$

Treat r_1 as a new data set, and perform the sifting process to obtain c_2.

Continuing the sifting process we obtain

$$r_2 = r_1 - c_2, \ \ldots, \ r_{n-1} - c_n = r_n.$$

Finally, the original signal is decomposed in terms of IMFs:

$$X(t) = \sum_{i=1}^{n} c_i + r_n$$

Hilbert-Huang Transform (HHT) for Nonlinearity Detection

The Hilbert-Hunag Transform(HHT) is perhaps the most notable development during the last decade. In a nutshell, the HHT consists of the following steps:

1. sifting, that is, empirical adoption of the Principal Component Analysis(PCA) for multi-components in the signal to rearrange the signal in terms of local bases that are nearly orthogonal each other;

2. physically based construction of instantaneous frequency whose concept is applicable to nonstationary and nonlinear signals;

3. complexification of the signal via the Hilbert Transform to characterize the signal in terms of the modulated amplitude and the associated instantaneous frequencies that appear to represent both interwaves and intrawaves;

4. reconstruction of the signal and the Hilbert spectrum (energy-frequency) and the multi-component frequency-time relations.

HHT - Cont'd

The nonlinear and non-stationary time series signal reconstructed via the HHT are:

complete by employing its stopping criterion;

nearly orthogonal;

local; and,

highly adaptive.

We label these properties into one phrase **COLA**.

We will now introduce the HHT via the example problem used by Huang and et al.

The Sifting Process

Assumptions introduced (Huang et al, 1998):

1. The signal has at least one maximum and one minimum;

2. The characteristic time scale is defined by the time lapse between the extrema;

3. If the data were totally devoid of extrema but contained only inflection points, then it can be differentiated once or more times to reveal extrema; and, final results can be obtained by integration(s) of the components.

The Sifting Process

Restriction imposed (Huang et al, 1998):

In addition to the above assumptions, they imposed a restriction that the the resulting intrinsic mode functions (**IMF**) be **symmetric locally with respect to the zero mean level.** This restriction implies the IMFs have the same numbers of zero-crossings and extrema. **This restriction then allows one to define the instantaneous frequency** for each of the decomposed IMFs. In other words, an **IMF** satisfies:

(1) in the whole data set, the number of extrema and the number of zero-crossings must be either the same or differ at most by one;

(2) at any point, the mean value of the envelope defined by the local maxima and the envelope defined by the local minima is zero.

The Sifting Process

Restriction imposed (Huang et al, 1998):

The first condition is similar to the narrow band requirement for a stationary Gaussian process. The second one, however, modifies the classical global zero-mean requirement to a local one. It is this very second property that goes with the concept of the **instantaneous frequency** that is valid for nonstationary process and nonlinear signals. From the context of signal processing, the second property allows us to avoid a local-averaging time scale altogether.

Invoking the above assumptions and restriction, Huang et al showed that their **empirical mode decomposition (EMD)** can identify the intrinsic oscillatory modes by their characteristic time scales in the data.

For a given data the maxima are marked with blue bullets and interpolated by the cubic spline function. The same is done for the minima with red bullets followed by cubic spline interpoaltion. The local mean $(m_1(t)))$ is then constructed between the blue and red figures and plotted with pink color. The first IMF $(h_1(t))$ is thus obtained by

$$h_1(t) = x(t) - m_1(1) \tag{12}$$

If $h_1(t)$ is not symmetric, then repeat the sifting process such that

$$\begin{aligned} &h_1 - m_{11} = h_{11} \\ &\quad .. \\ &h_{1(k-1)} = m_{1k} = h_{1k} \\ &\quad .. \\ &\text{Let} \;\; c_1 = h_{1k} \\ &r_1 = x(t) - c_1 \end{aligned} \tag{13}$$

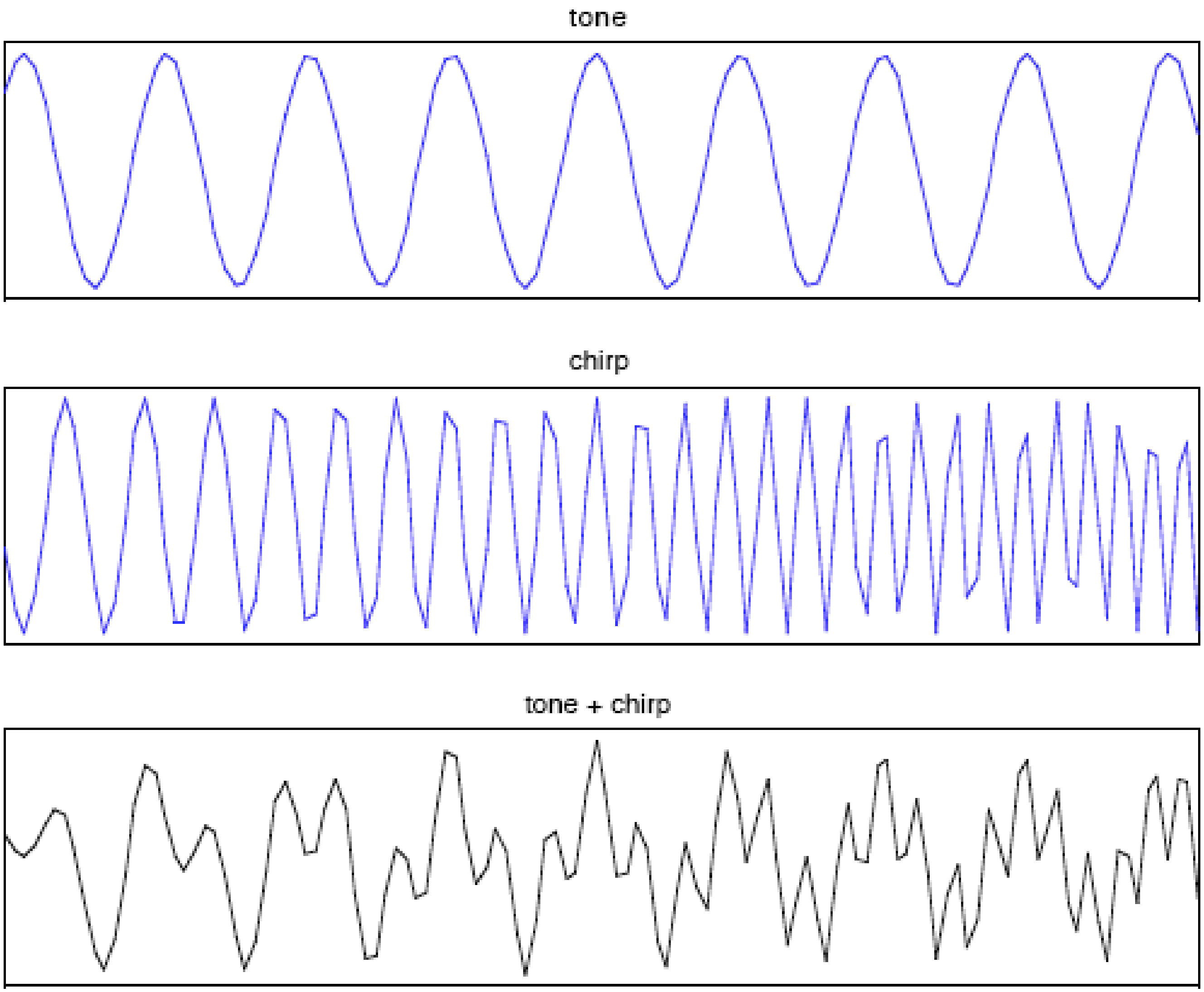
and the next sifting process starts with r_1. The sifting process is completed "when the residual r_n is so small that it is less than the predetermined value of substantial consequence, or when the residue, r_n, becomes a monotonic function from which no IMF can be extracted. Now the original signal, $x(t)$, is rearranged in terms of the IMFs:

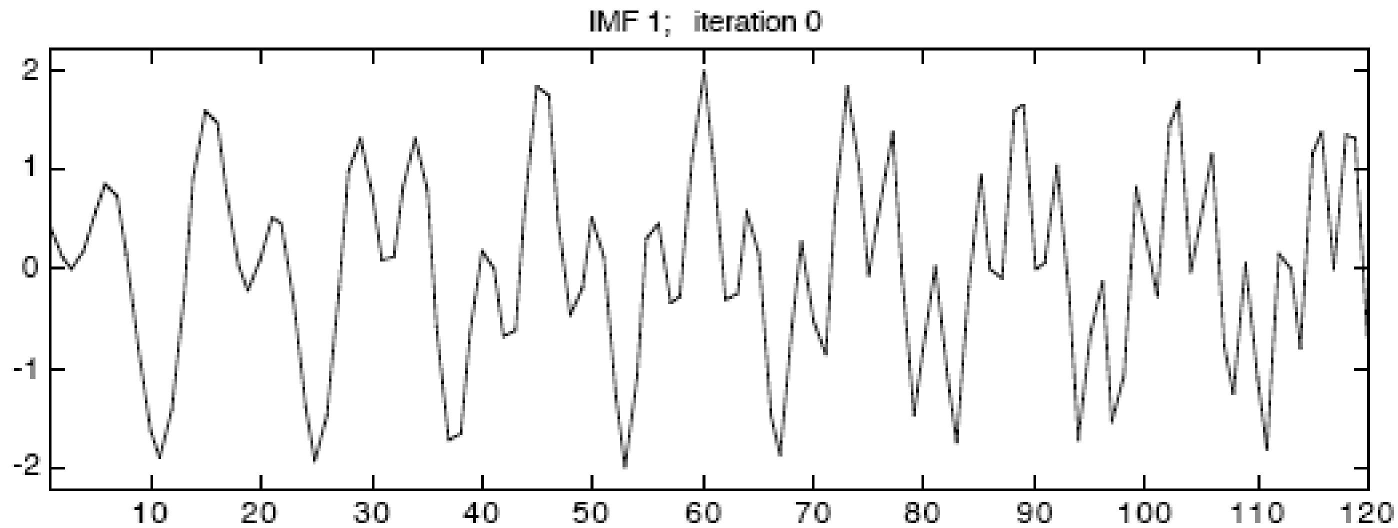
$$x(t) = \sum_{i=1}^{n} c_i(t) + r_n(t) \tag{14}$$

Note that c_1 contains the finest local mode; c_2 the next the second finest, etc. However, it is not recommended to carry out the subsequent iterations to the extreme lest the resulting $\{c_j, \quad i \geq 2\}$ obliterate the physically meaningful amplitude fluctuations. A recommended iteration stopping criterion suggested is the following measure of the standard deviation(SD):

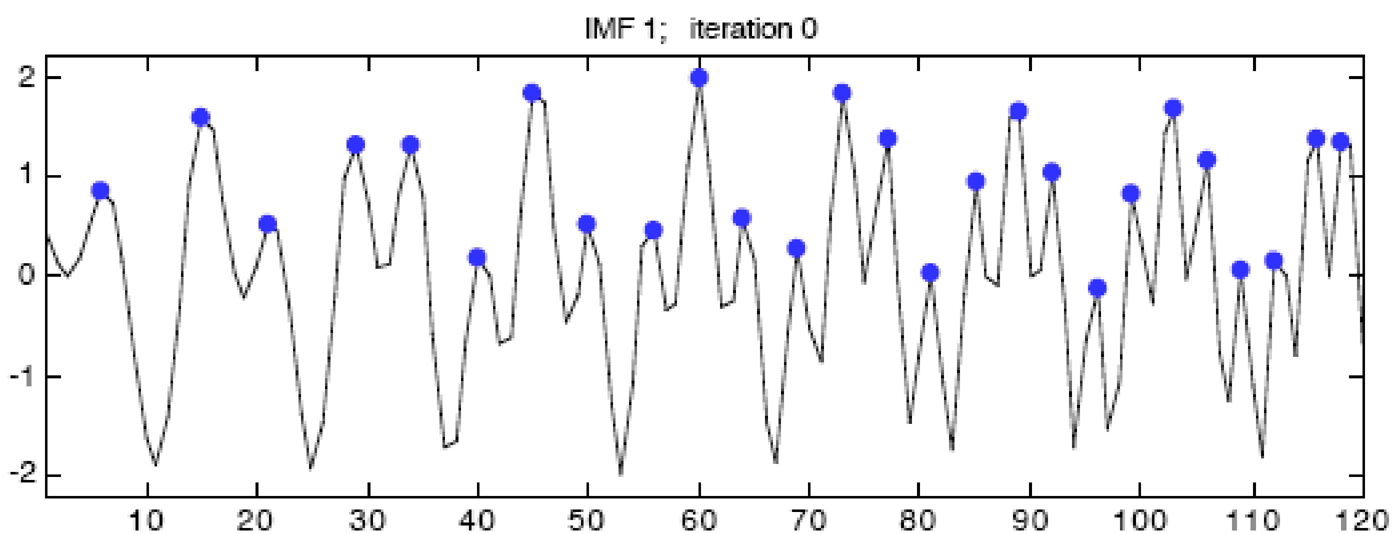
$$SD = \sum_{t=0}^{T} \left[\frac{|h_{1(k-1)}(t) - h_{1k}(t)|^2}{h_{1(k-1)}^2(t)} \right], \quad 0.2 \leq SD \leq 0.3 \tag{15}$$

Example Problem: Tone plus Chirp Oscillation
(Source: Gabriel.Rilling (at) ens-lyon.fr
http://perso.ens-lyon.fr/patrick.flandrin/emd.html)

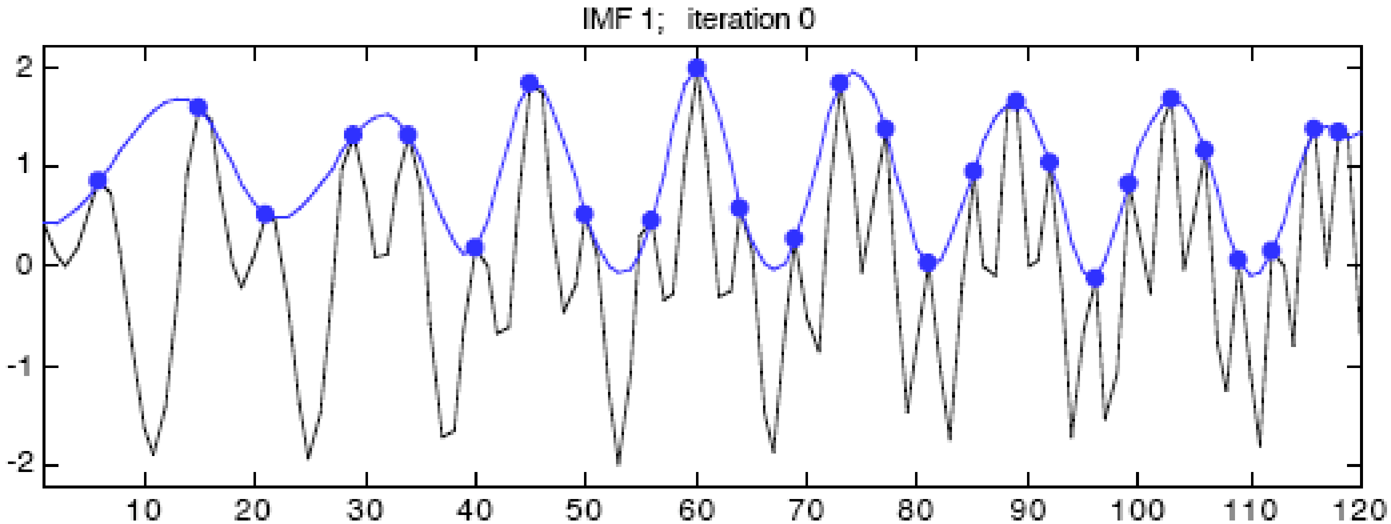




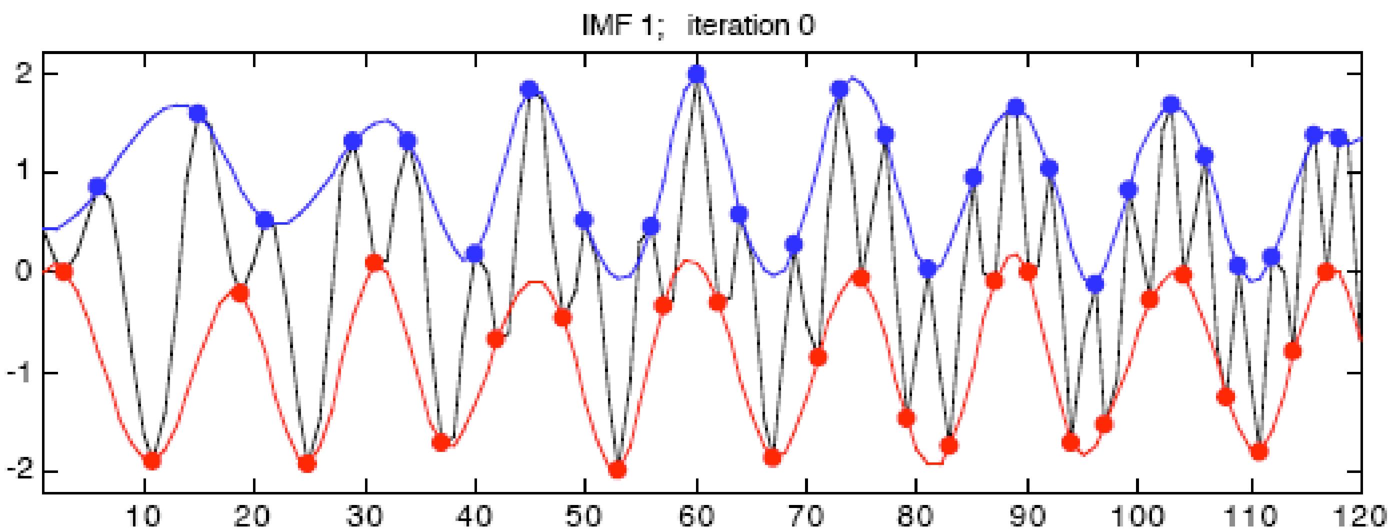
Mark the maxima



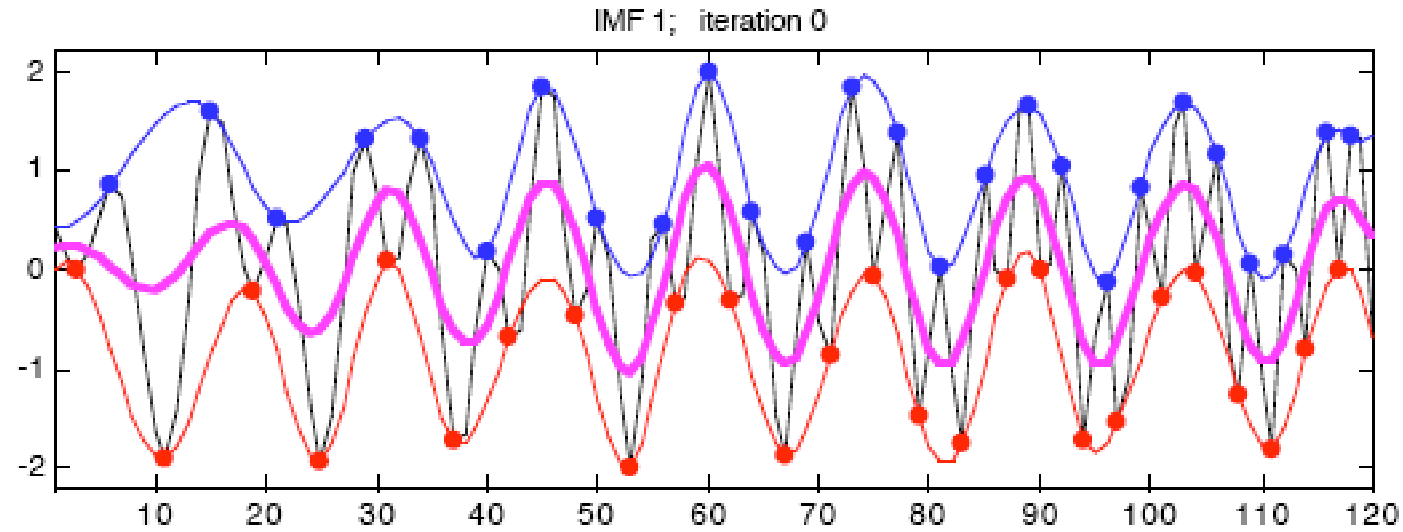
Interpolate the maxima by cubic splines



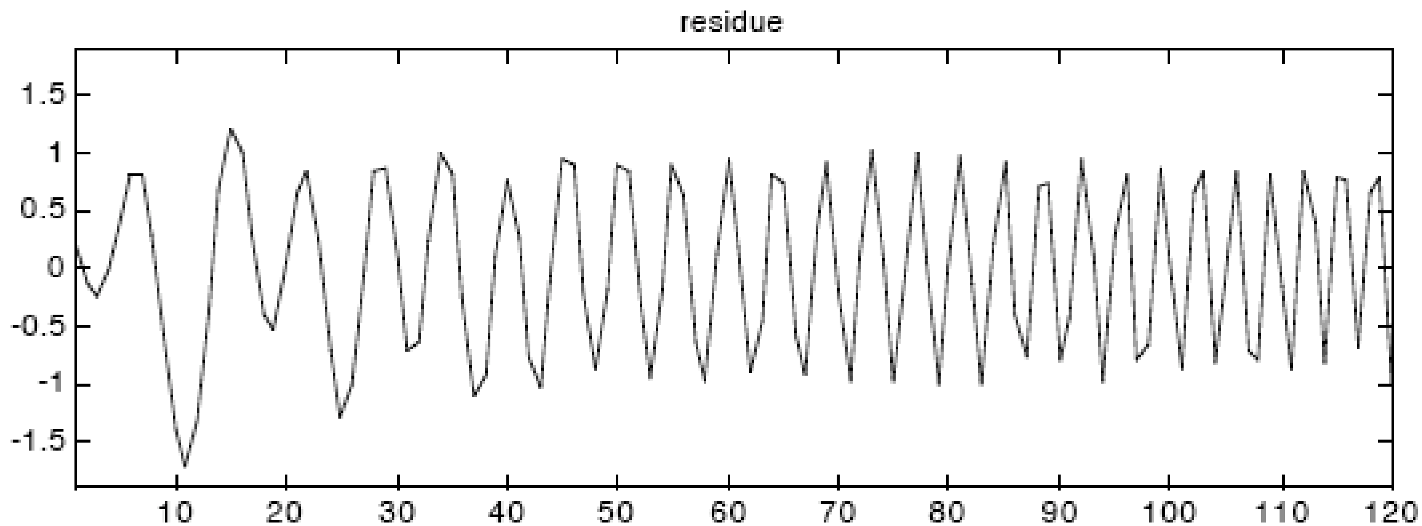
Repeat the minima by cubic splines



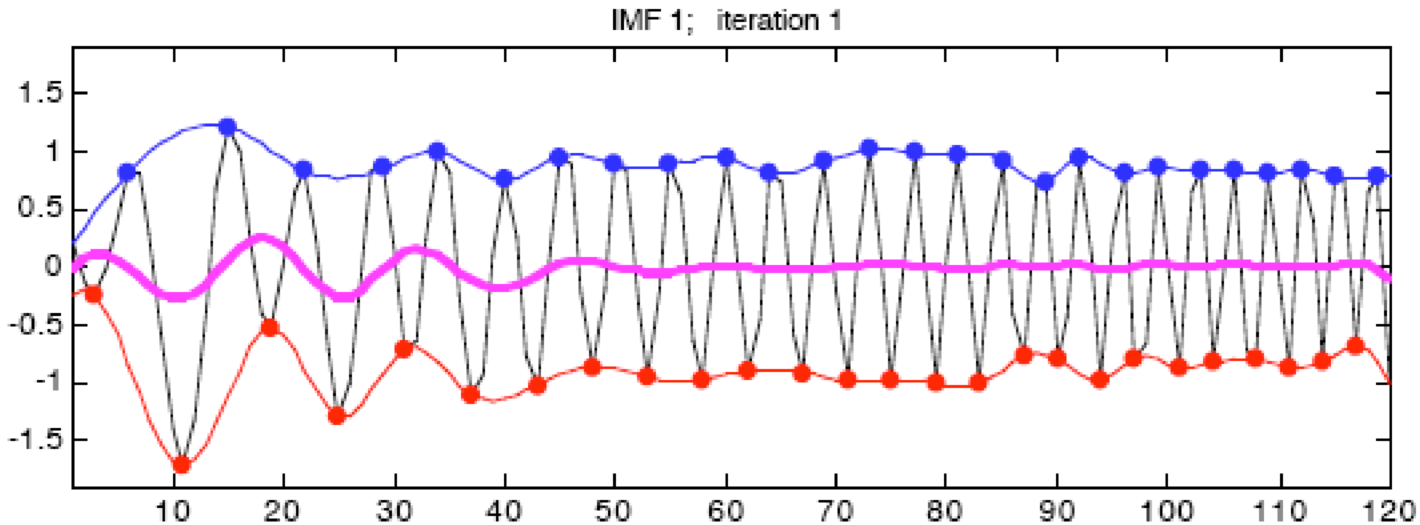
Obtain the local mean curve, m_1

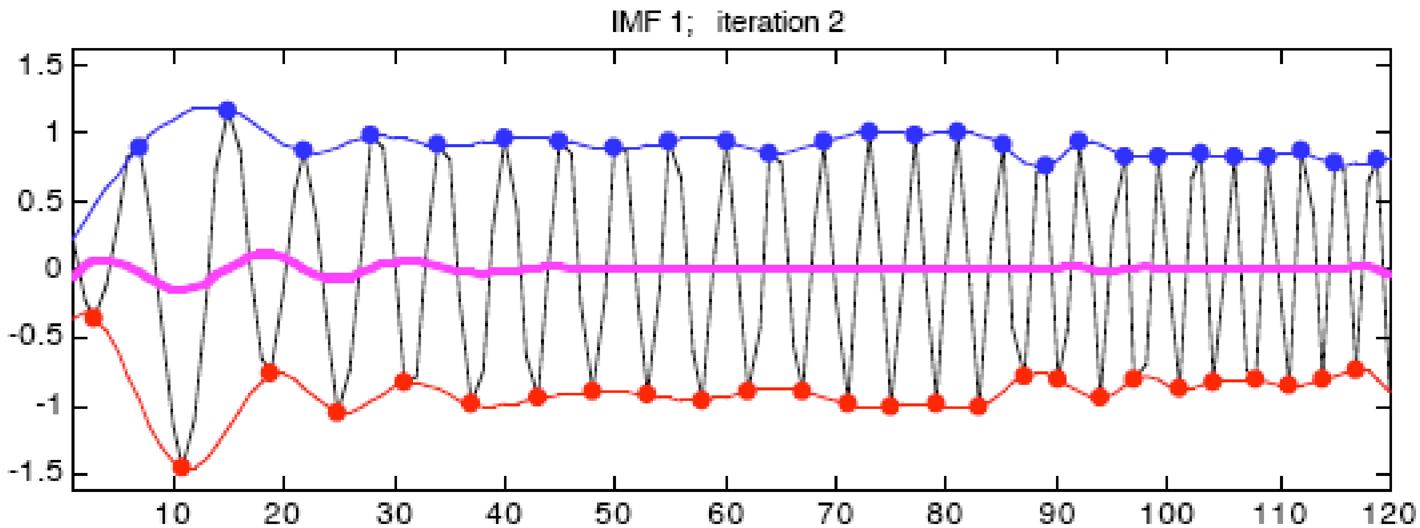


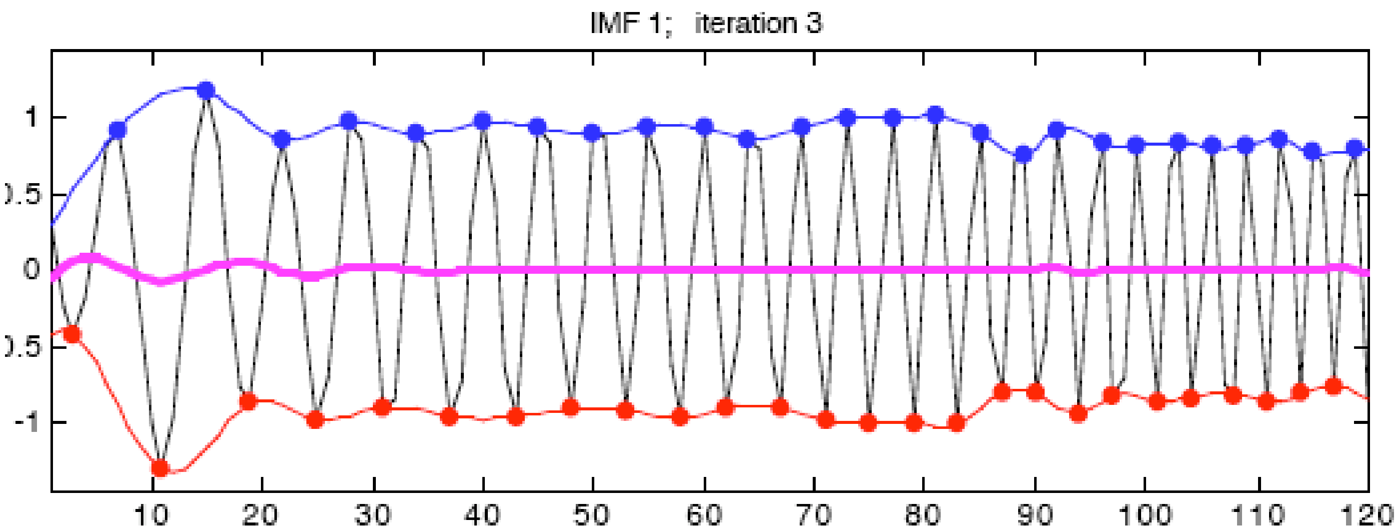
Obtain the residue, $r_1 = x - m_1$

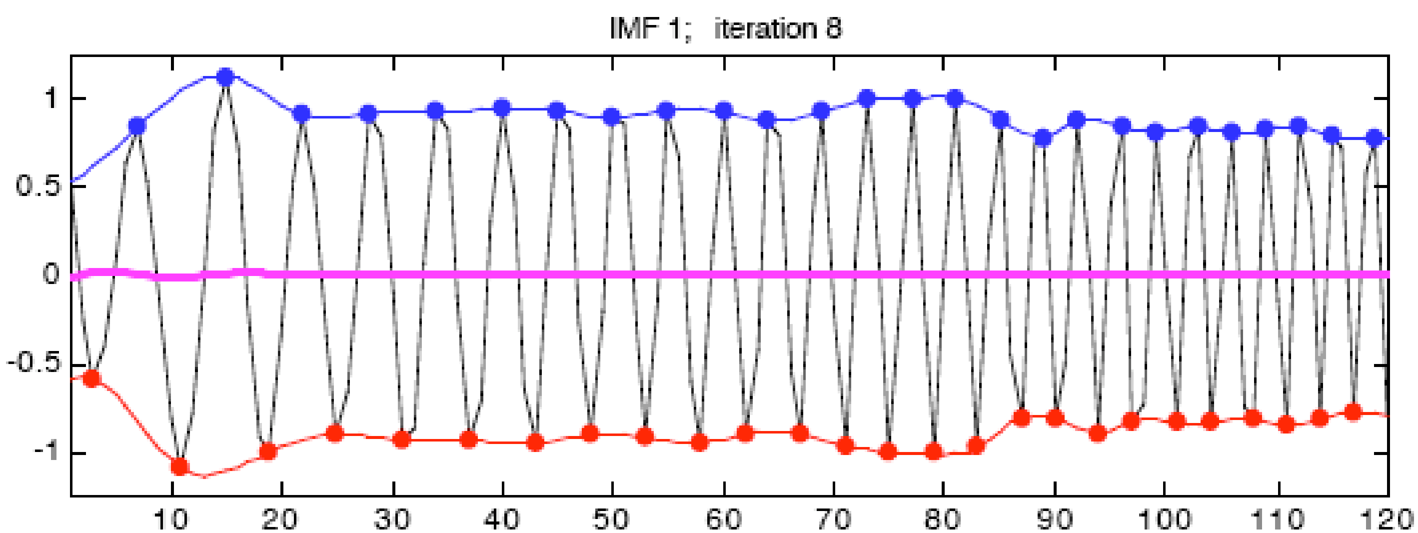


Iterate on h_1 if it violates the assumptions and restrictions






IMF 1; iteration 3
1
0.5
0
-0.5
-1
10
20
30
40
50
60
70
80
90
100
110
120


IMF 1; iteration 8
1
0.5
0
-0.5
-1
10
20
30
40
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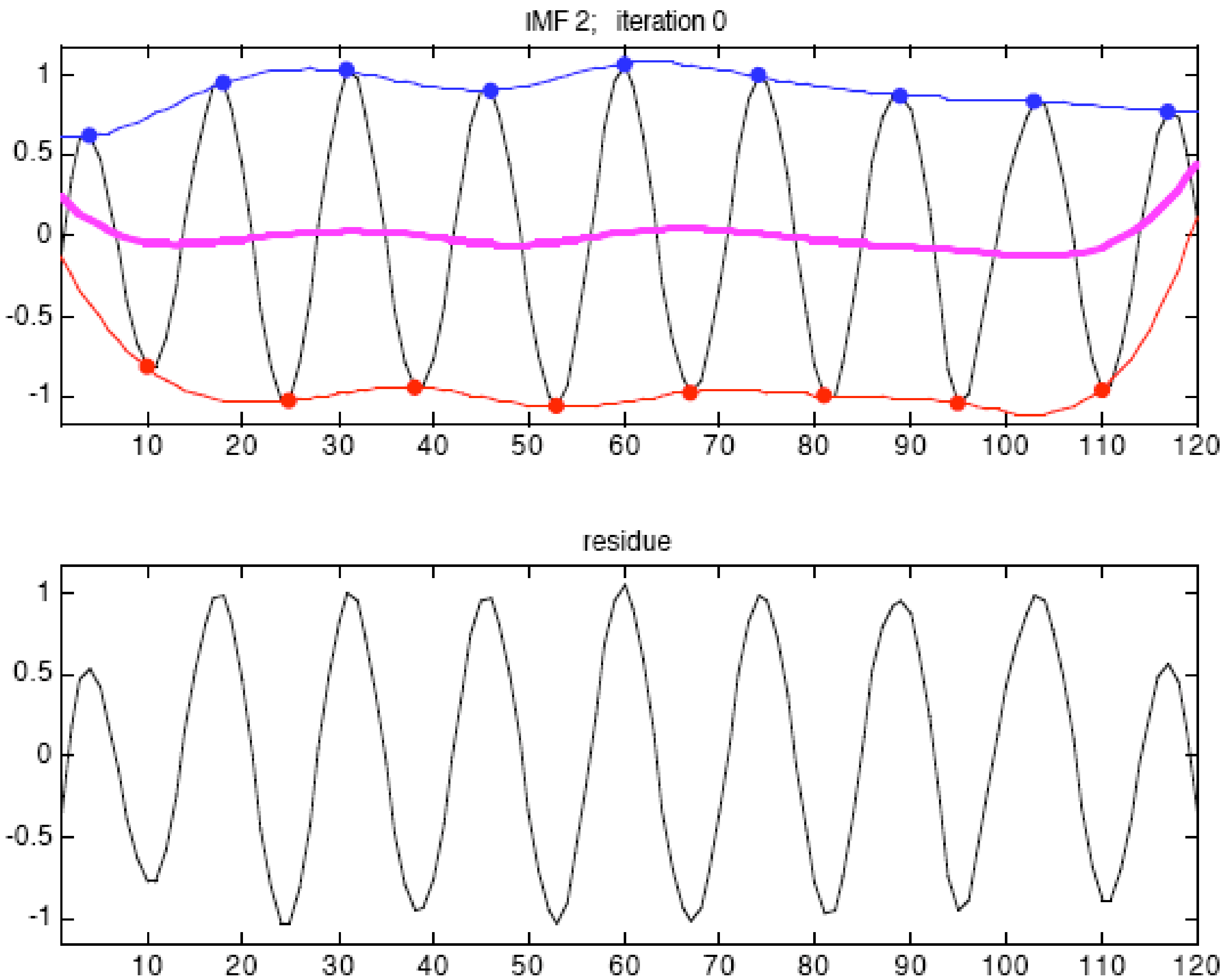
100
110
120



IMF 2; iteration 0
1
0.5
0
-0.5
-1
10
20
30
40
50
60
70
80
90
100
110
120
residue
1
0.5
0
-0.5
-1
10
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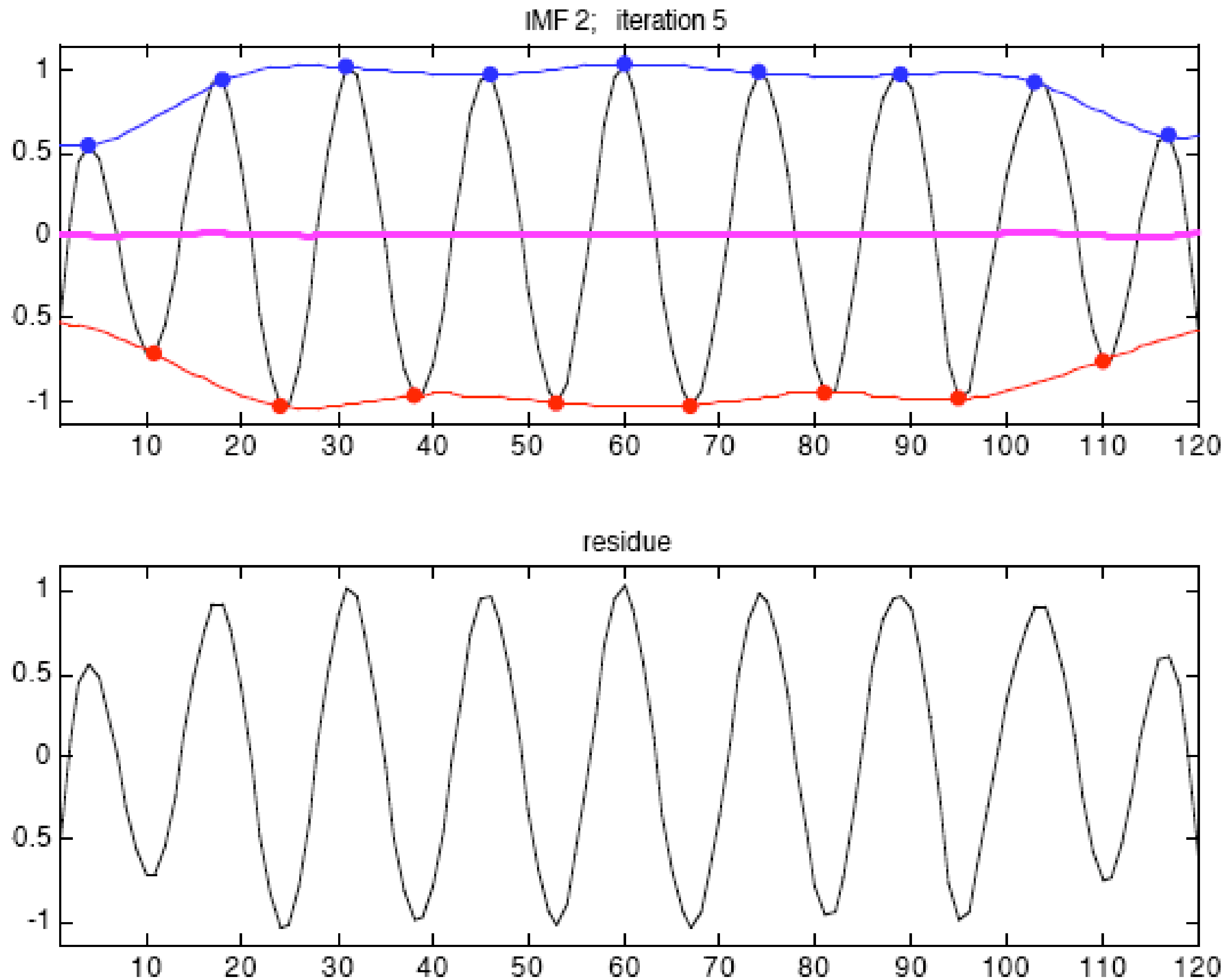
IMF 2; iteration 5
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0
-0.5
-1
10
20
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40
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60
70
80
90
100
110
120
residue
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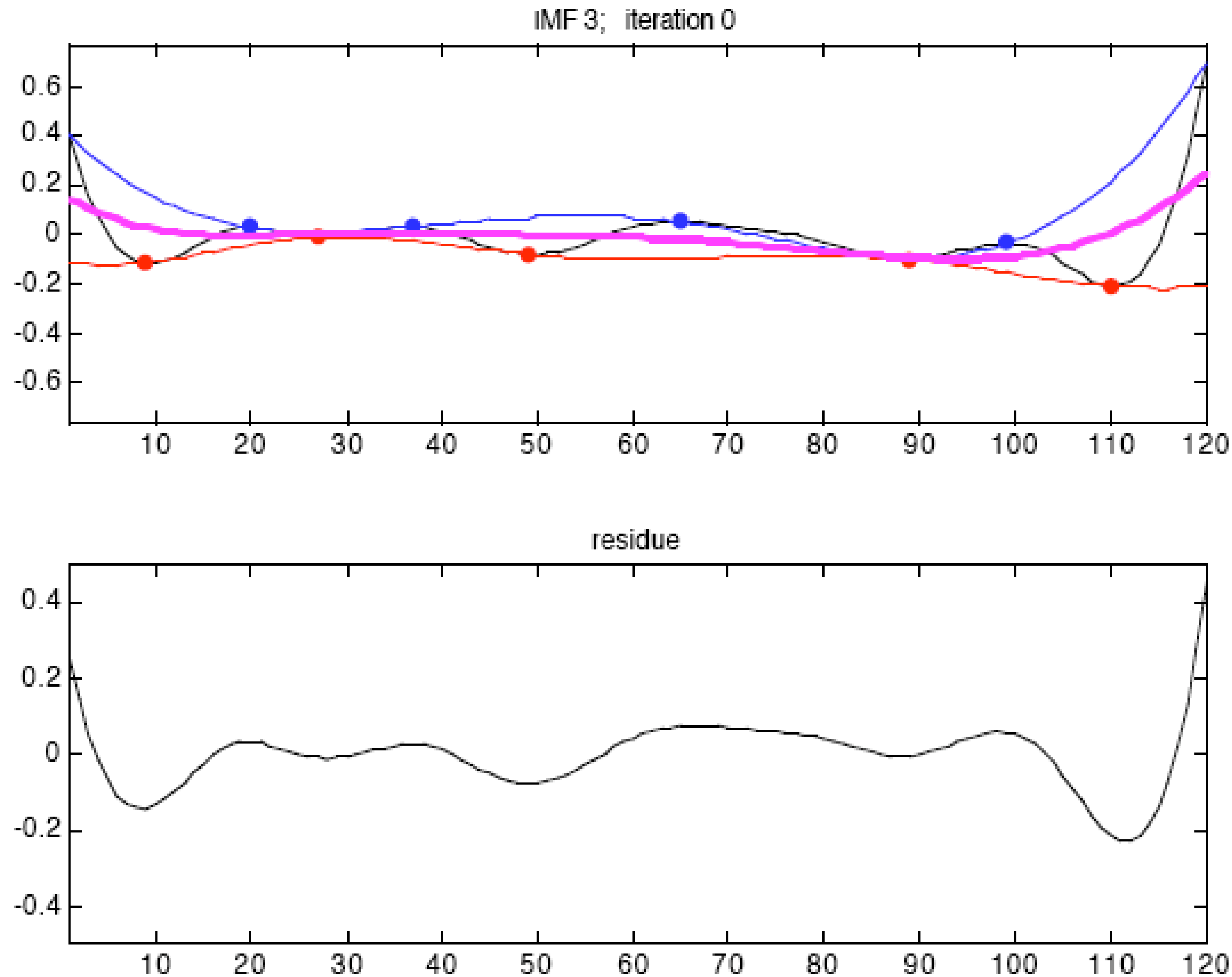

IMF 3; iteration 0
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0.4
0.2
0
-0.2
-0.4
-0.6
10
20
30
40
50
60
70
80
90
100
110
120
residue
0.4
0.2
0
-0.2
-0.4
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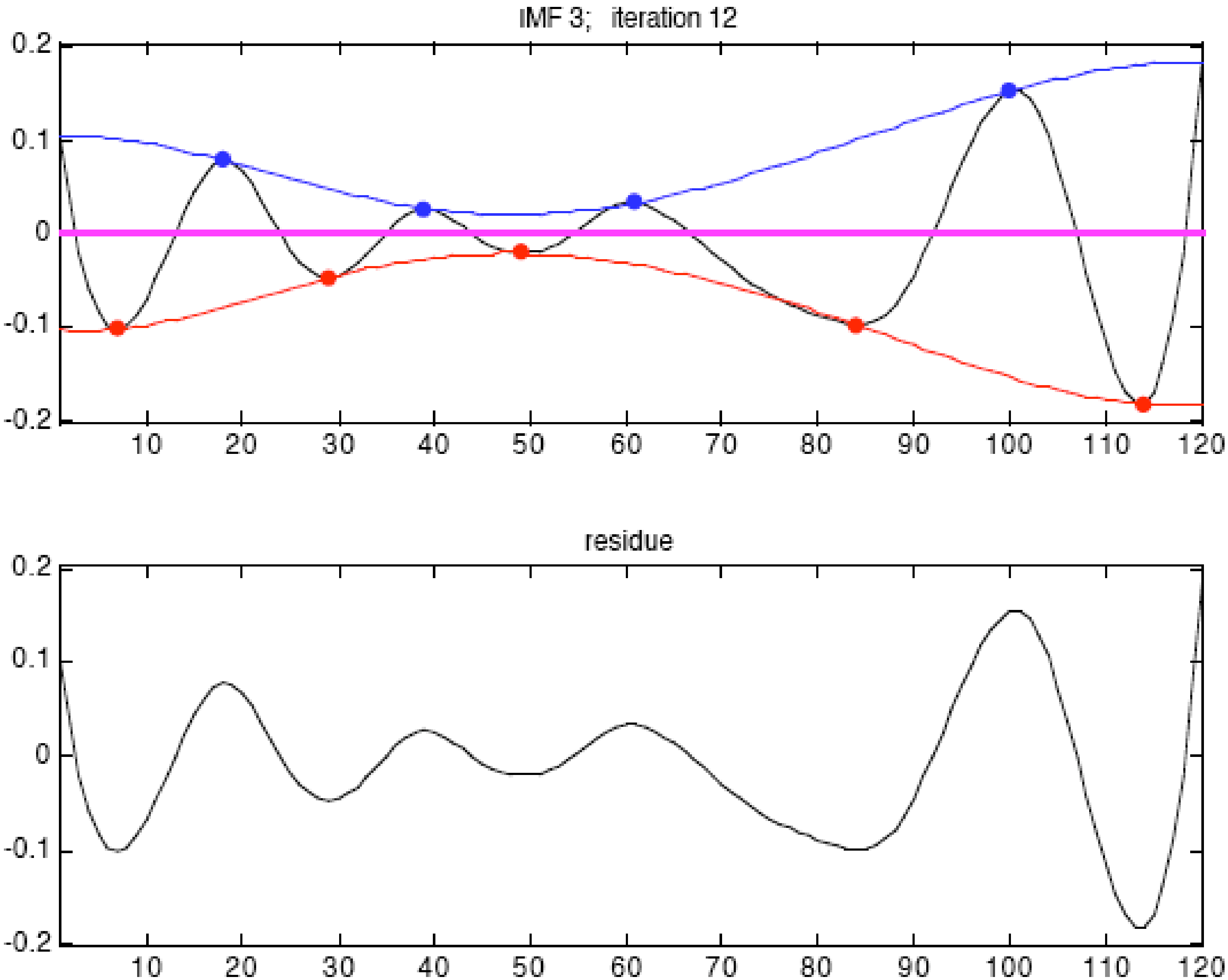
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0.2
0.1
0
-0.1
-0.2
10
20
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40
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60
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80
90
100
110
120
residue
0.2
0.1
0
-0.1
-0.2
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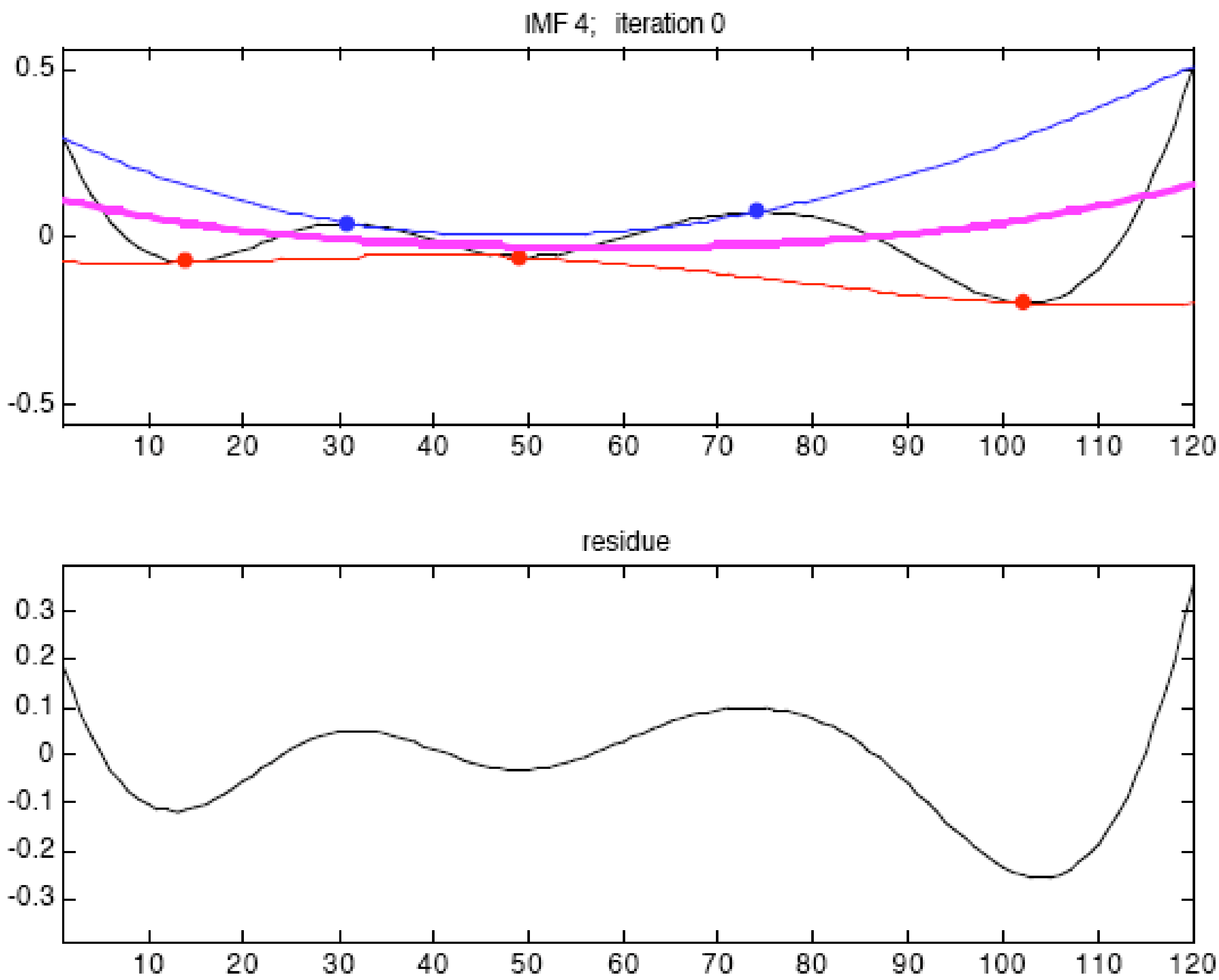

IMF 4; iteration 0
0.5
0
-0.5
10
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100
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120
residue
0.3
0.2
0.1
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-0.1
-0.2
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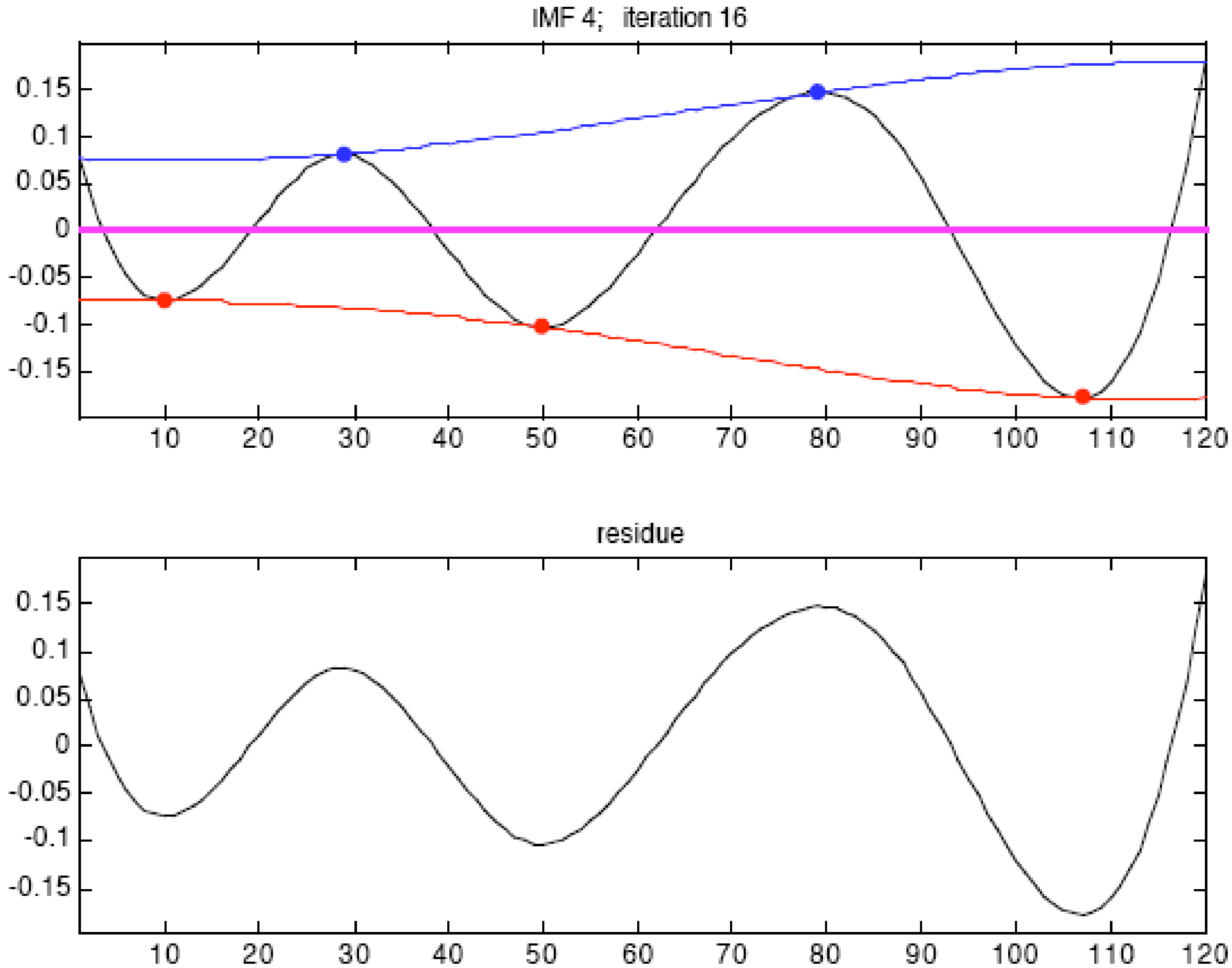

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0.1
0.05
0
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-0.15
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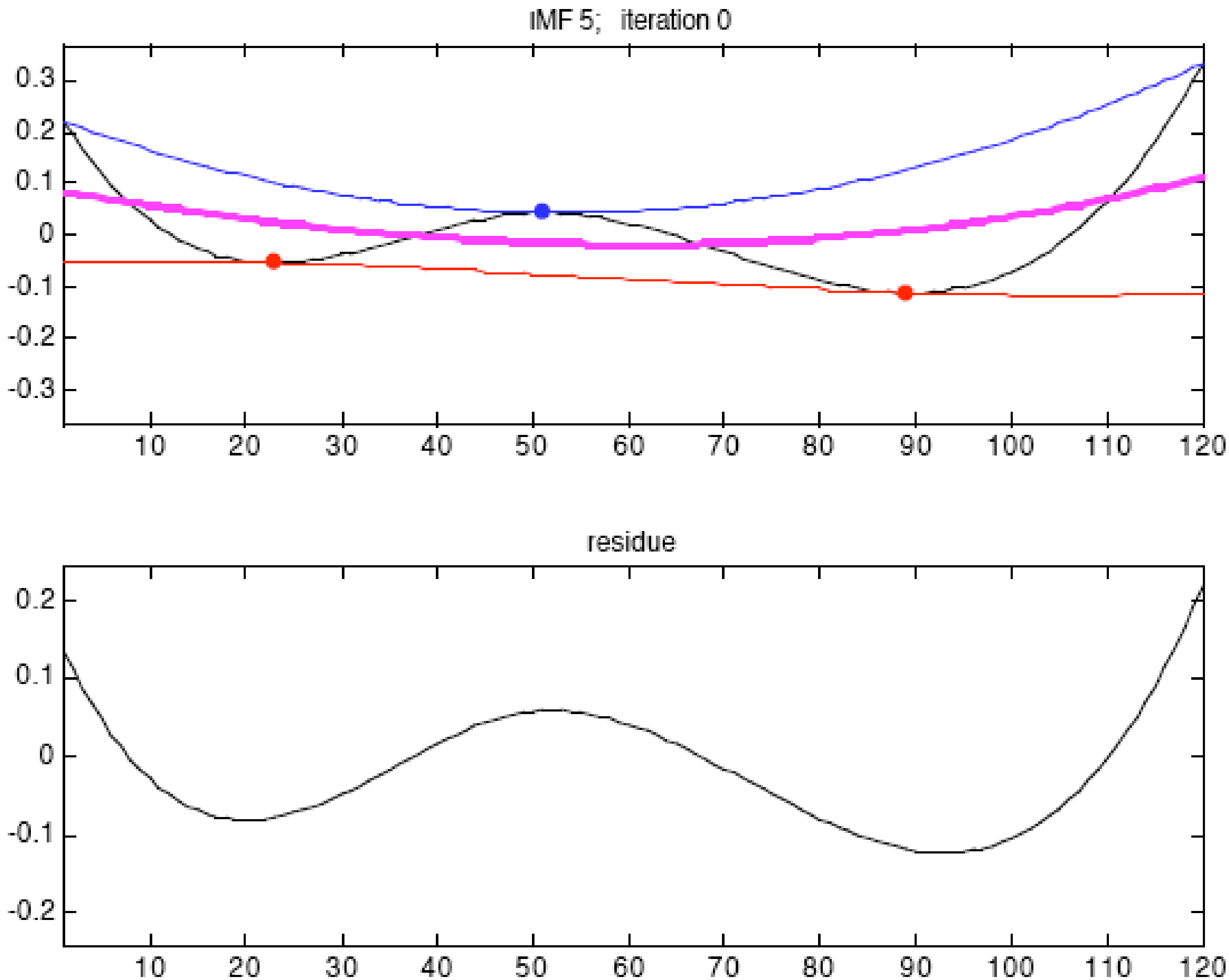
residue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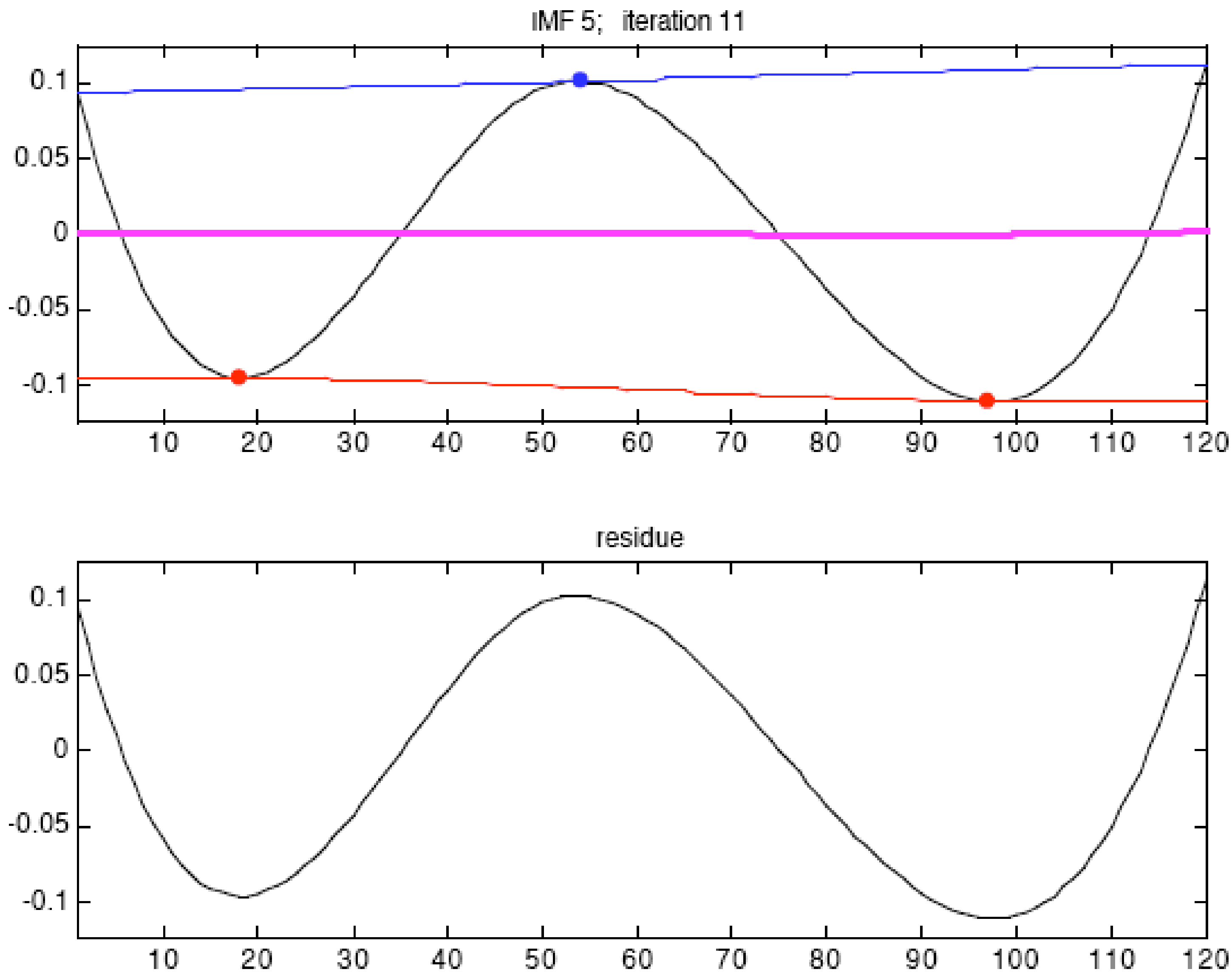
0.2
0.1
0
-0.1
-0.2
-0.3
10 20 30 40 50 60 70 80 90 100 110 120
residue
0.2
0.1
0
-0.1
-0.2
10 20 30 40 50 60 70 80 90 100 110 120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0.05
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residue
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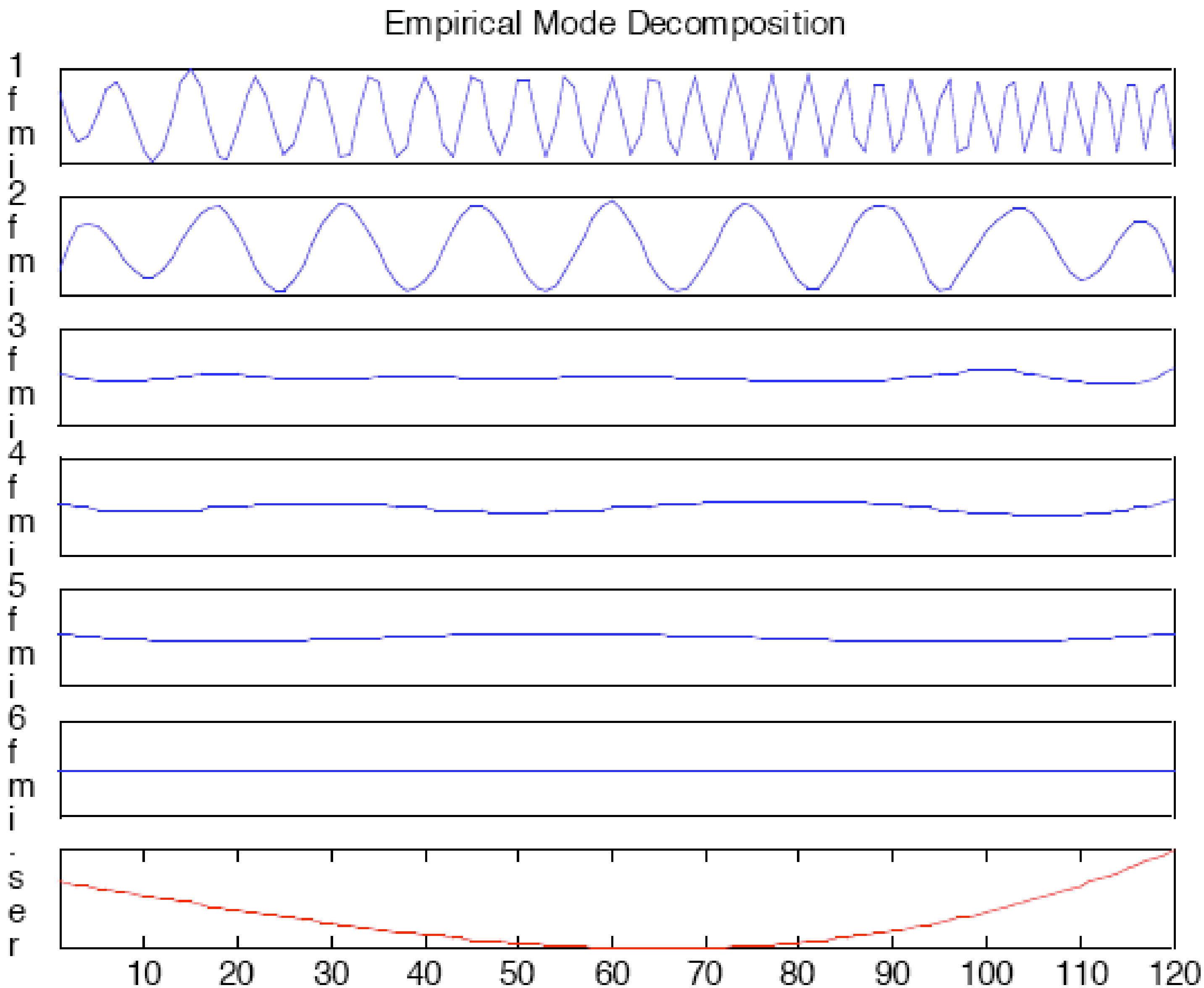
Empirical Mode Decomposition
imf1
imf2
imf3
imf4
imf5
imf6
res.
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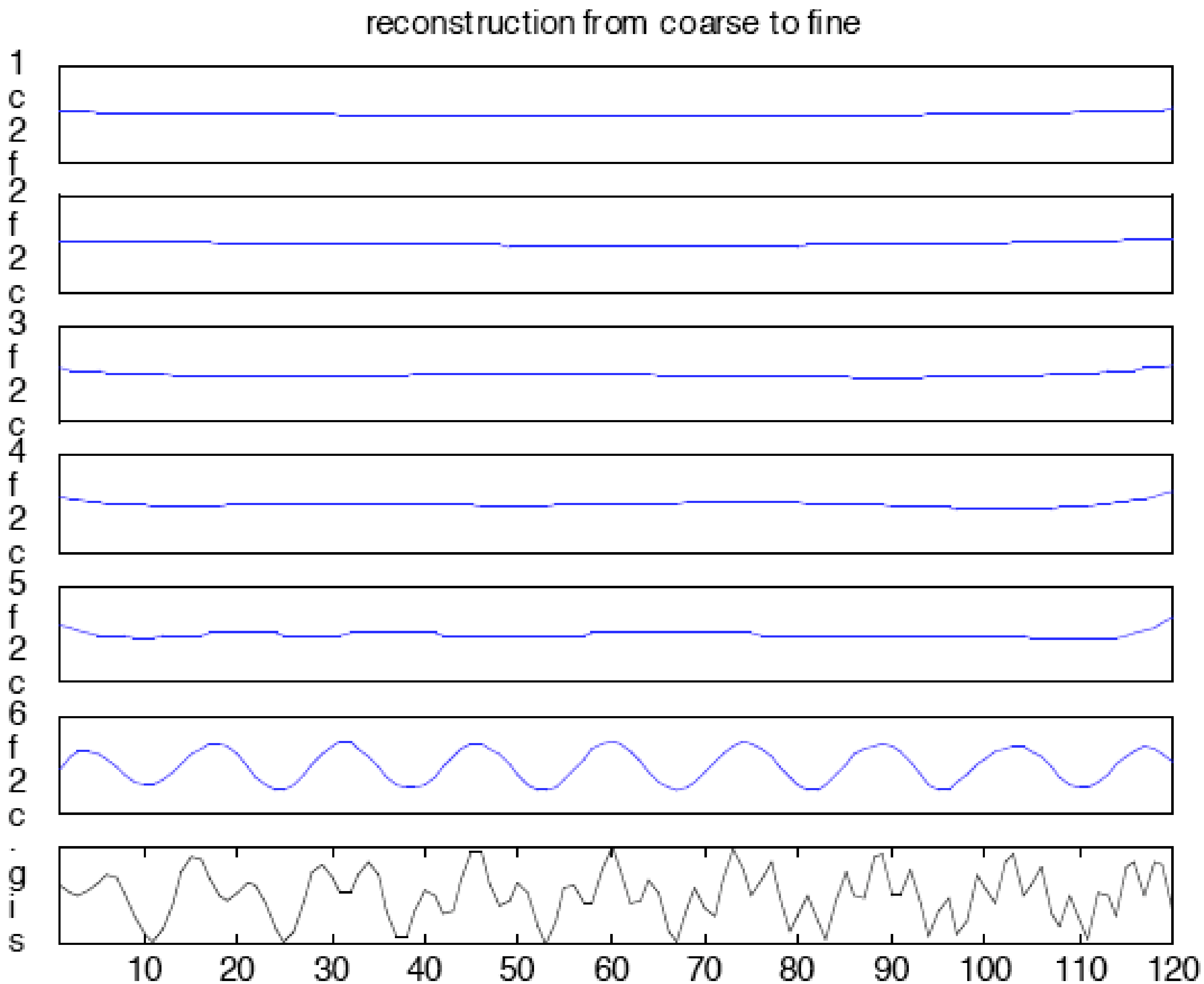
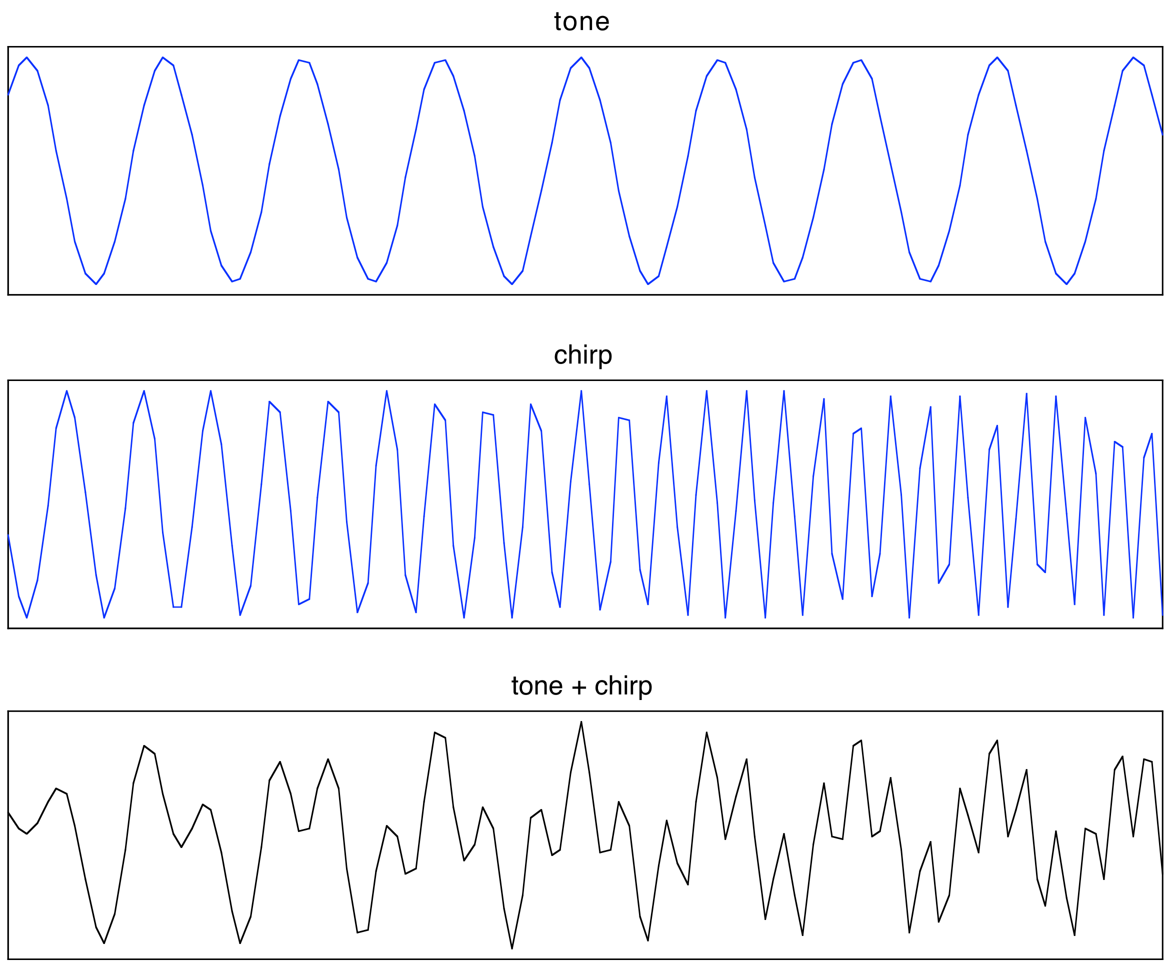
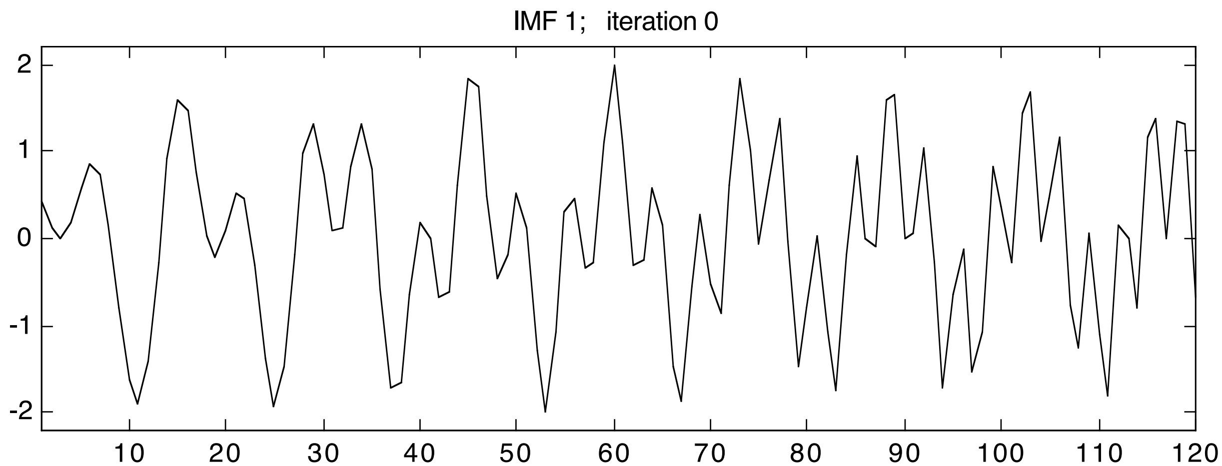


reconstruction from coarse to fine
1c2f
2f2c
3f2c
4f2c
5f2c
6f2c
.gis
10
20
30
40
50
60
70
80
90
100
110
120


tone
chirp
tone + chirp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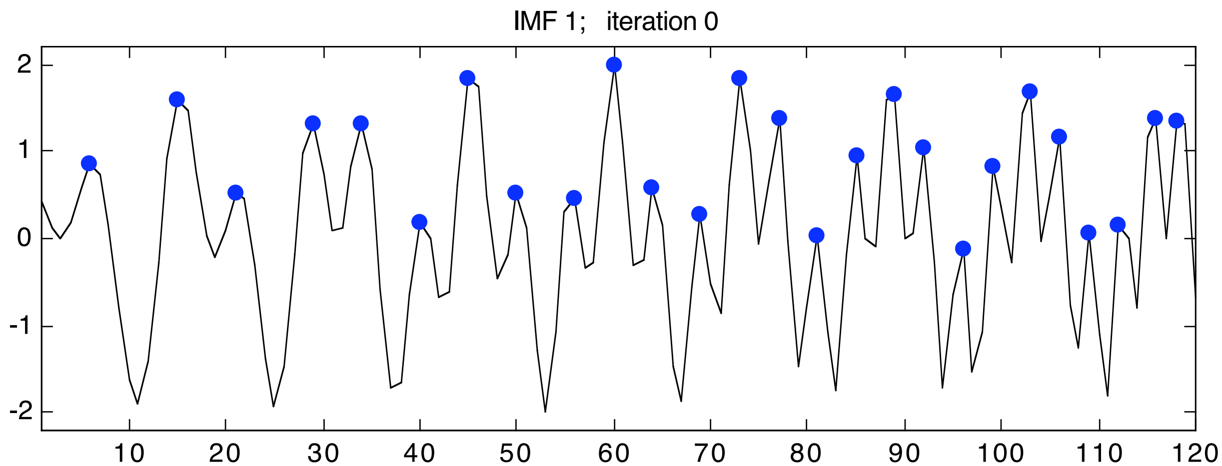

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-1
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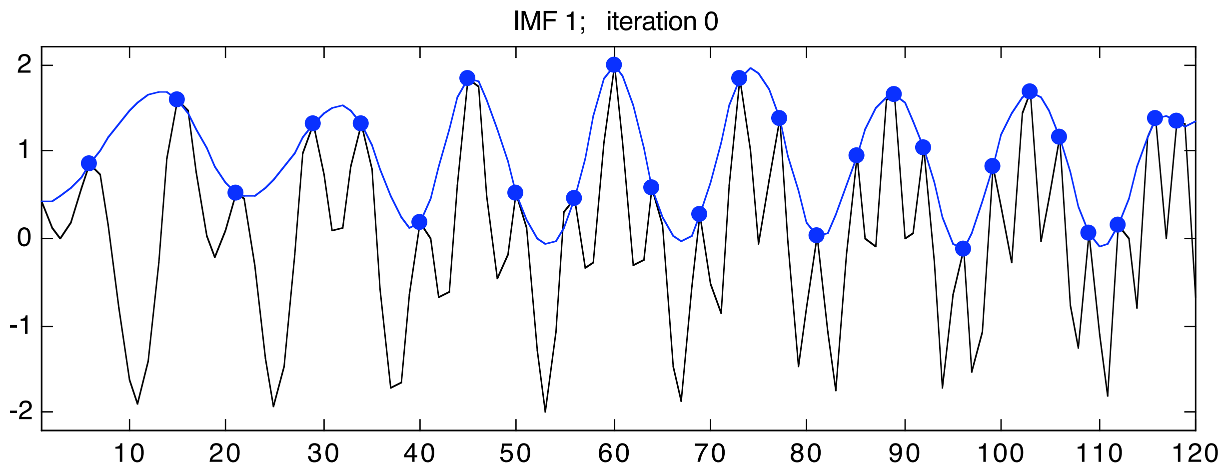

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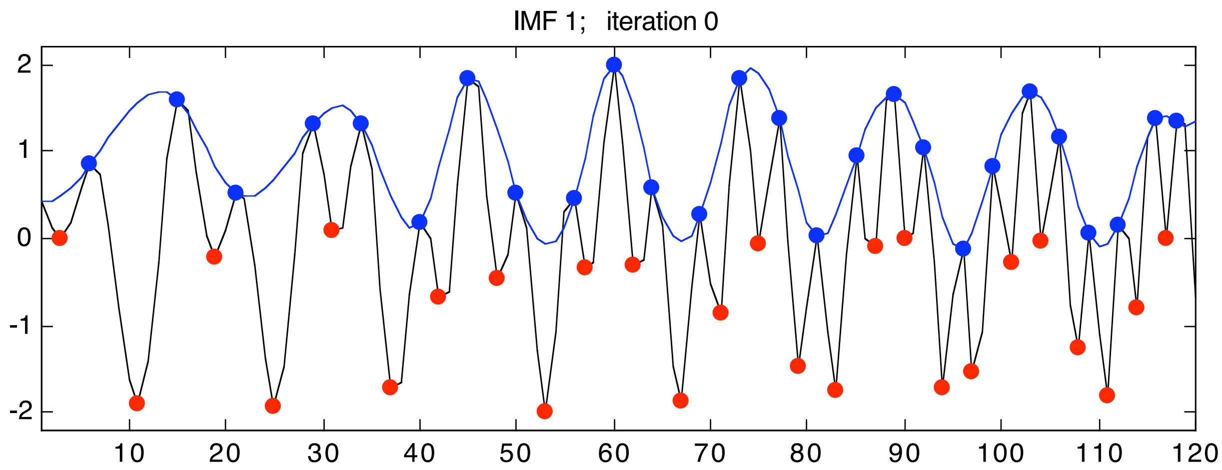
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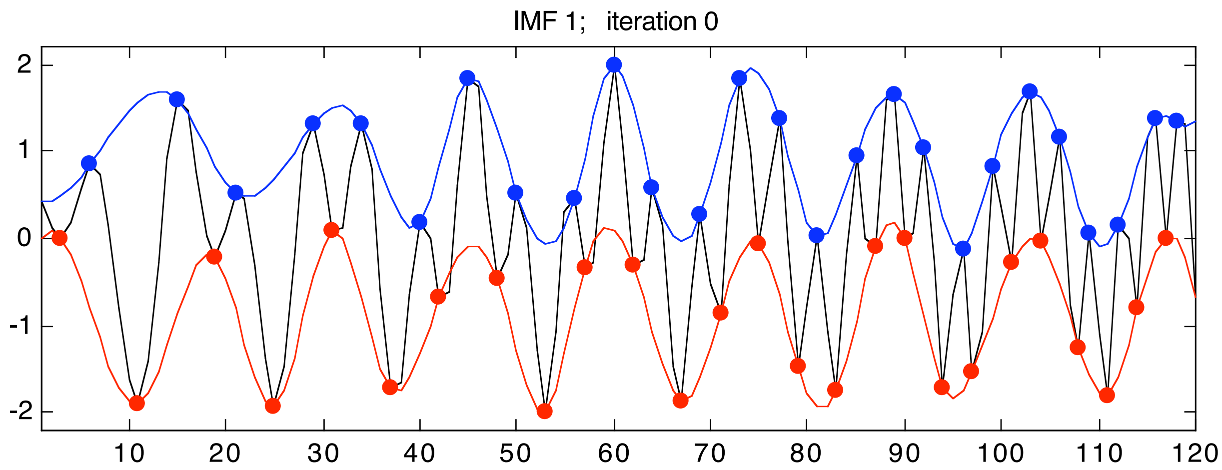

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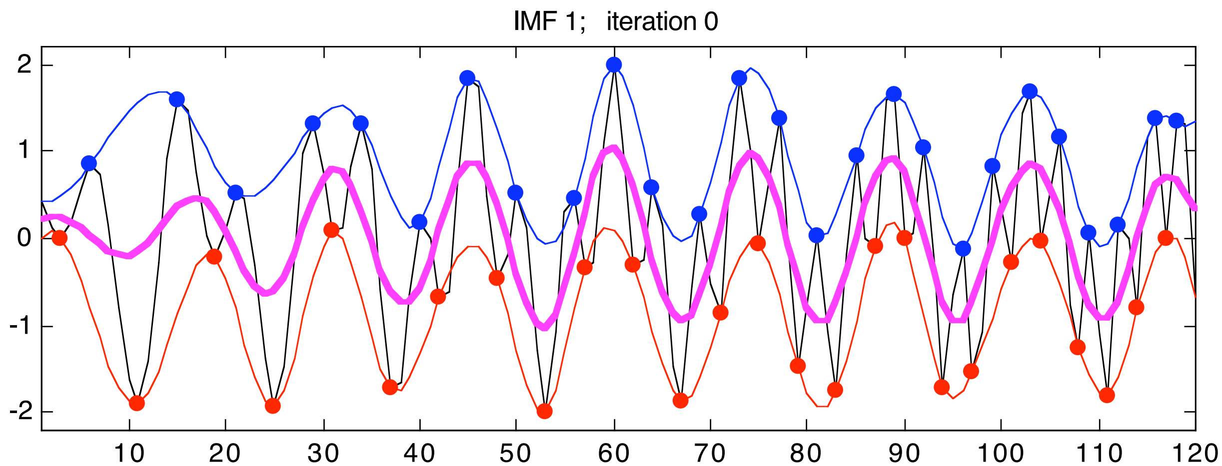

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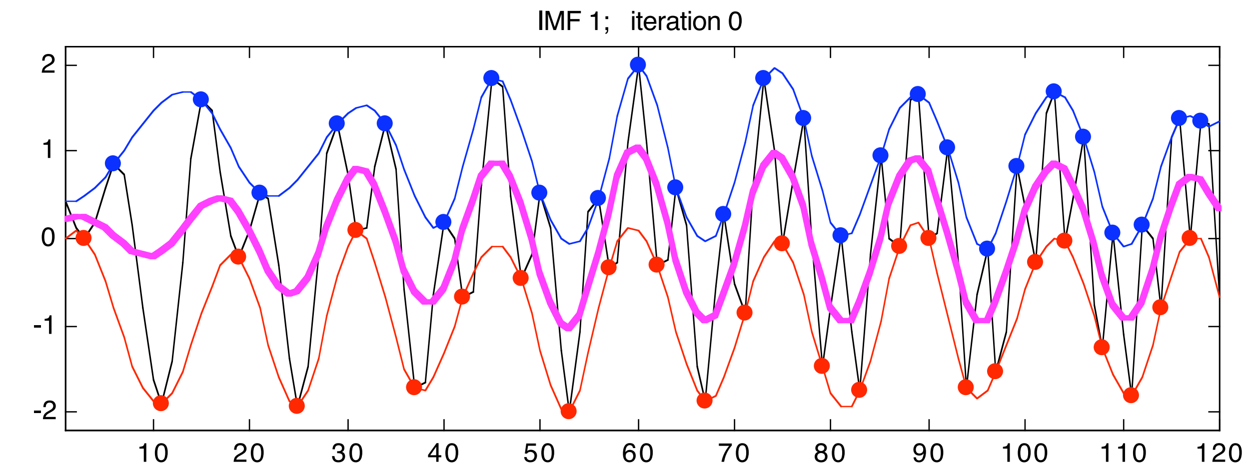
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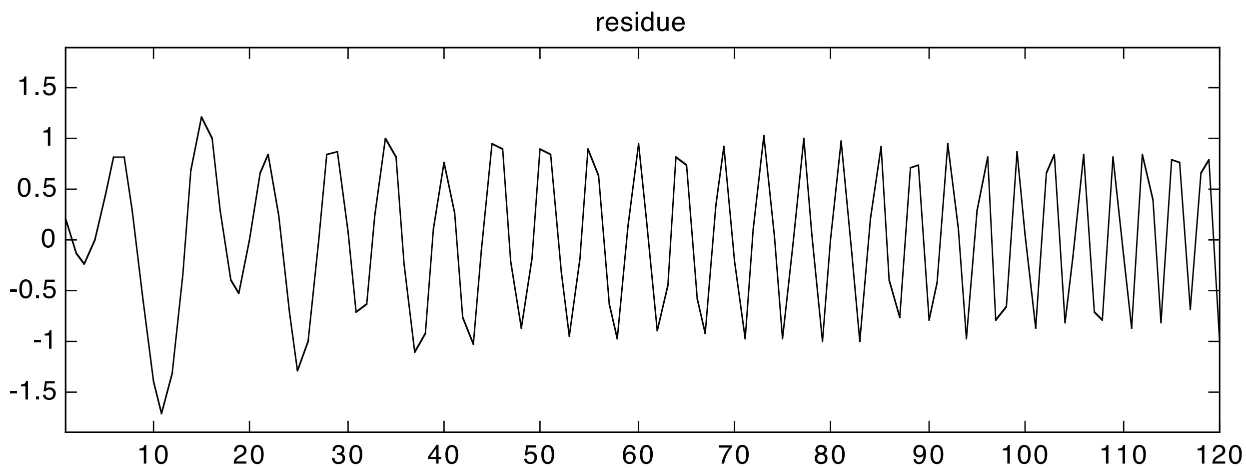
residue
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-1
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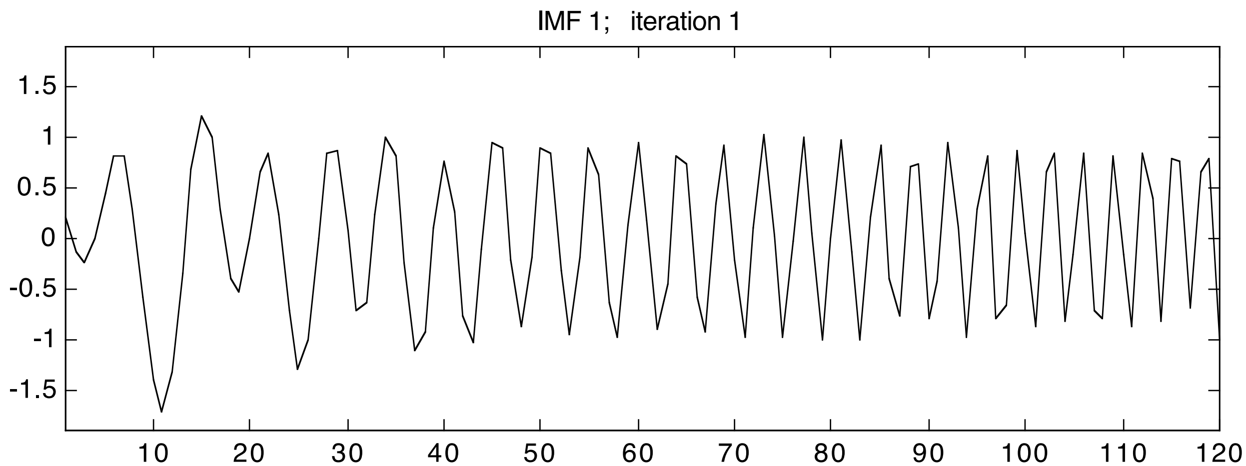
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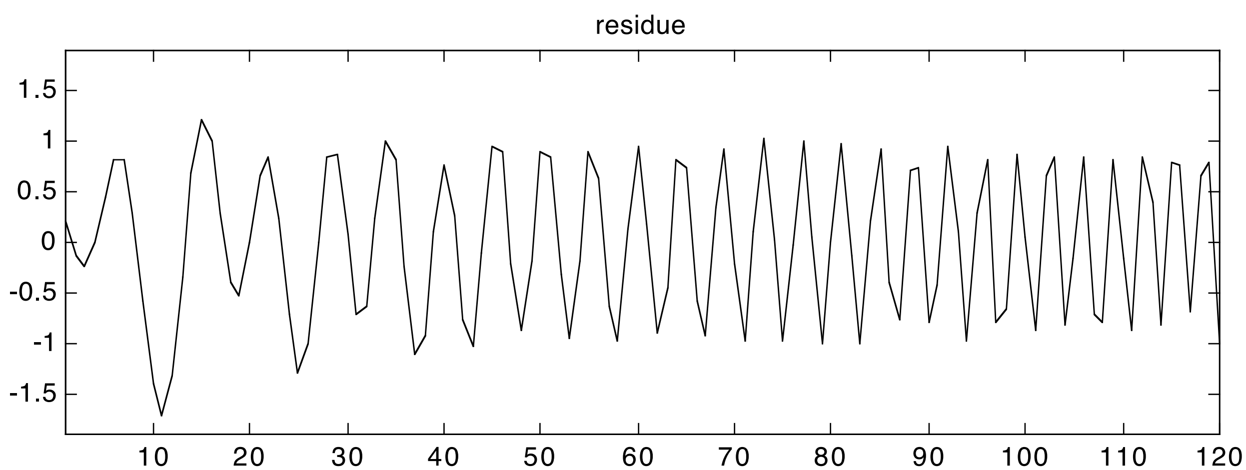
-1
-1.5
10 20 30 40 50 60 70 80 90 100 110 120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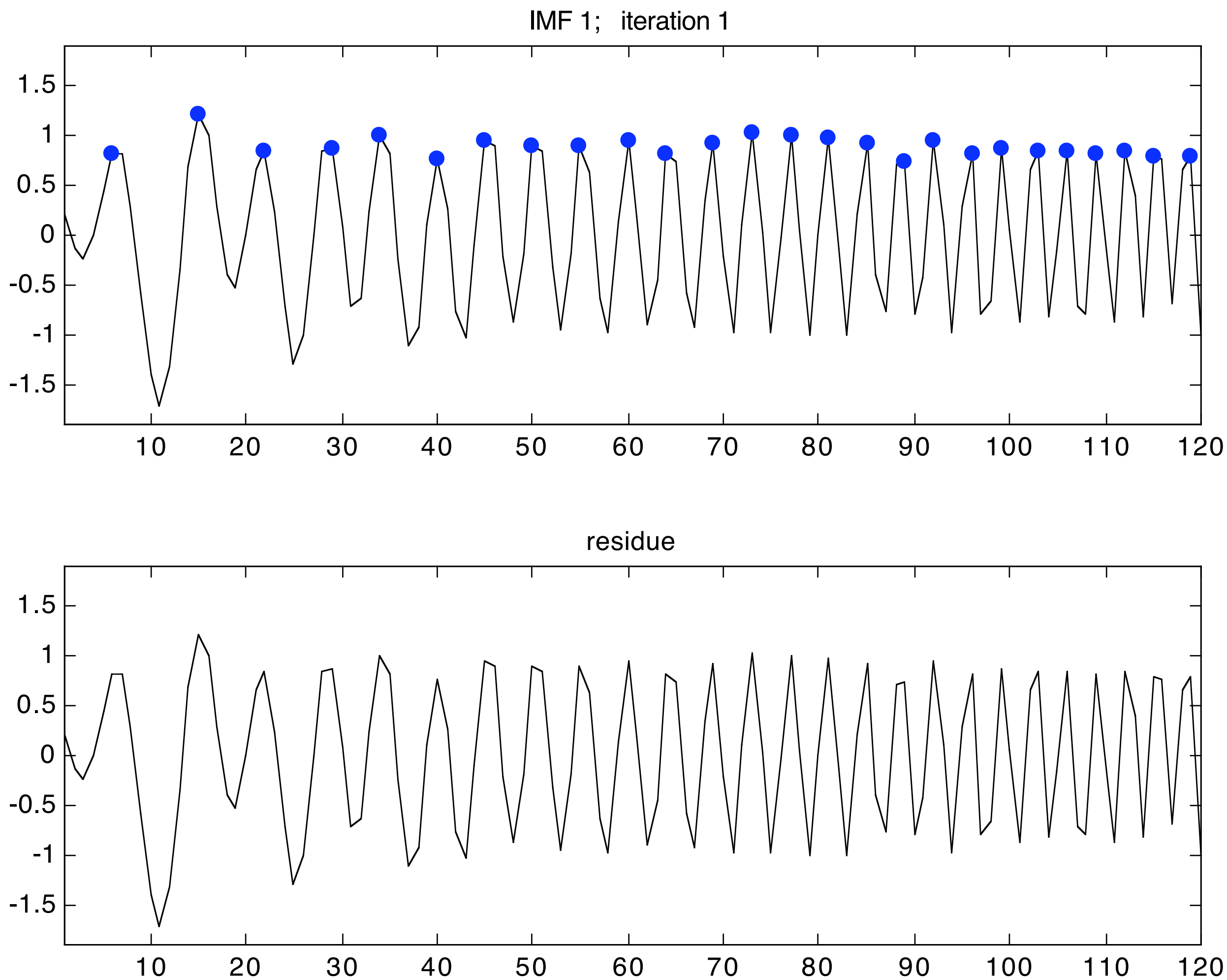
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110
120
residue
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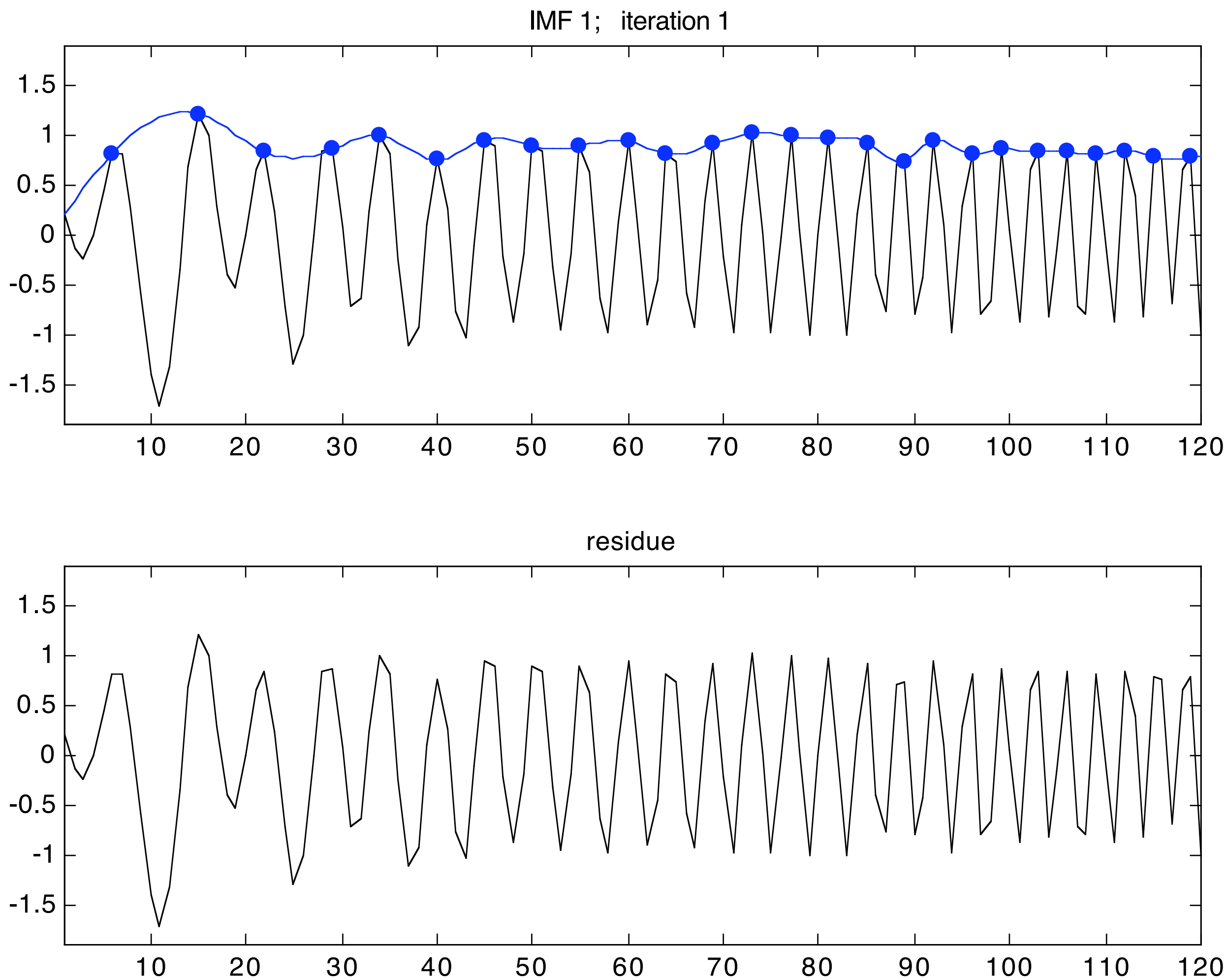
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residue
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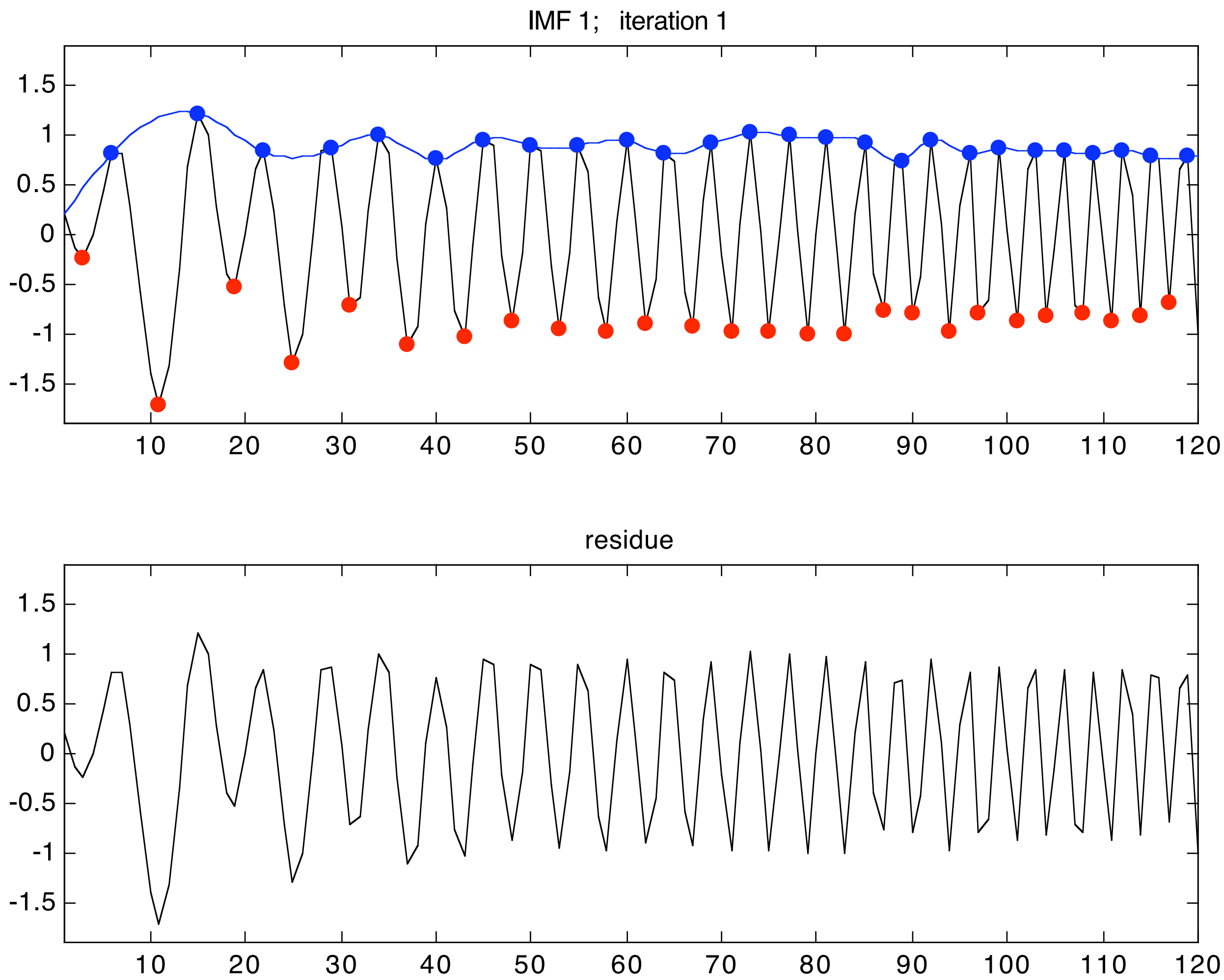
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residue
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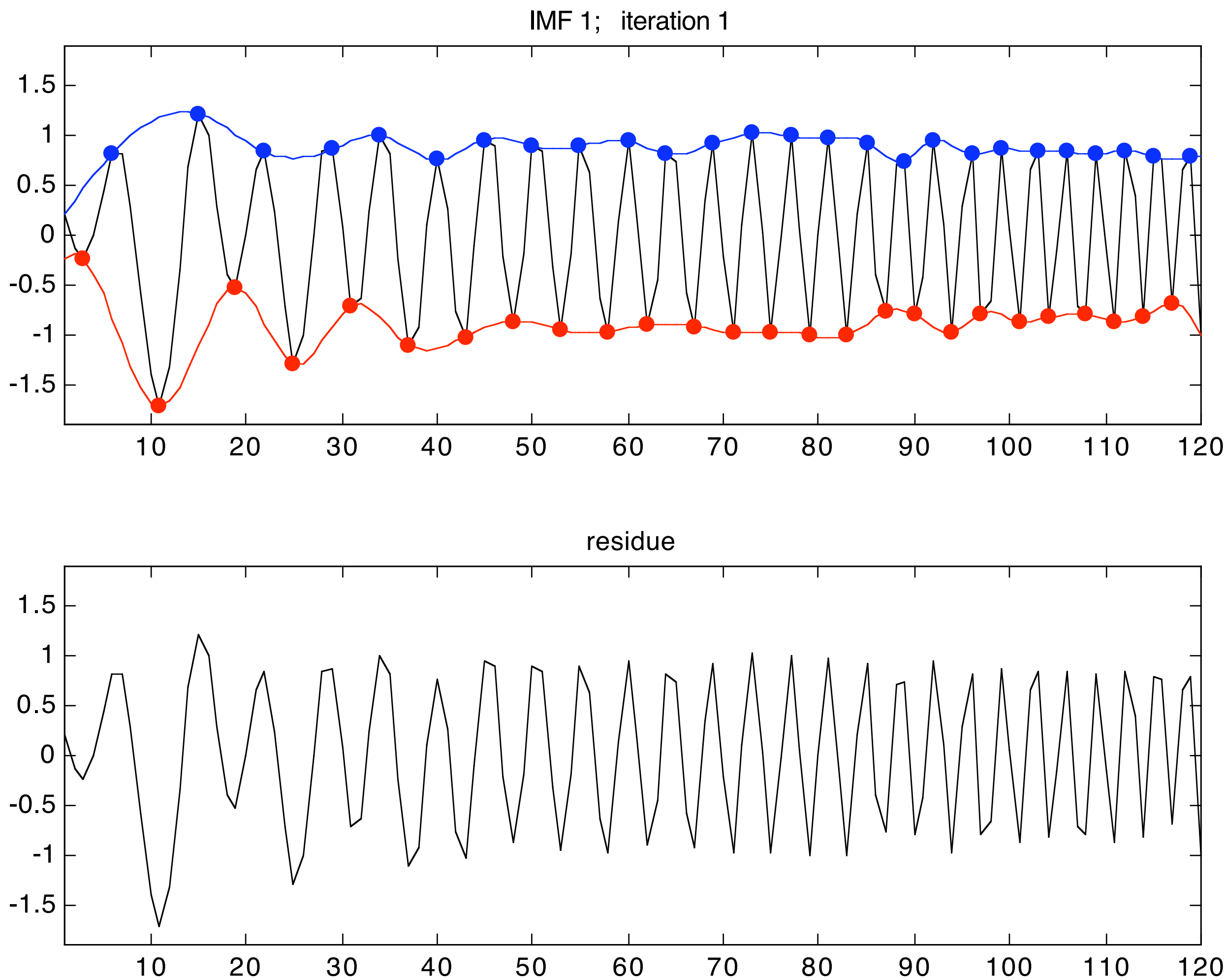

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residue
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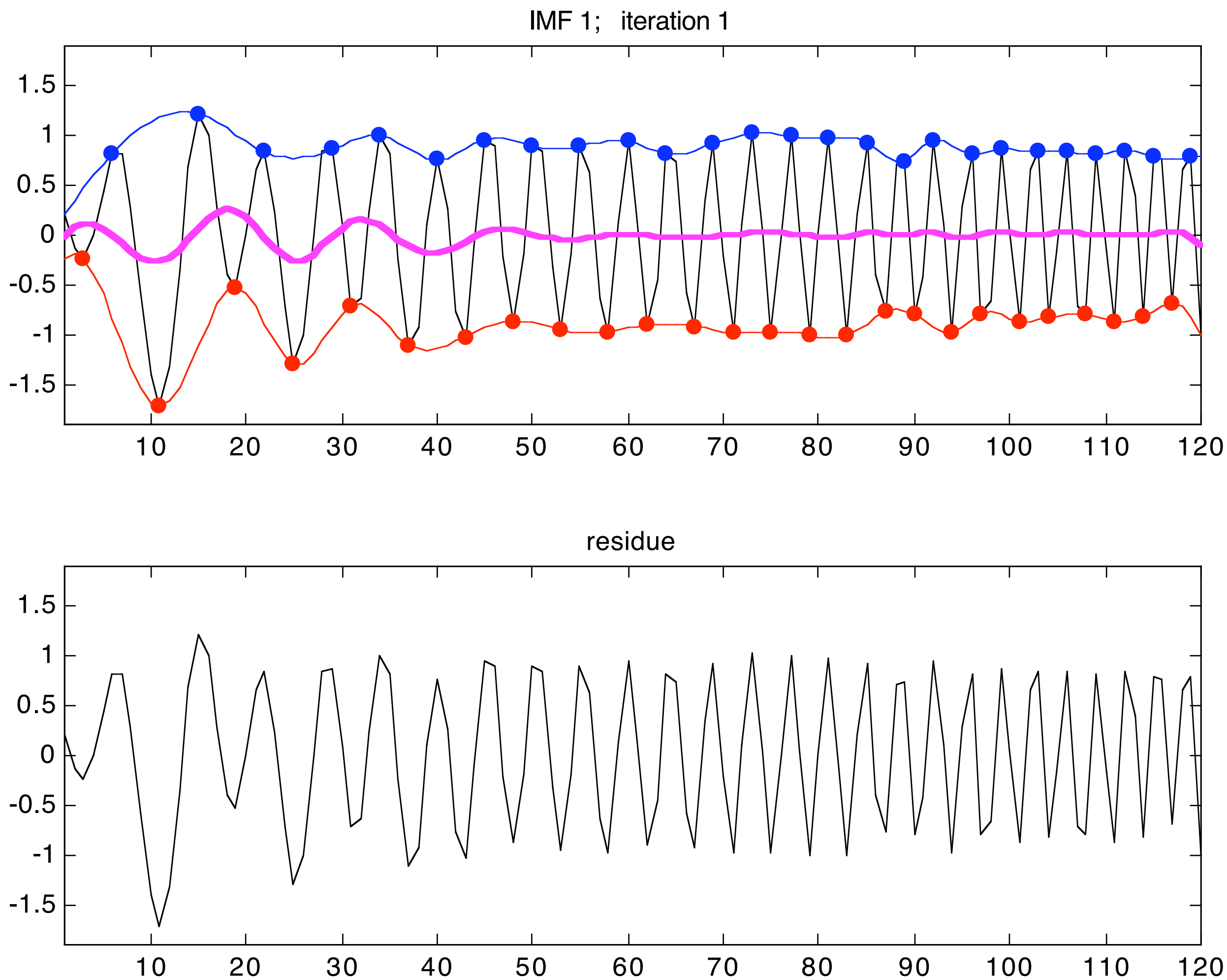
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residue
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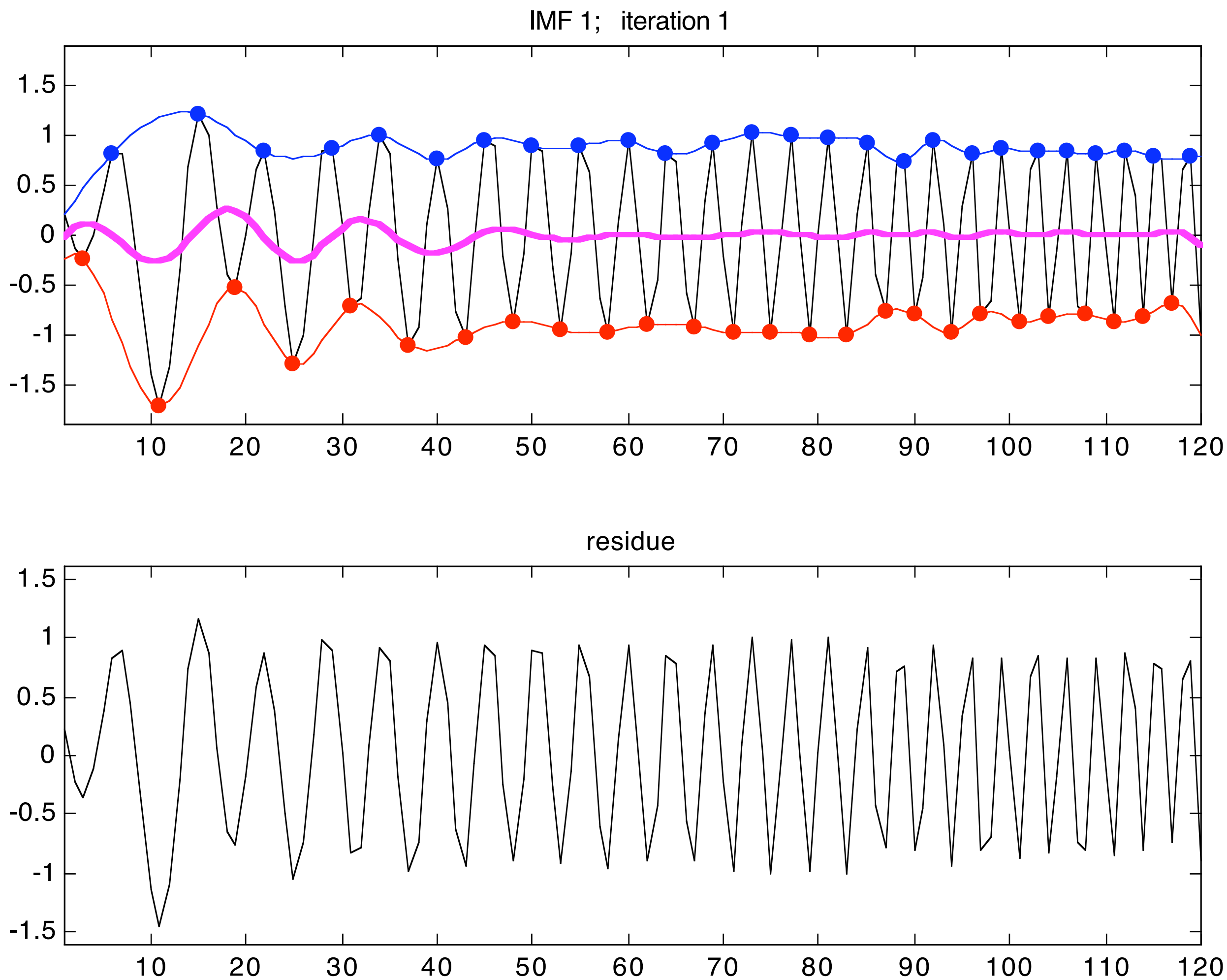
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residue
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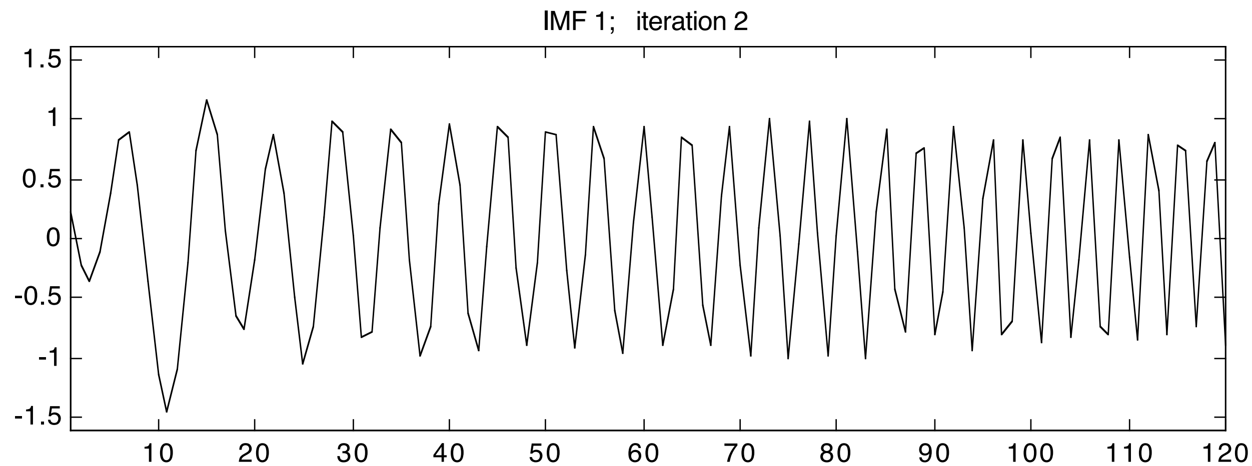
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-1.5
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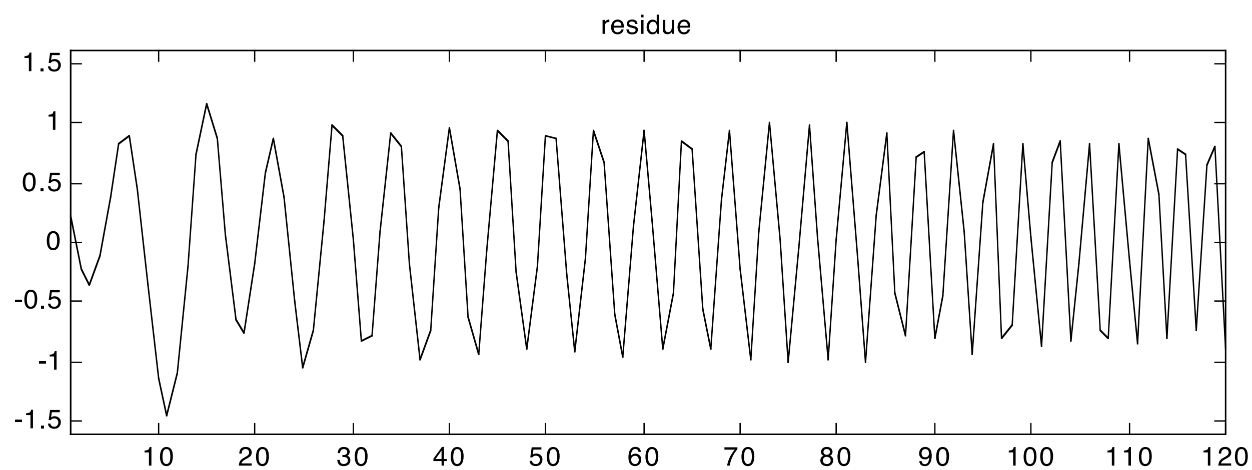
residue
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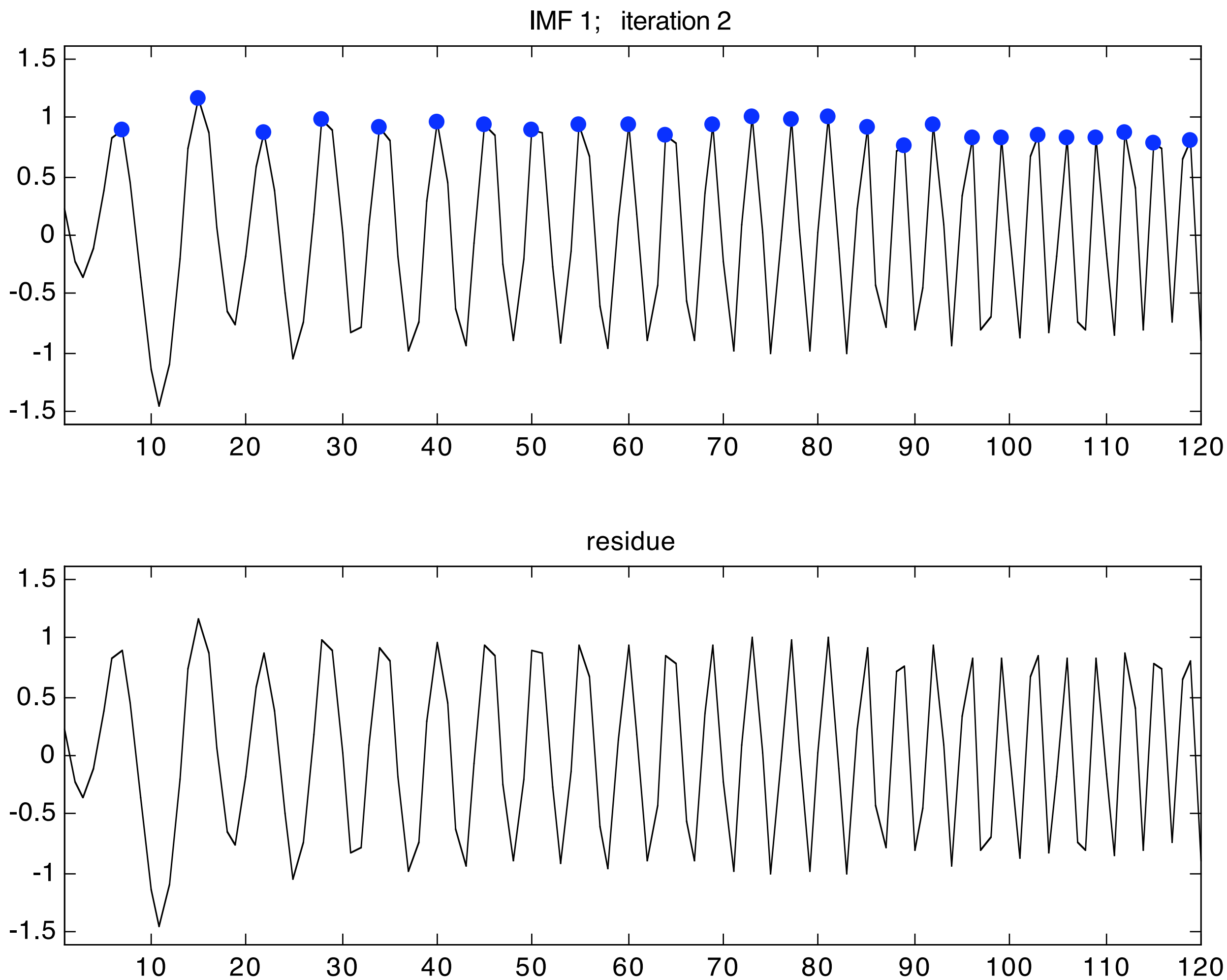
IMF 1;   iteration 2
1.5
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-1.5
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residue
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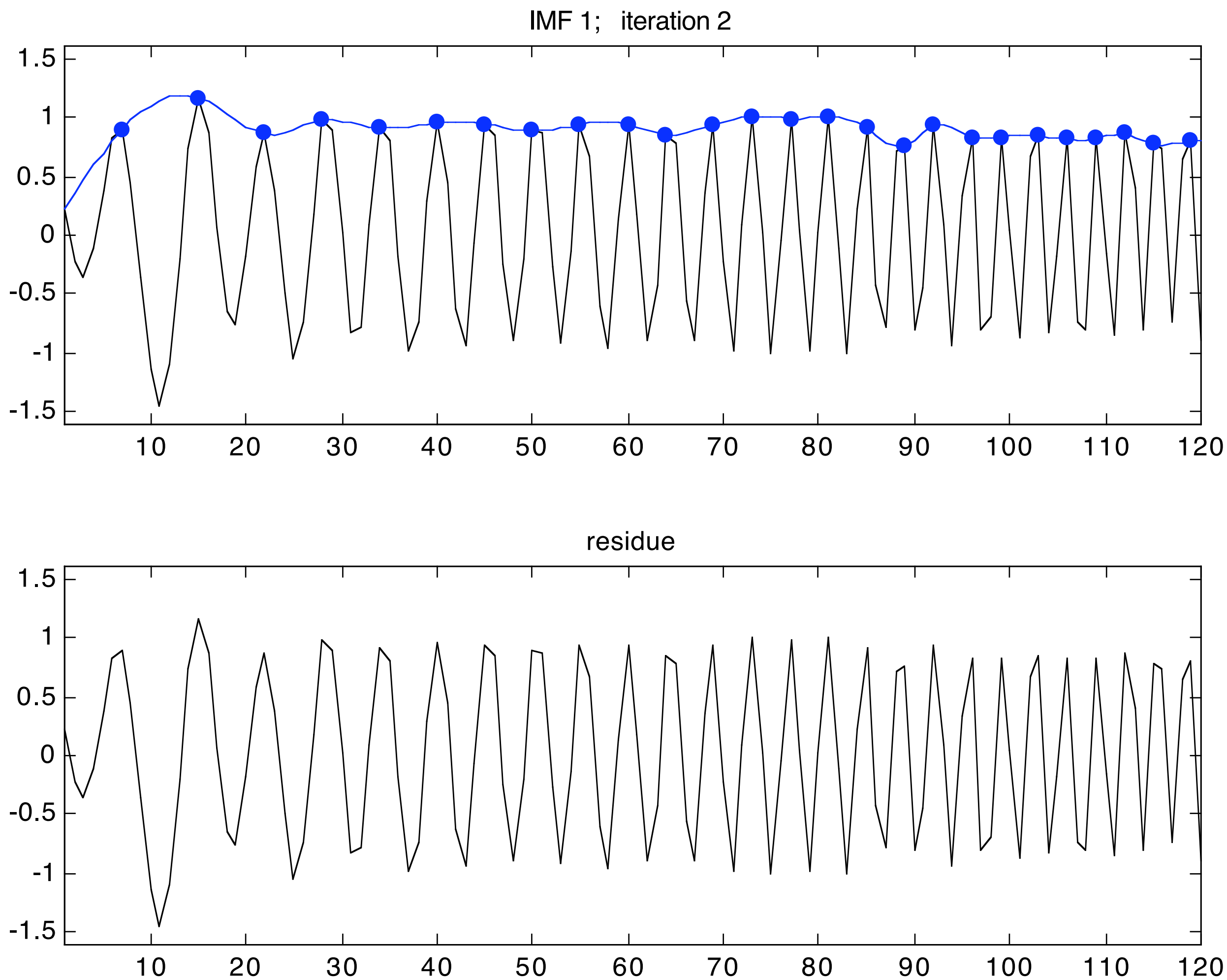
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-1
-1.5
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residue
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-1.5
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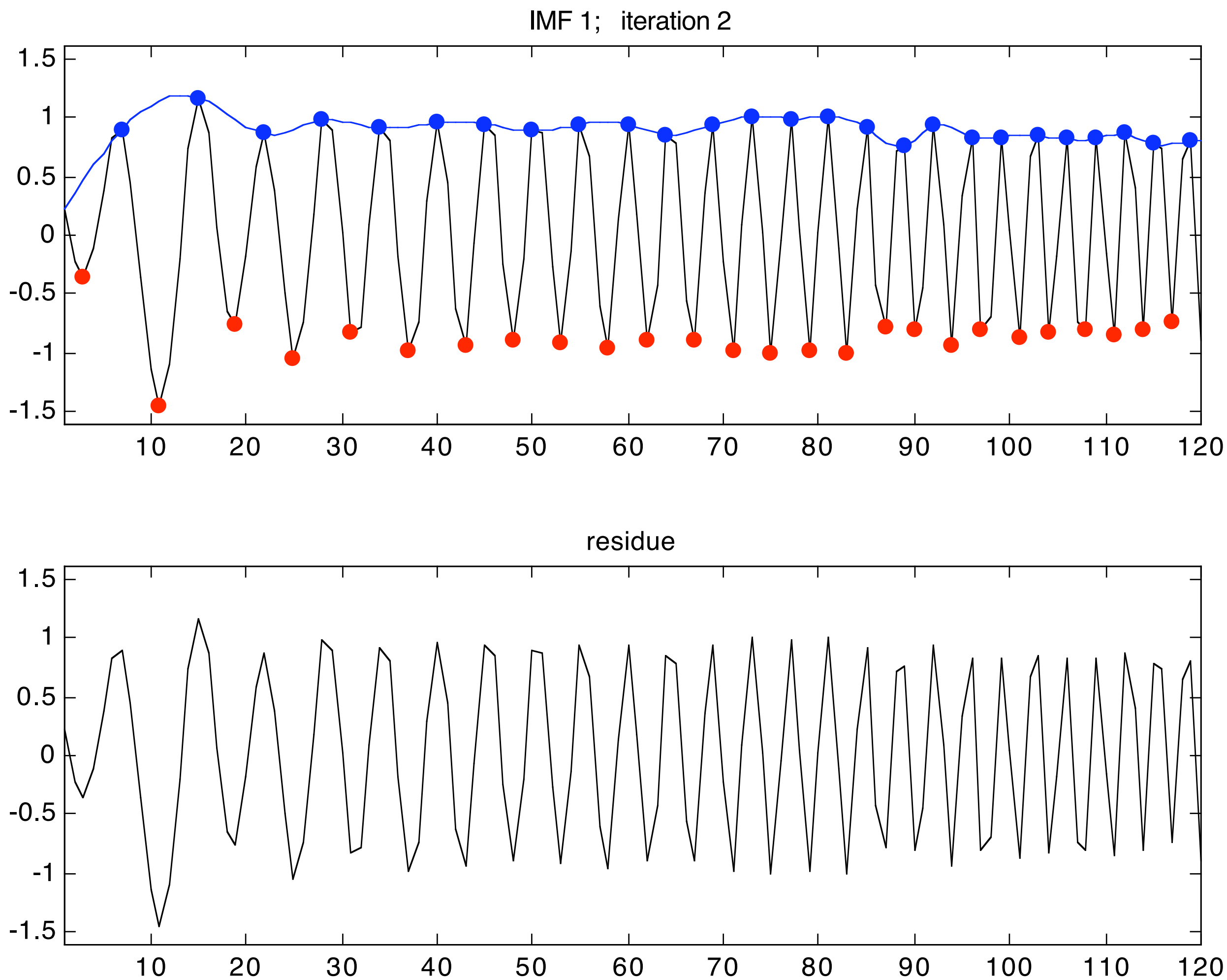
IMF 1; iteration 2
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-0.5
-1
-1.5
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100
110
120
residue
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-1
-1.5
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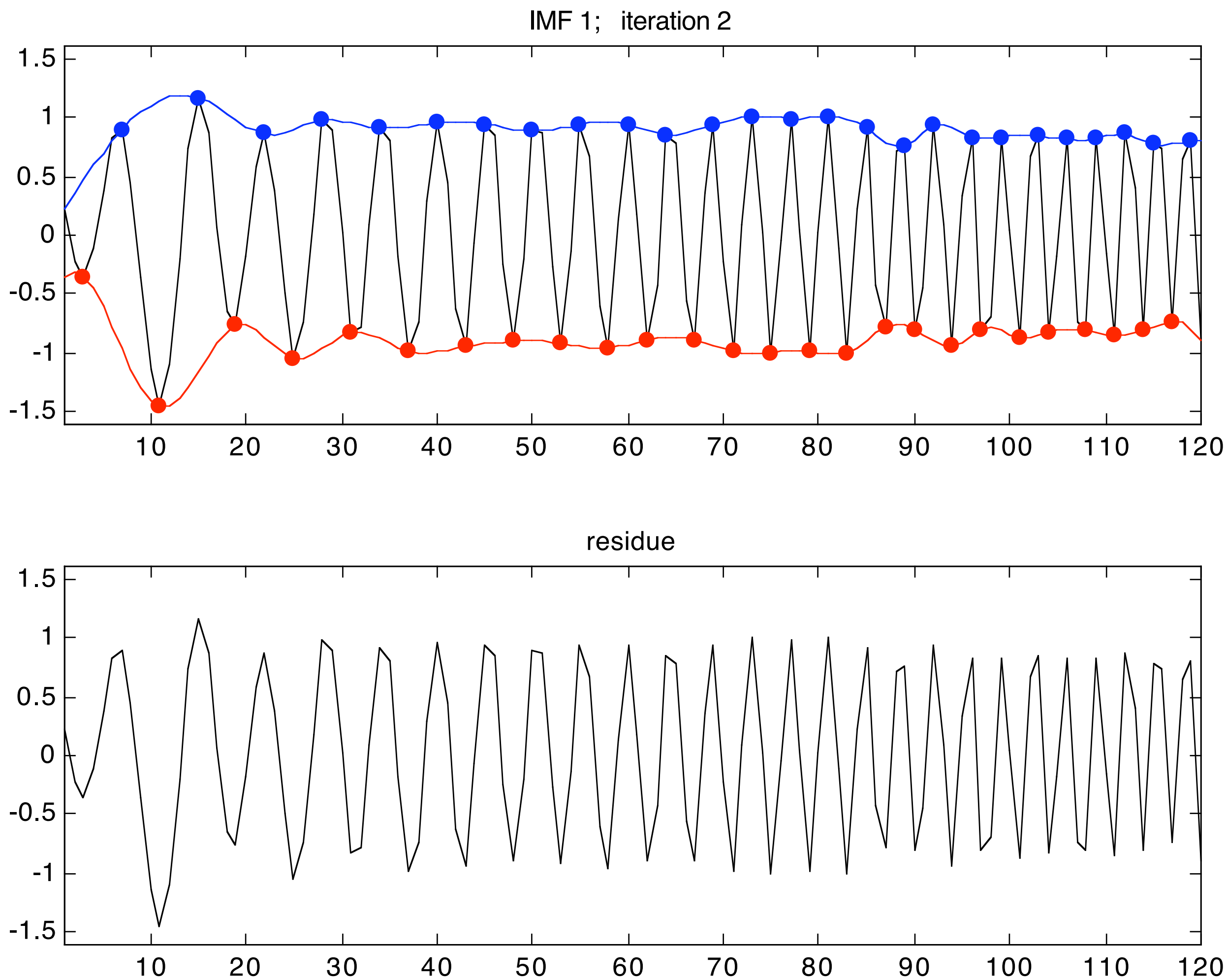
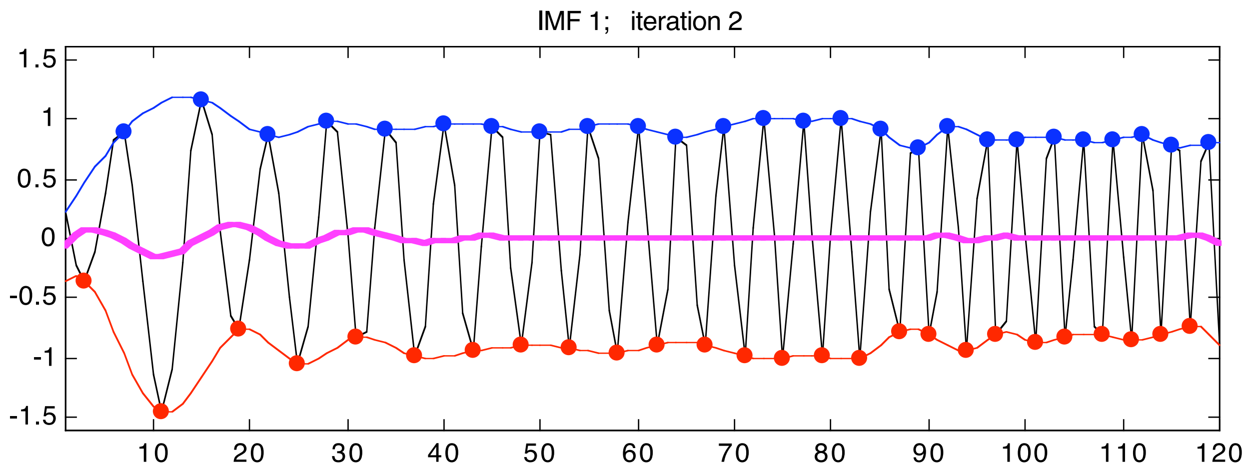

IMF 1;   iteration 2
1.5
1
0.5
0
-0.5
-1
-1.5
10 20 30 40 50 60 70 80 90 100 110 120
residue
1.5
1
0.5
0
-0.5
-1
-1.5
10 20 30 40 50 60 70 80 90 100 110 120



IMF 1;   iteration 2
1.5
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0.5
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-0.5
-1
-1.5
10
20
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50
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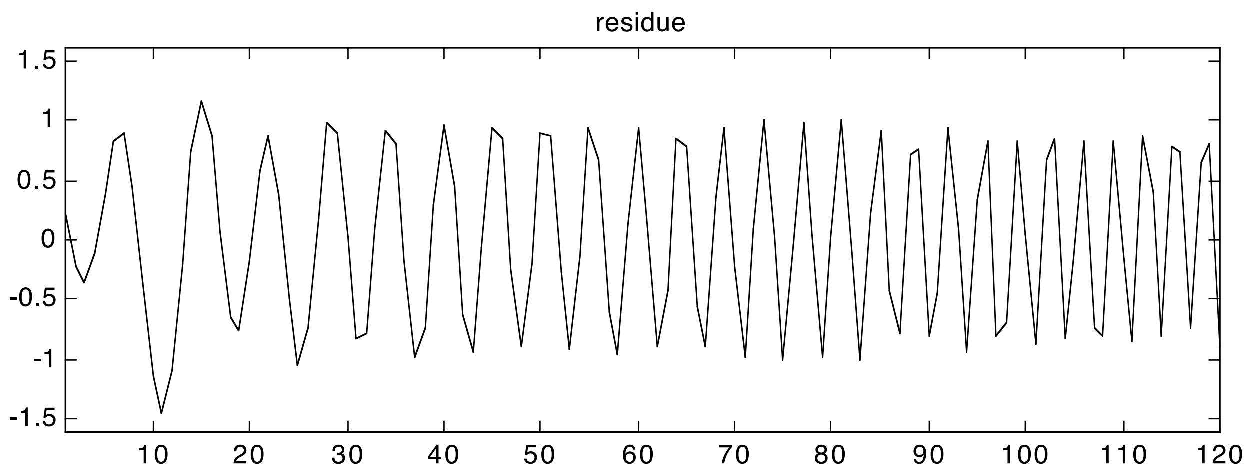
110
120



residue
1.5
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0
-0.5
-1
-1.5
10
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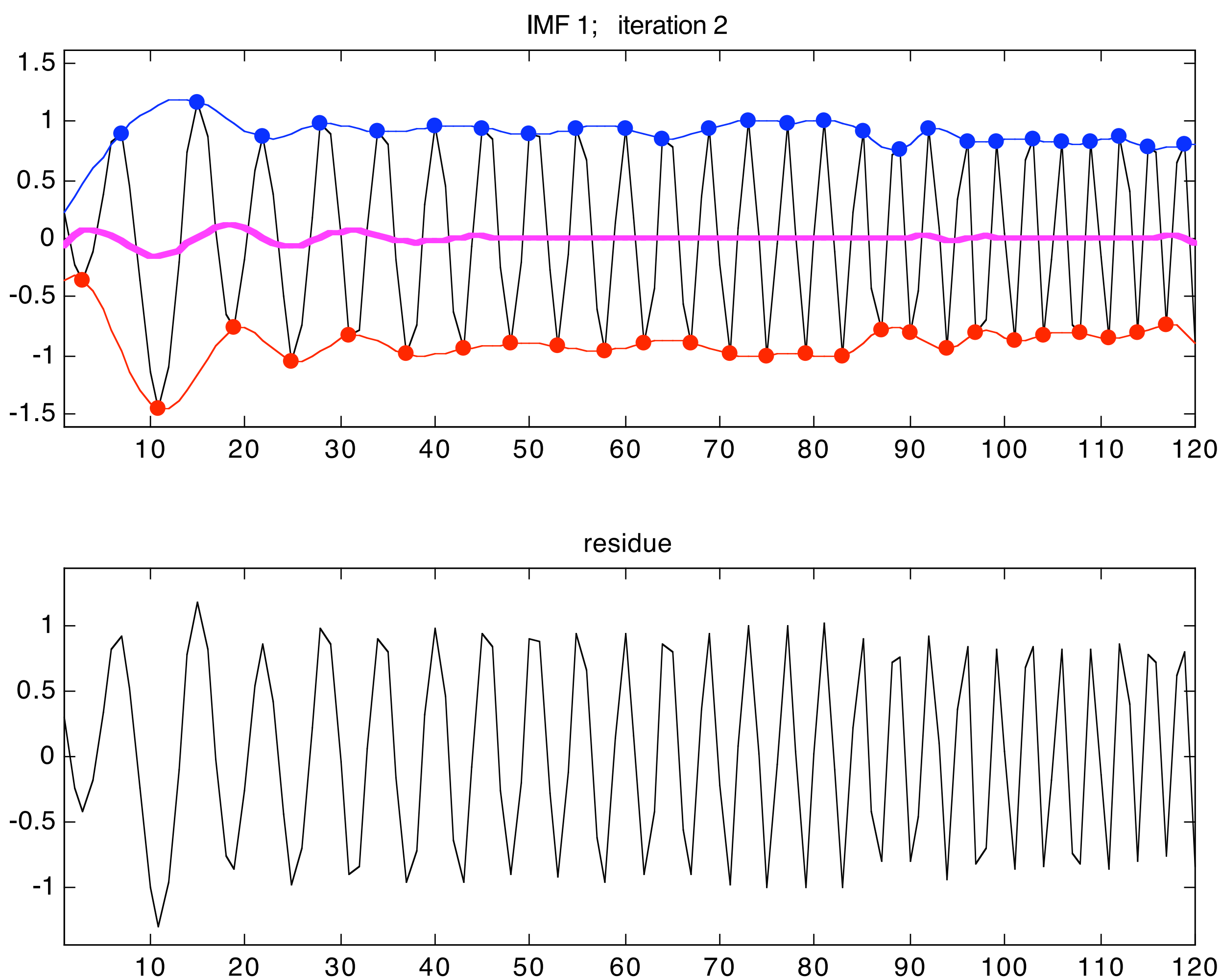
100
110
120


IMF 1;   iteration 2
1.5
1
0.5
0
-0.5
-1
-1.5
10
20
30
40
50
60
70
80
90
100
110
120
residue
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0.5
0
-0.5
-1
10
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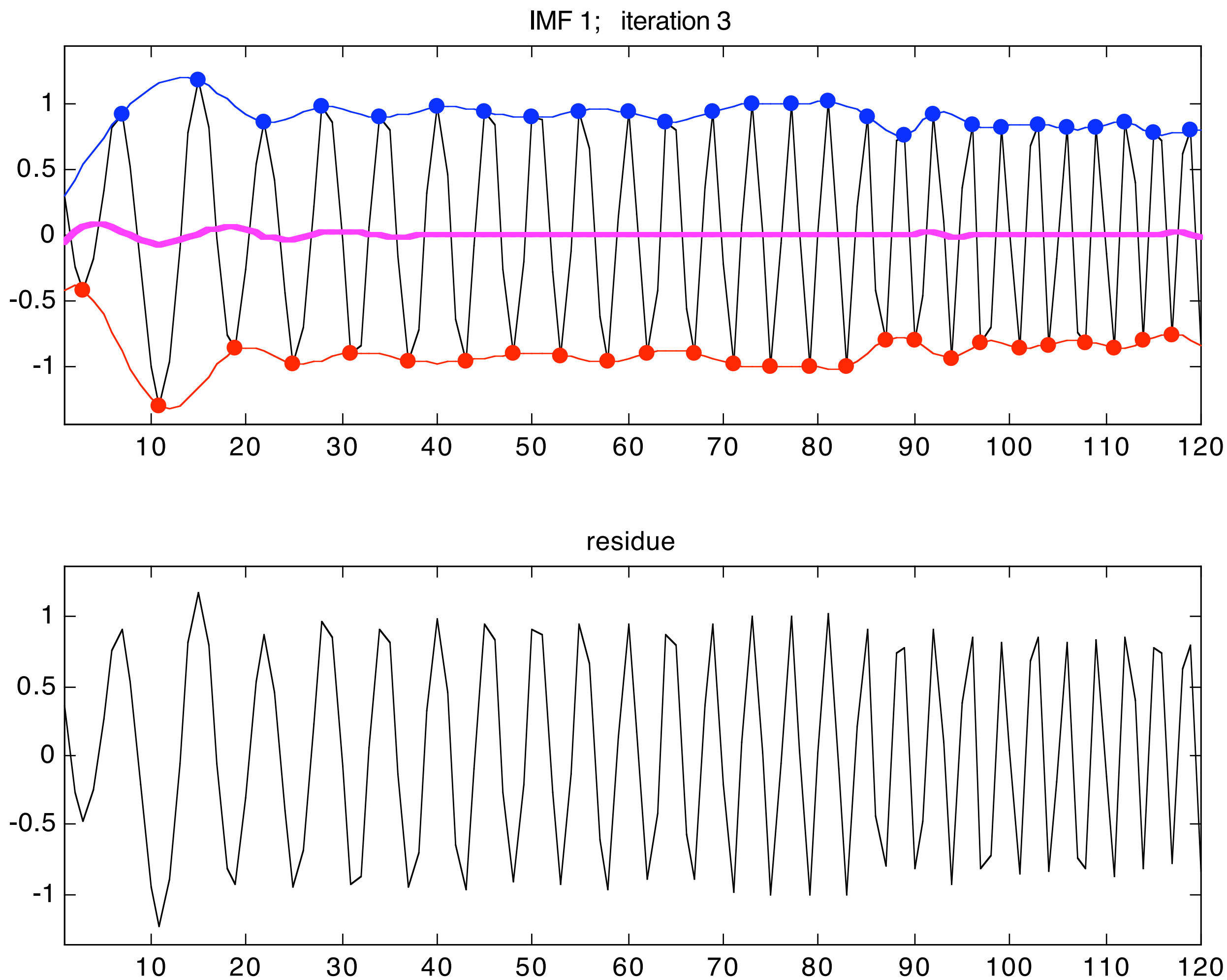
110
120


IMF 1; iteration 3
1
0.5
0
-0.5
-1
10 20 30 40 50 60 70 80 90 100 110 120
residue
1
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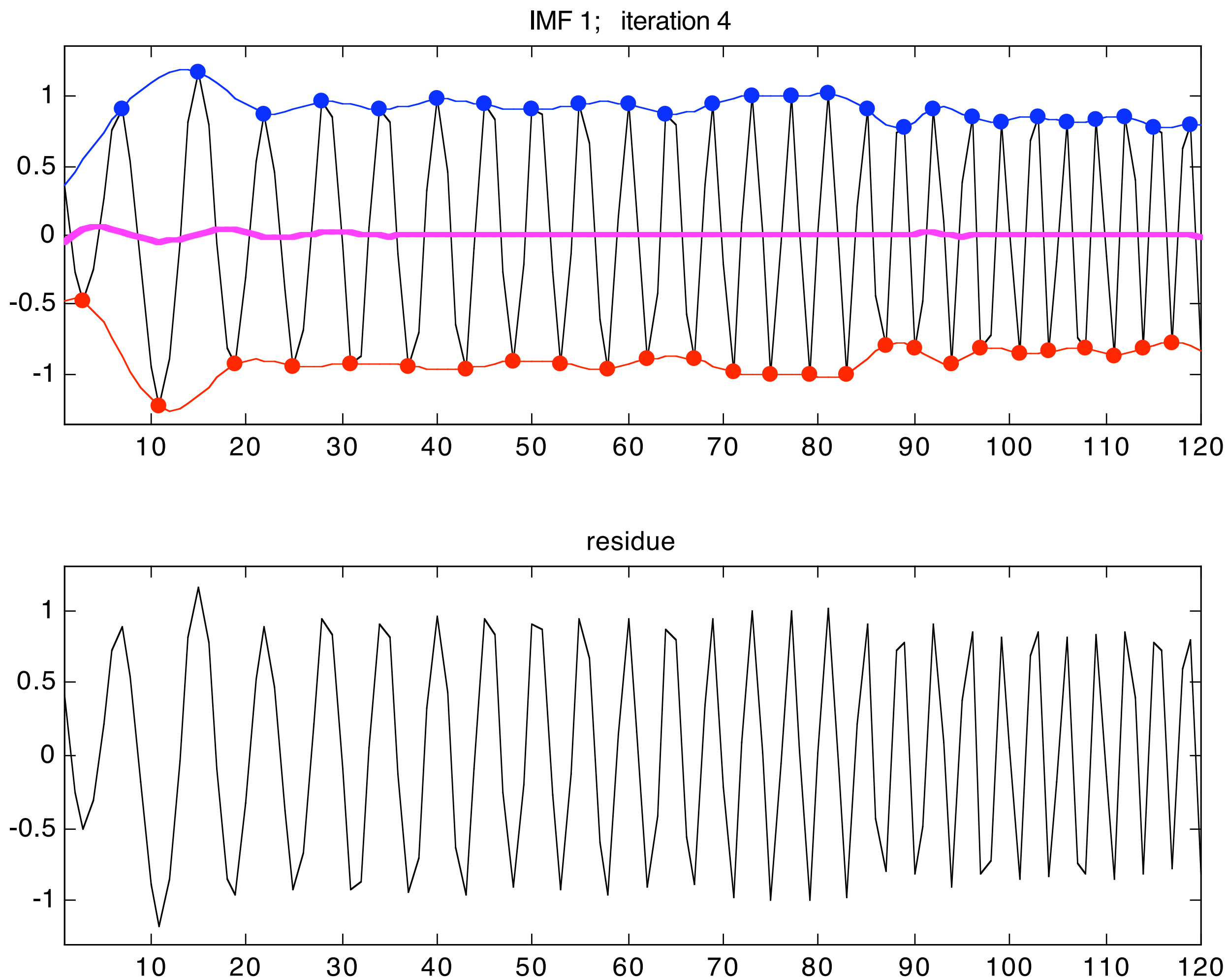
0
-0.5
-1
10 20 30 40 50 60 70 80 90 100 110 120


IMF 1; iteration 4
1
0.5
0
-0.5
-1
10
20
30
40
50
60
70
80
90
100
110
120
residue
1
0.5
0
-0.5
-1
10
20
30
40
50
60
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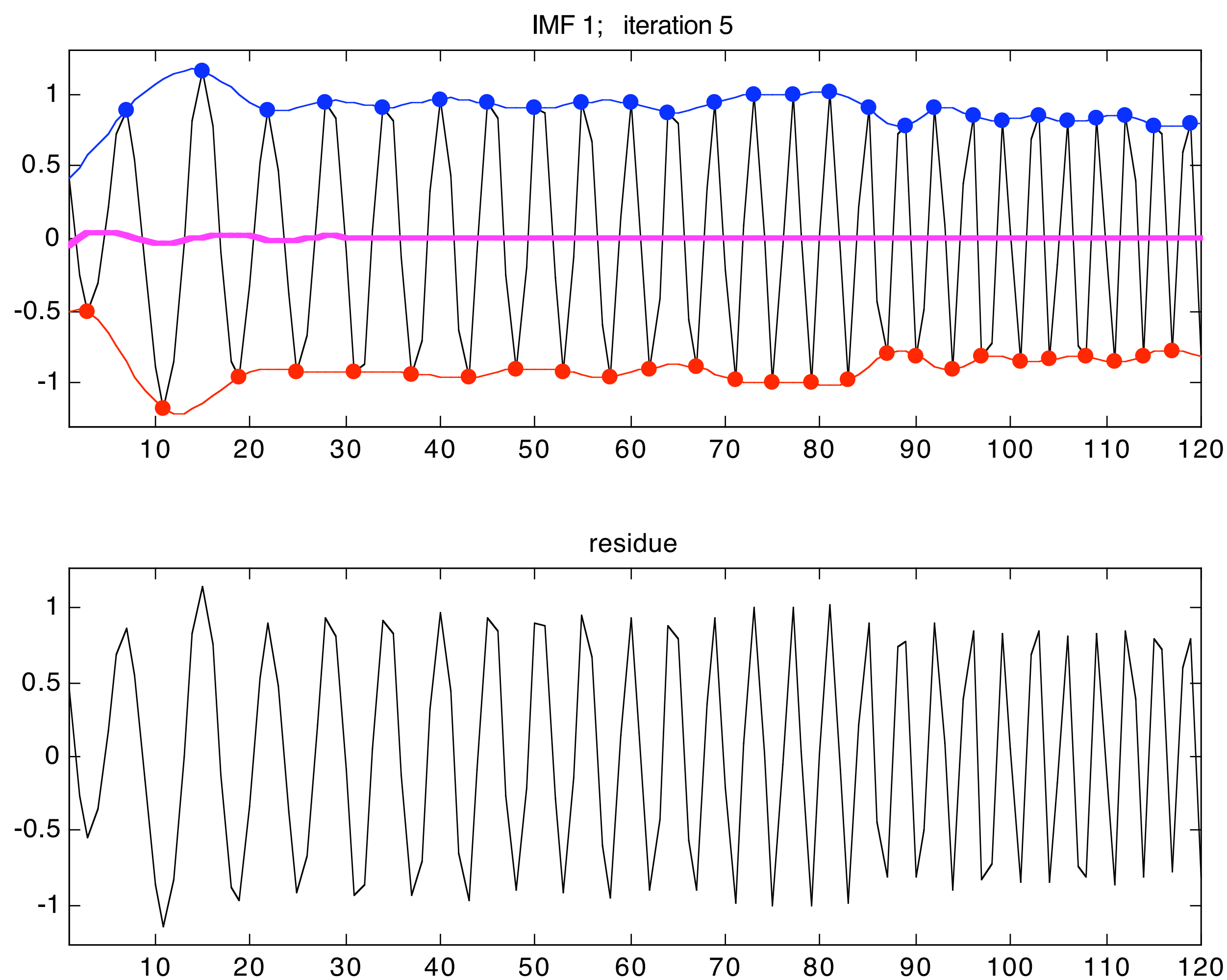
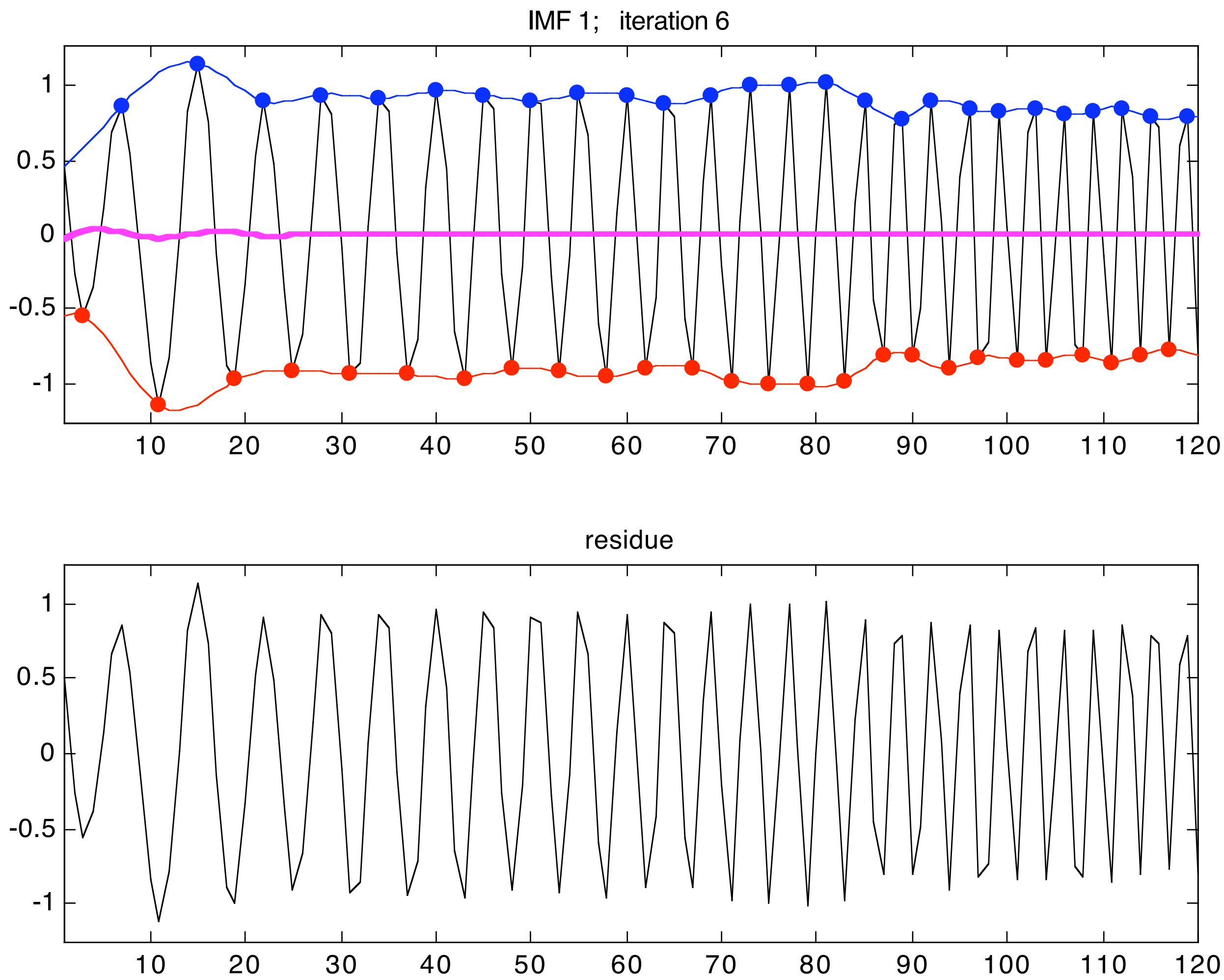
90
100
110
120


IMF 1; iteration 5
1
0.5
0
-0.5
-1
10
20
30
40
50
60
70
80
90
100
110
120
residue
1
0.5
0
-0.5
-1
10
20
30
40
50
60
70
80
90
100
110
120


IMF 1; iteration 6
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0.5
0
-0.5
-1
10
20
30
40
50
60
70
80
90
100
110
120
residue
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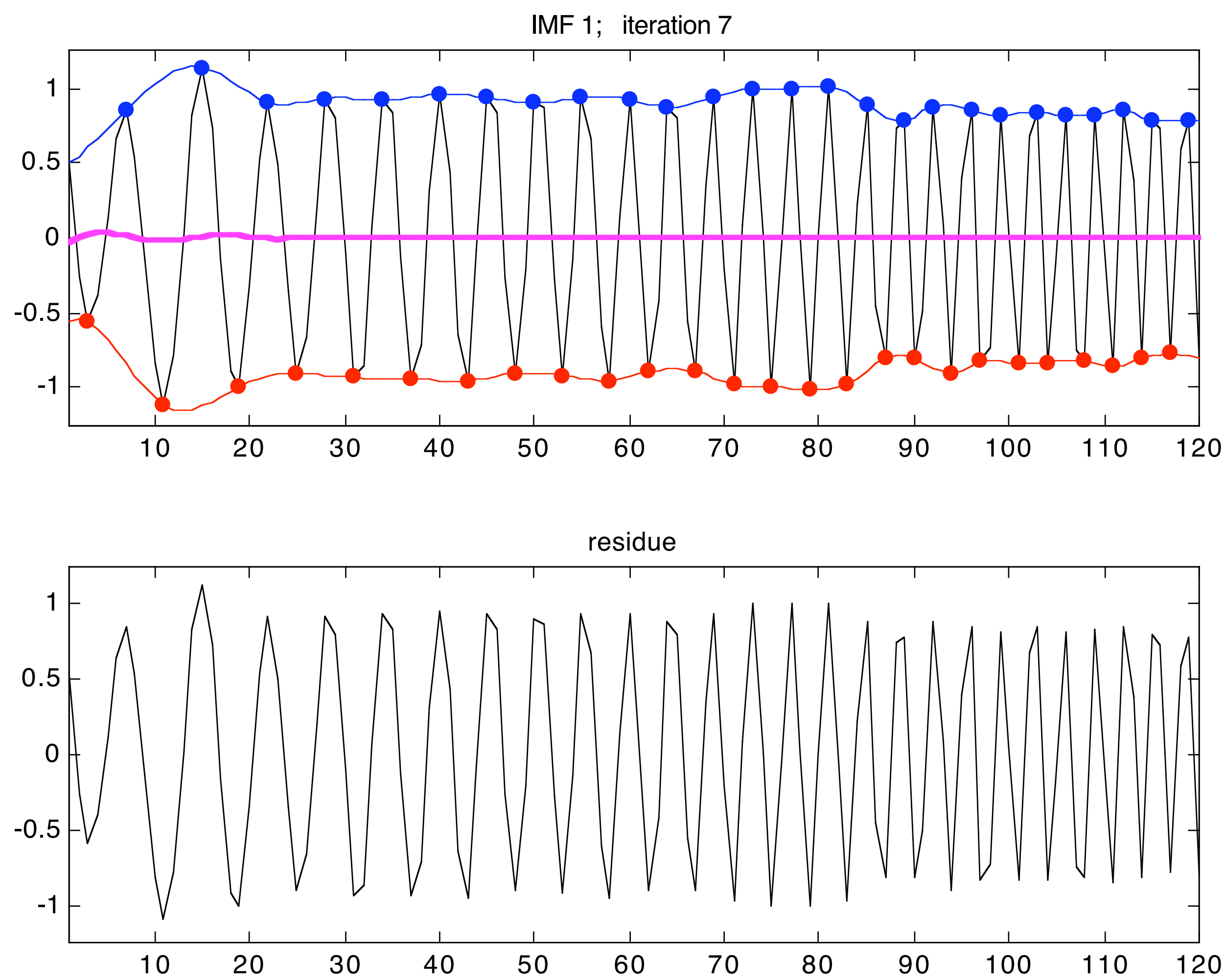
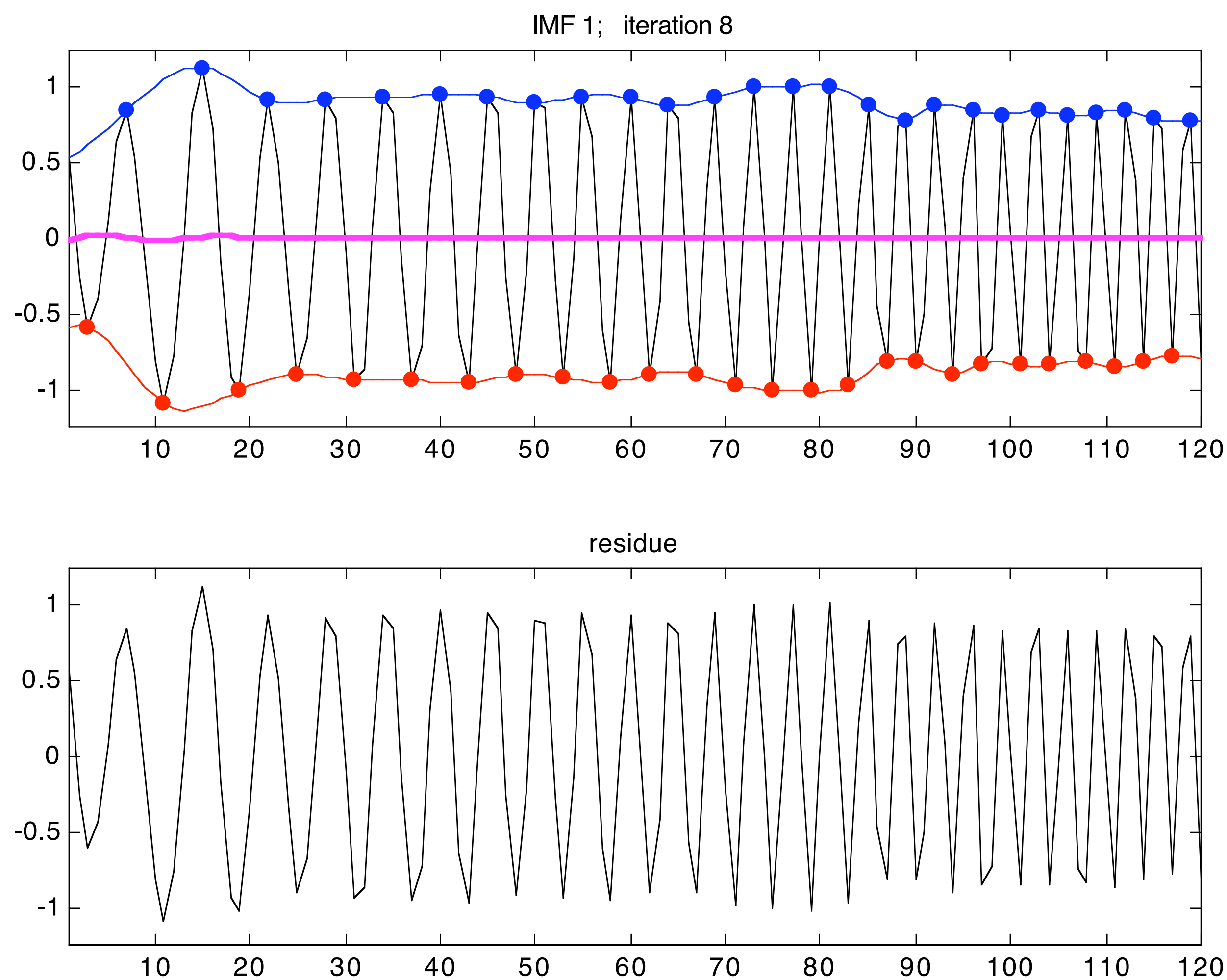
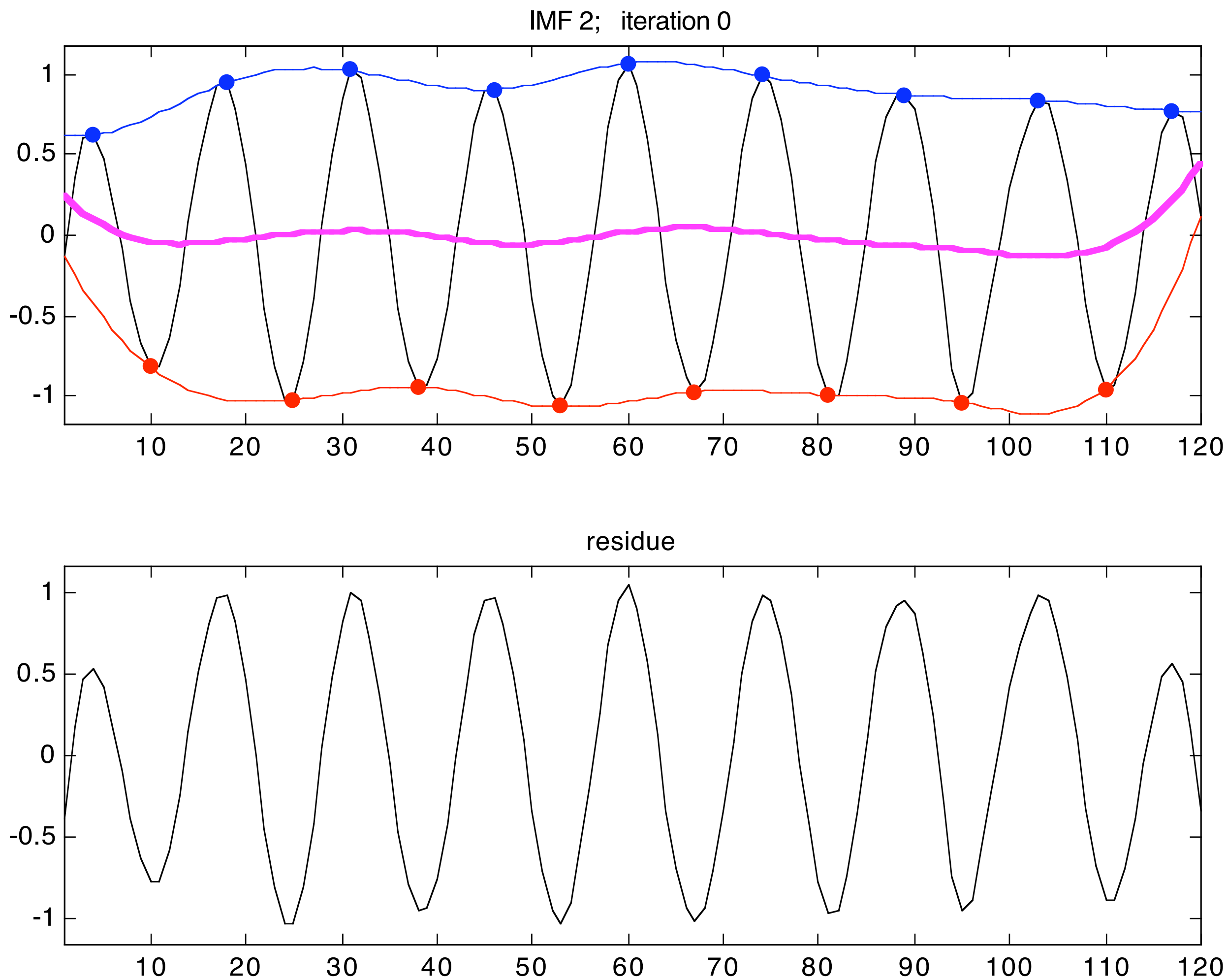
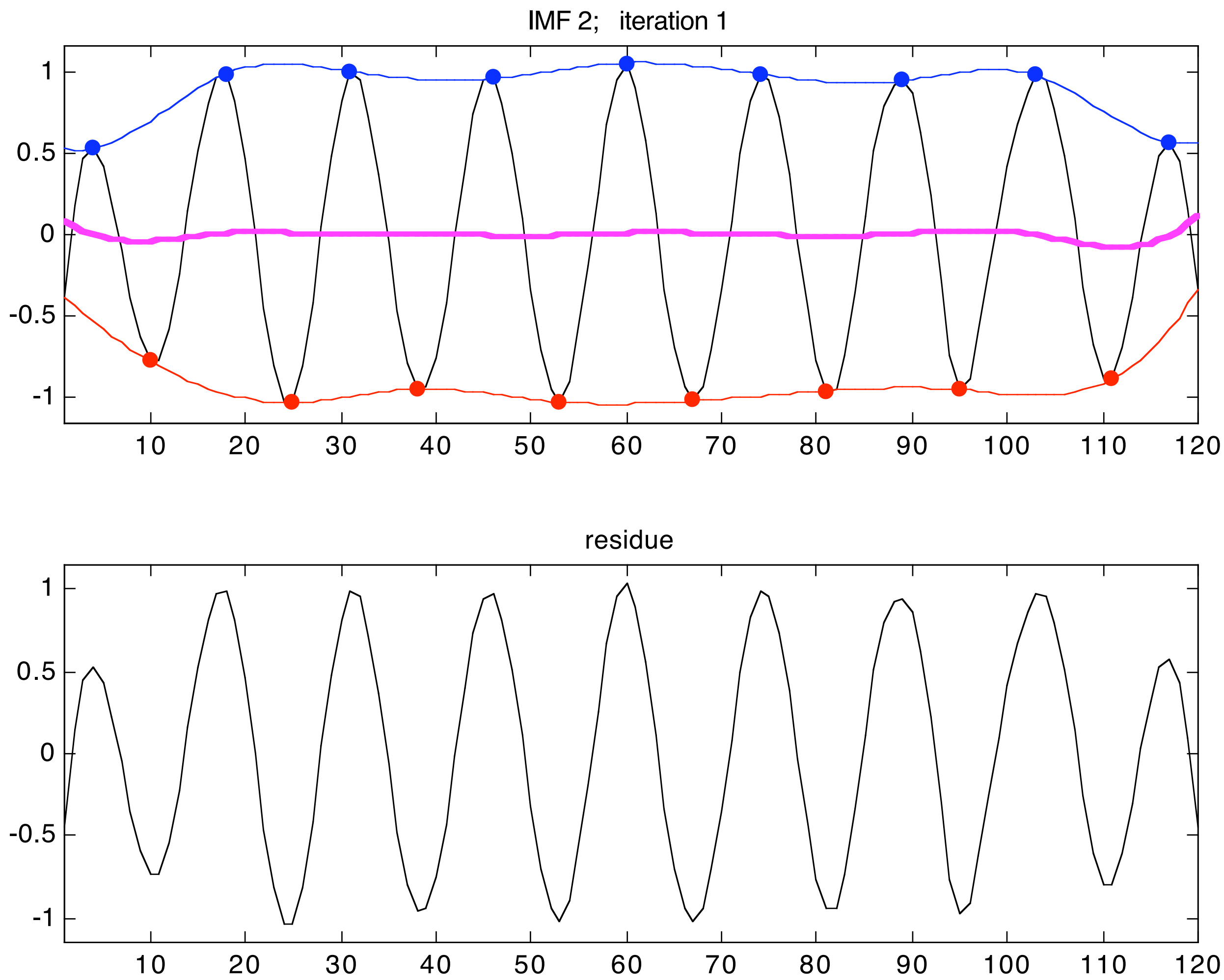
0.5
0
-0.5
-1
10
20
30
40
50
60
70
80
90
100
110
120

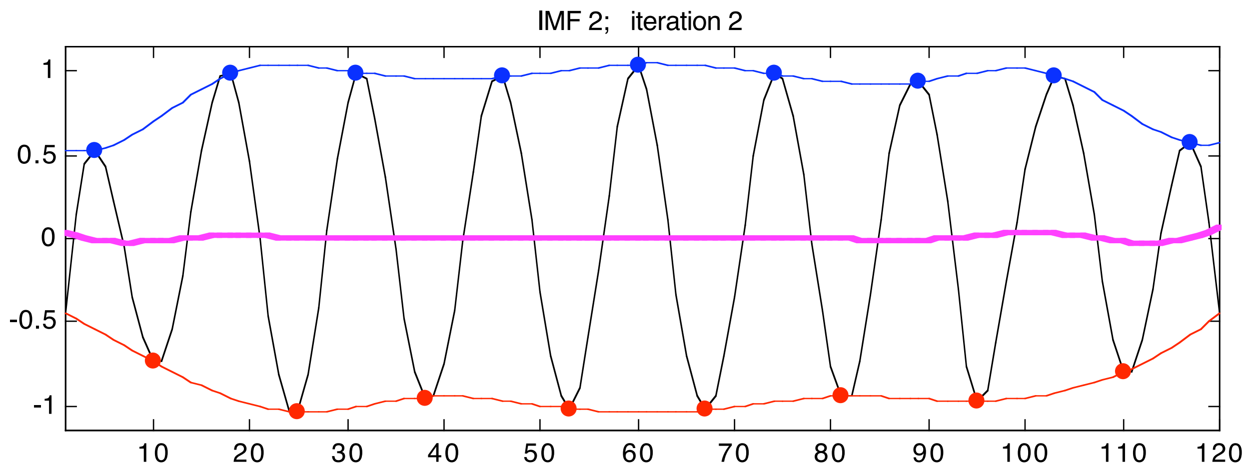

IMF 1;   iteration 7
1
0.5
0
-0.5
-1
10
20
30
40
50
60
70
80
90
100
110
120
residue
1
0.5
0
-0.5
-1
10
20
30
40
50
60
70
80
90
100
110
120


IMF 1;   iteration 8
1
0.5
0
-0.5
-1
10
20
30
40
50
60
70
80
90
100
110
120
residue
1
0.5
0
-0.5
-1
10
20
30
40
50
60
70
80
90
100
110
120



IMF 2; iteration 0
1
0.5
0
-0.5
-1
10 20 30 40 50 60 70 80 90 100 110 120
residue
1
0.5
0
-0.5
-1
10 20 30 40 50 60 70 80 90 100 110 120


IMF 2; iteration 1
1
0.5
0
-0.5
-1
10
20
30
40
50
60
70
80
90
100
110
120
residue
1
0.5
0
-0.5
-1
10
20
30
40
50
60
70
80
90
100
110
120


IMF 2; iteration 2
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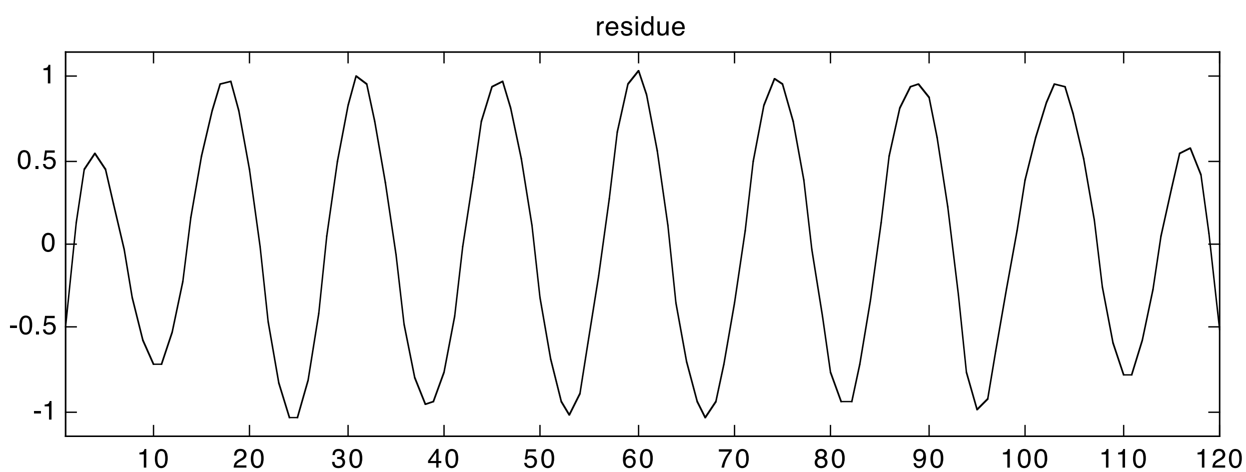
0
-0.5
-1
10
20
30
40
50
60
70
80
90
100
110
120


residue
1
0.5
0
-0.5
-1
10
20
30
40
50
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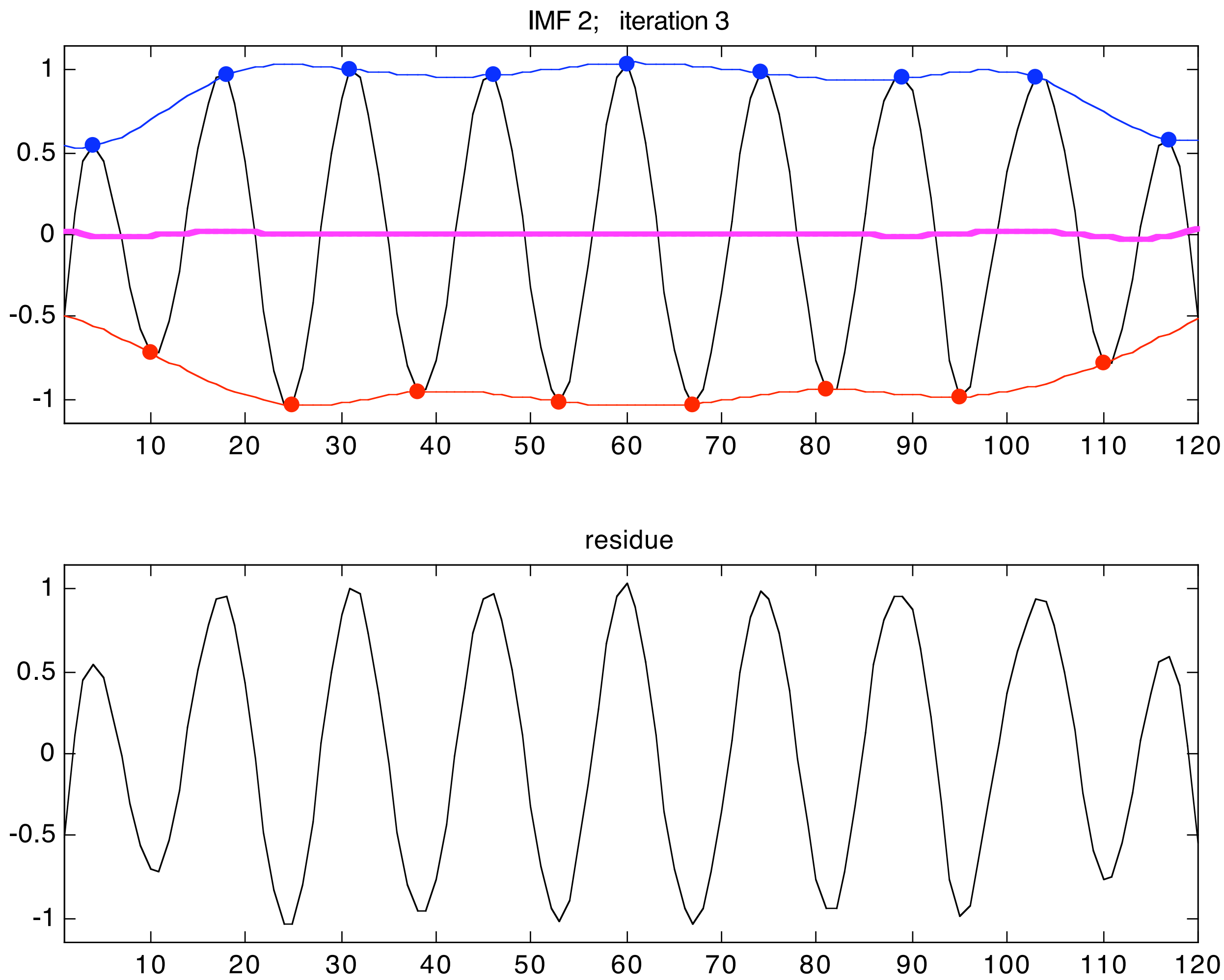
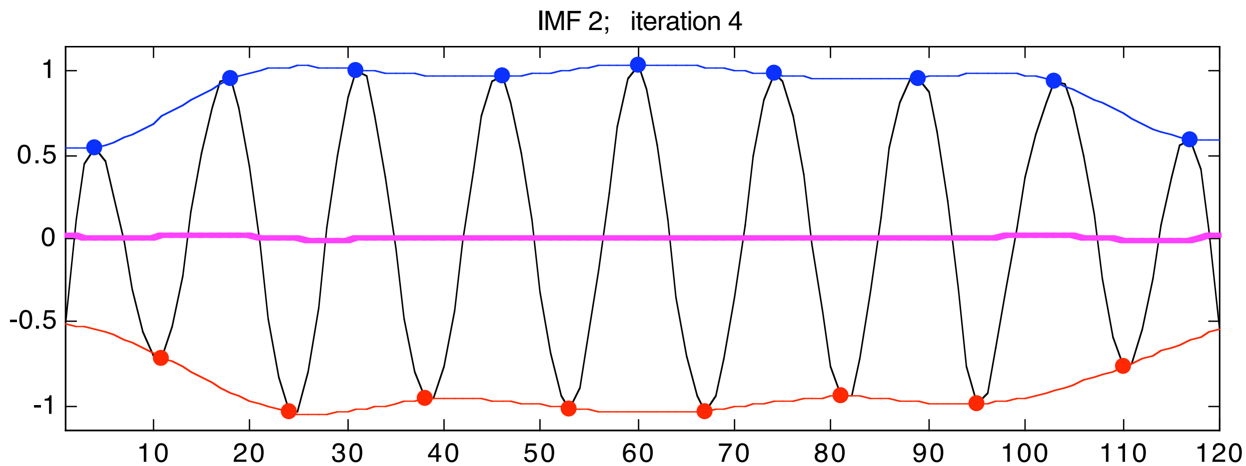
110
120



IMF 2; iteration 3
1
0.5
0
-0.5
-1
10
20
30
40
50
60
70
80
90
100
110
120
residue
1
0.5
0
-0.5
-1
10
20
30
40
50
60
70
80
90
100
110
120


IMF 2; iteration 4
1
0.5
0
-0.5
-1
10
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30
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60
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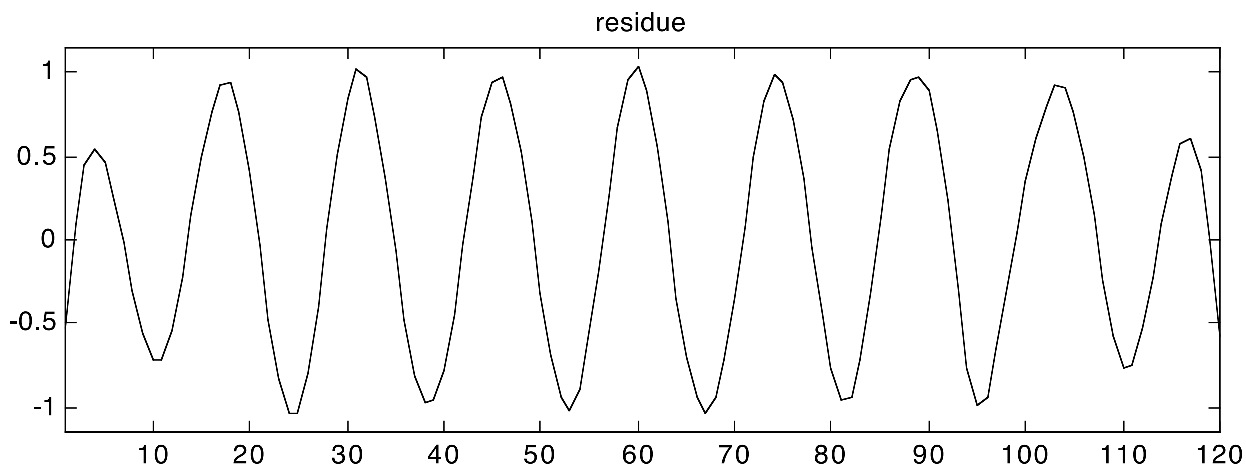
90
100
110
120


residue
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0
-0.5
-1
10
20
30
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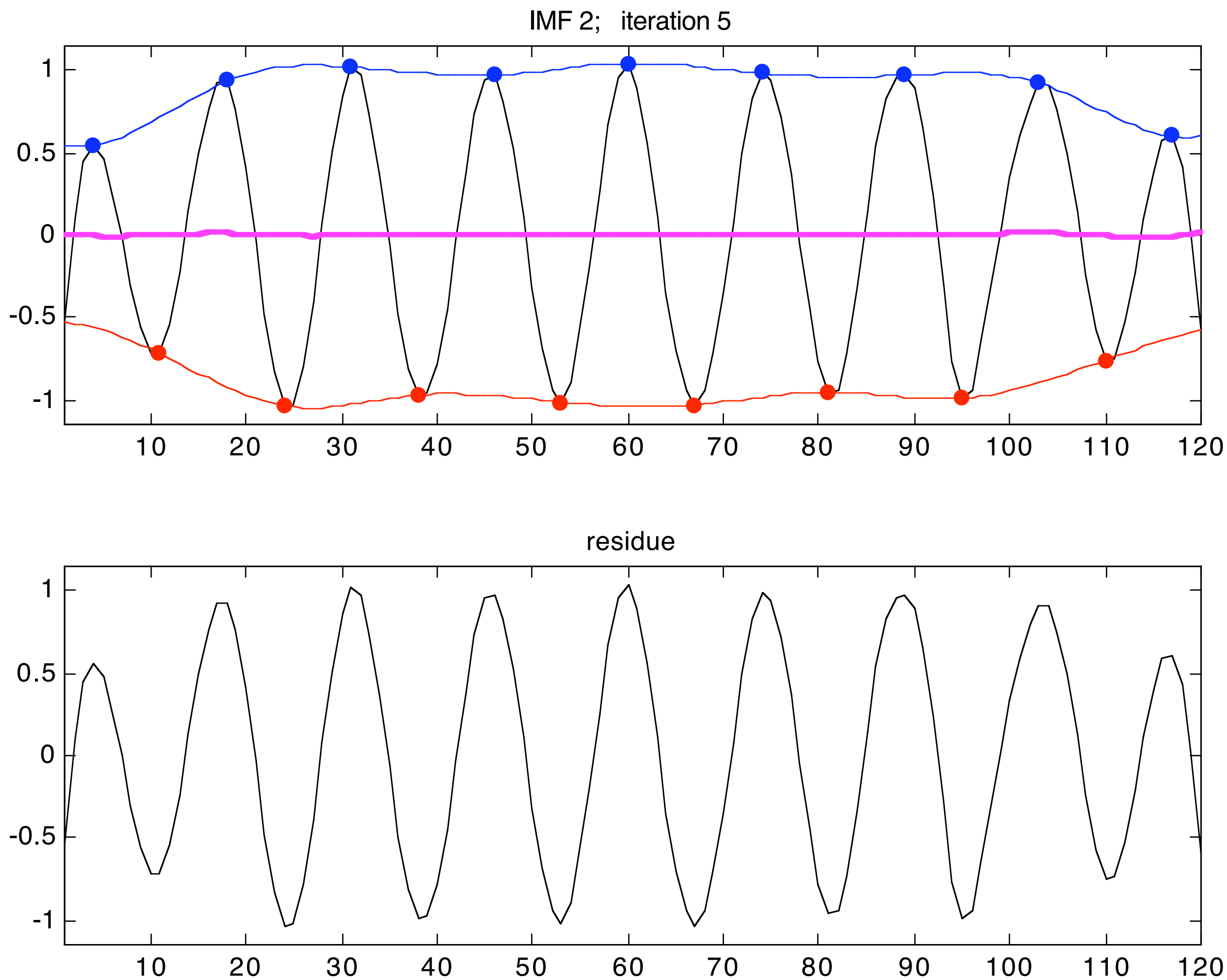
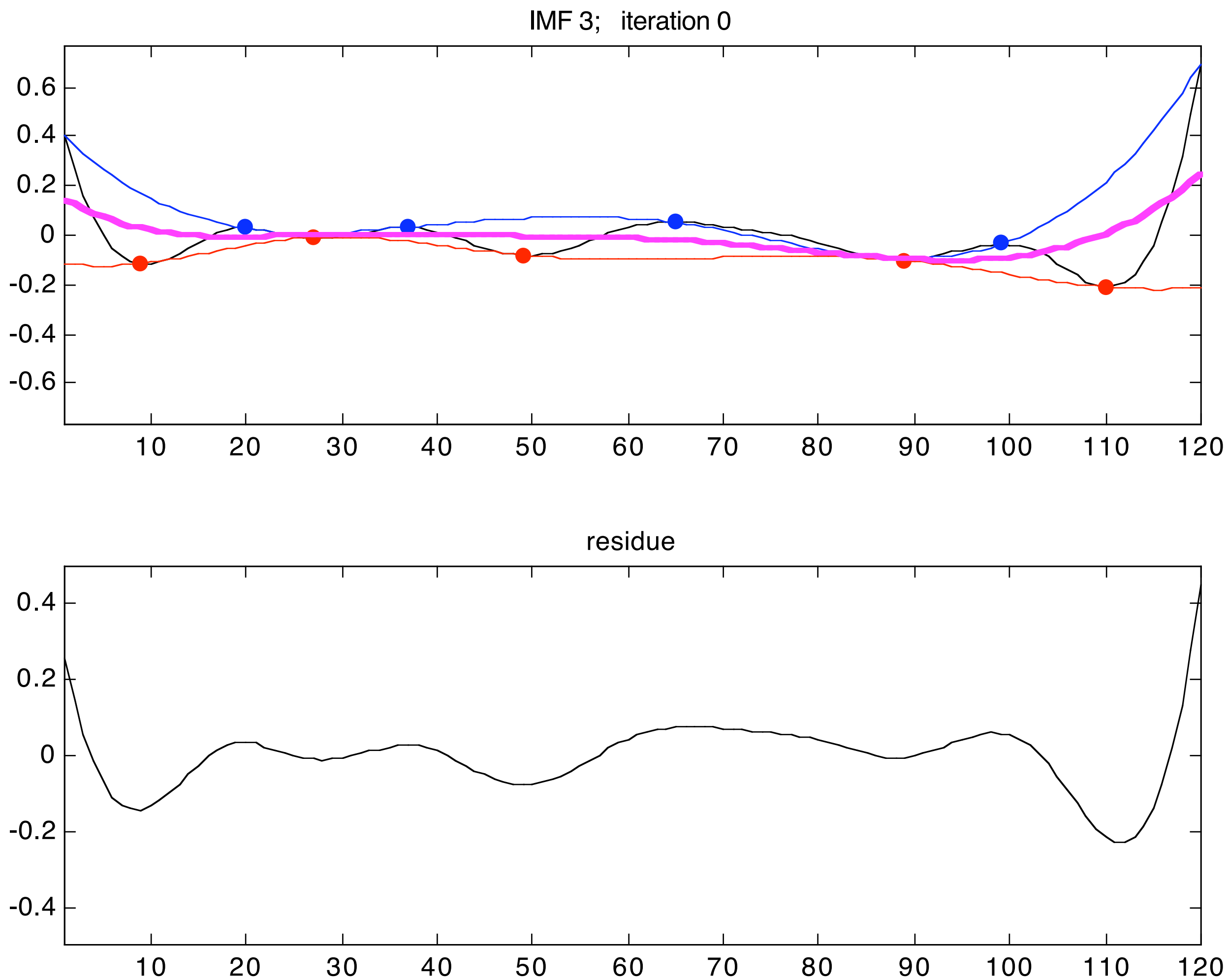
110
120


IMF 2;   iteration 5
1
0.5
0
-0.5
-1
10
20
30
40
50
60
70
80
90
100
110
120
residue
1
0.5
0
-0.5
-1
10
20
30
40
50
60
70
80
90
100
110
120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0.4
0.2
0
-0.2
-0.4
-0.6
10
20
30
40
50
60
70
80
90
100
110
120
residue
0.4
0.2
0
-0.2
-0.4
10
20
30
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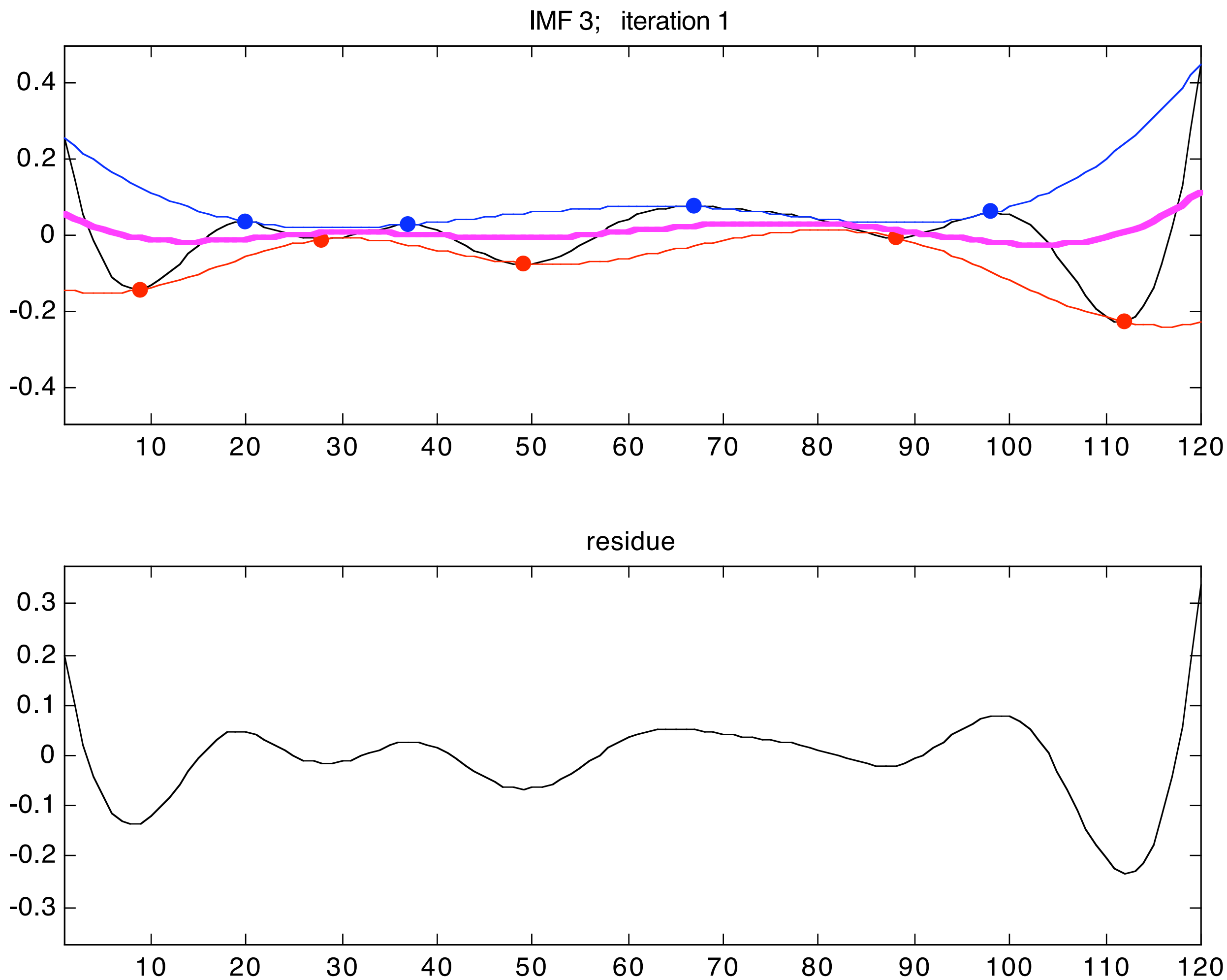
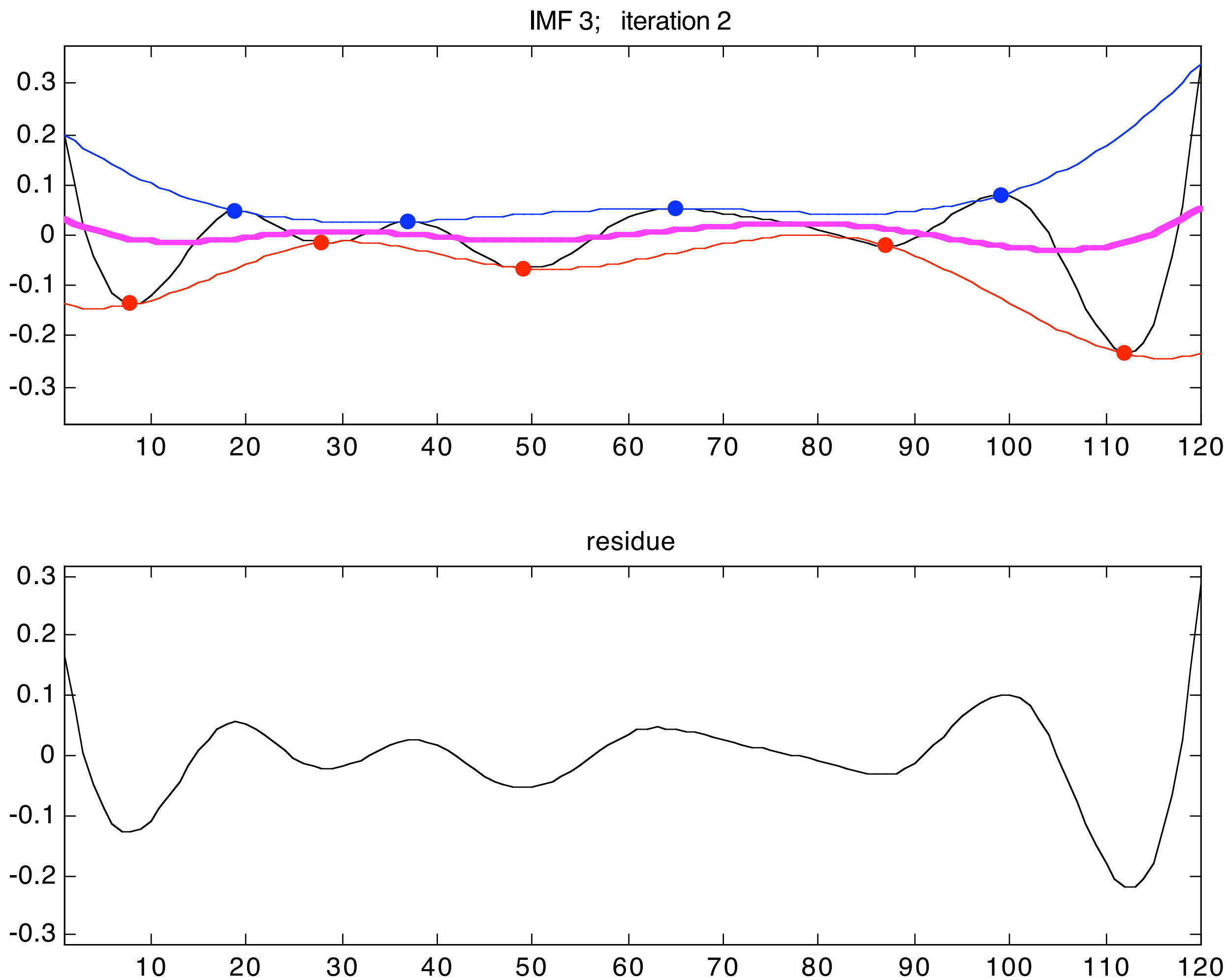
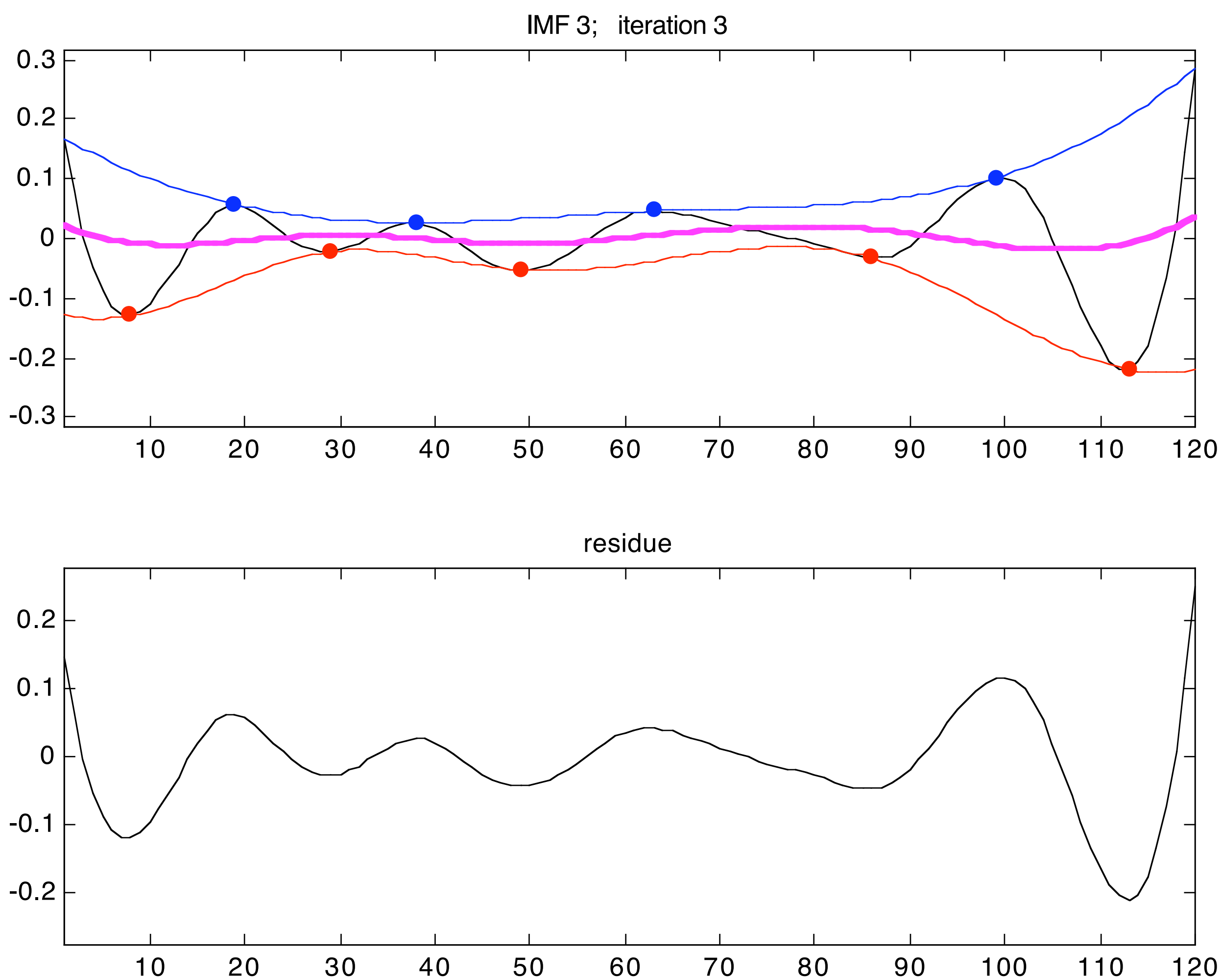
110
120

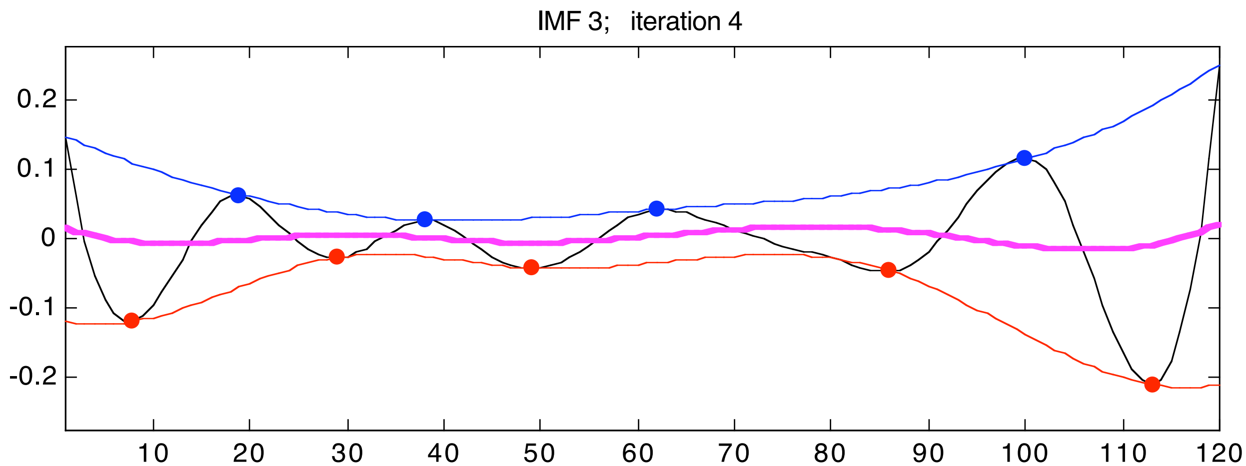

IMF 3; iteration 1
0.4
0.2
0
-0.2
-0.4
10 20 30 40 50 60 70 80 90 100 110 120
residue
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
10 20 30 40 50 60 70 80 90 100 110 120

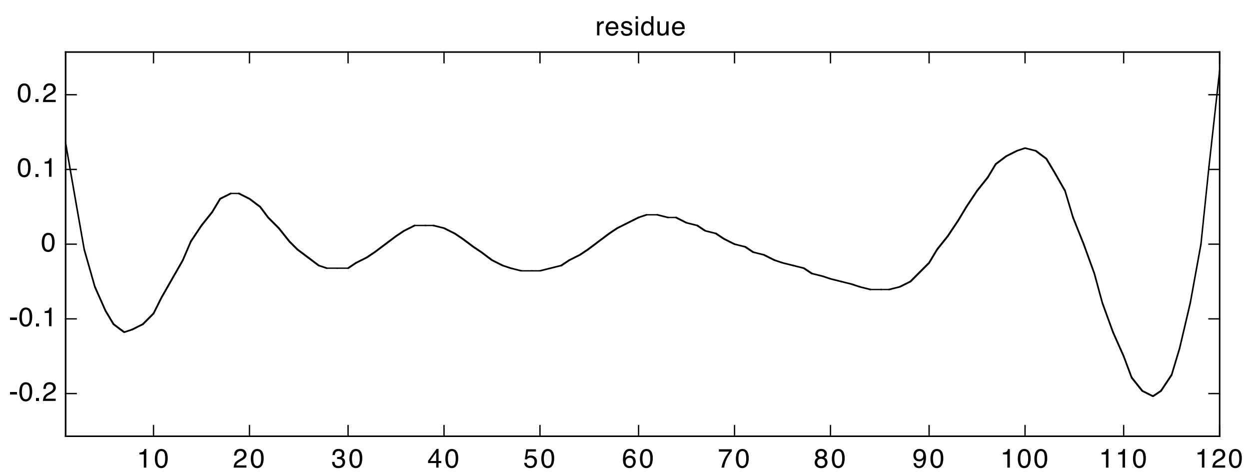


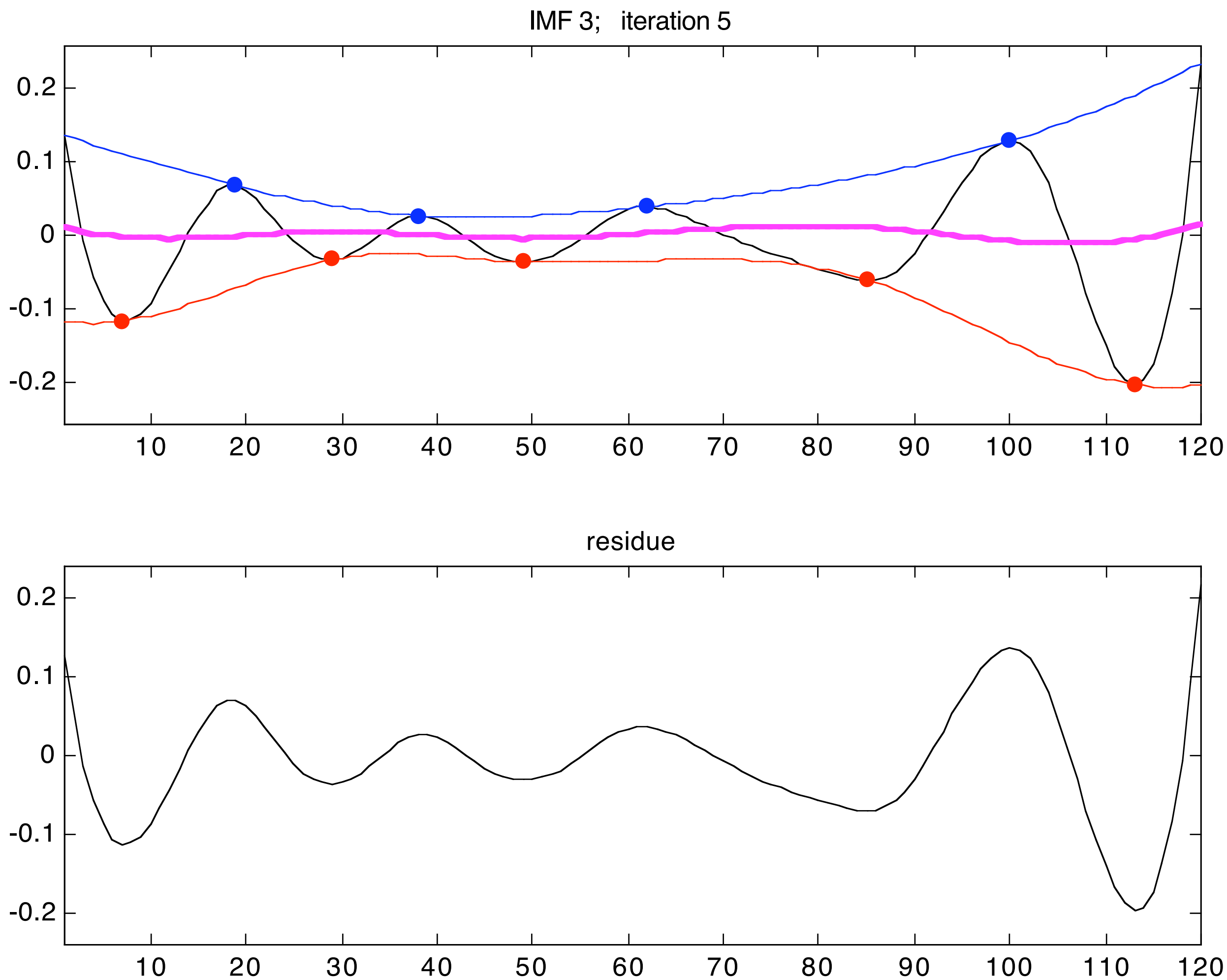
IMF 3; iteration 2
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
10
20
30
40
50
60
70
80
90
100
110
120
residue
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
10
20
30
40
50
60
70
80
90
100
110
120

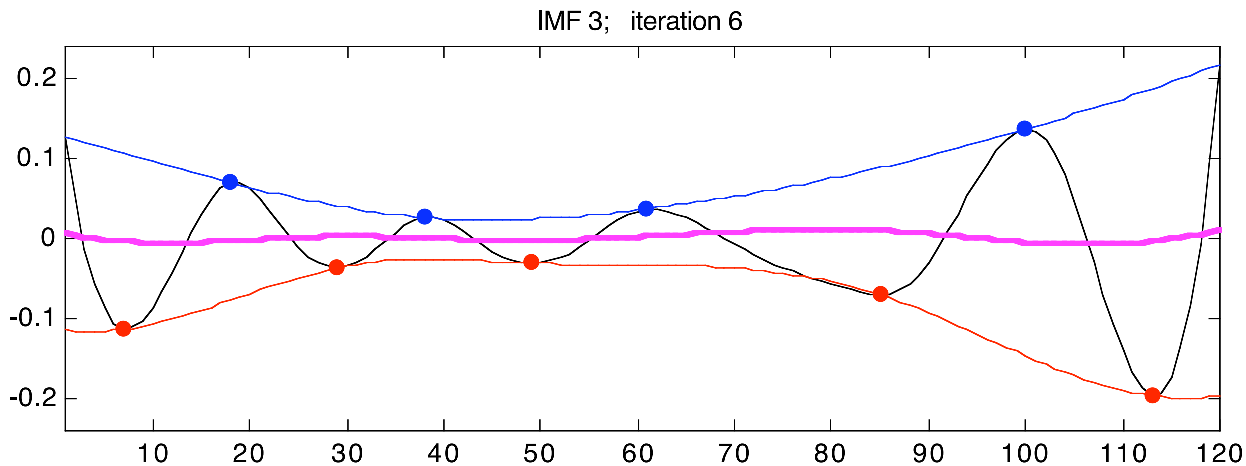


IMF 3; iteration 3
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
10
20
30
40
50
60
70
80
90
100
110
120
residue
0.2
0.1
0
-0.1
-0.2
10
20
30
40
50
60
70
80
90
100
110
120

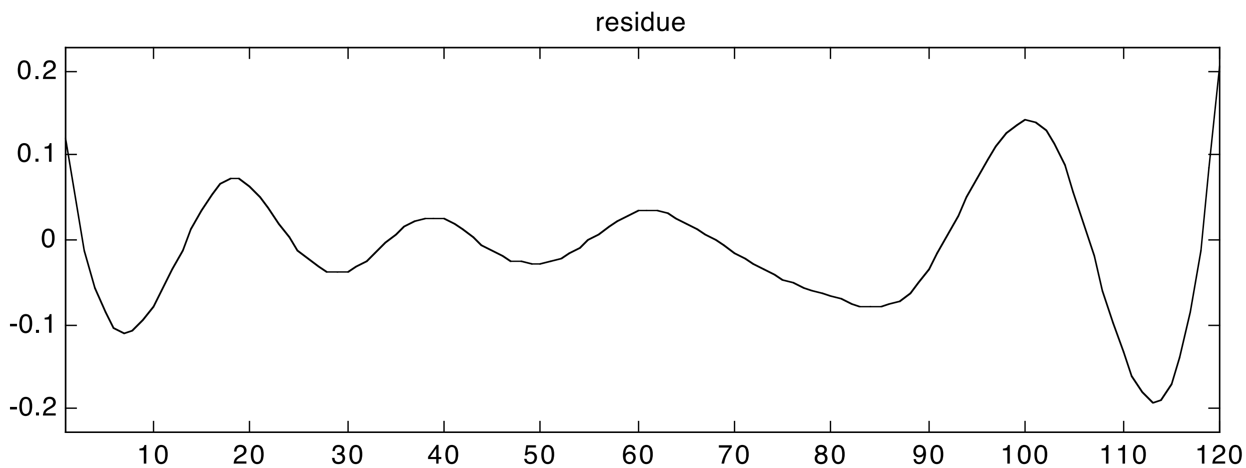

IMF 3;   iteration 4
0.2
0.1
0
-0.1
-0.2
10
20
30
40
50
60
70
80
90
100
110
120


residue
0.2
0.1
0
-0.1
-0.2
10
20
30
40
50
60
70
80
90
100
110
120

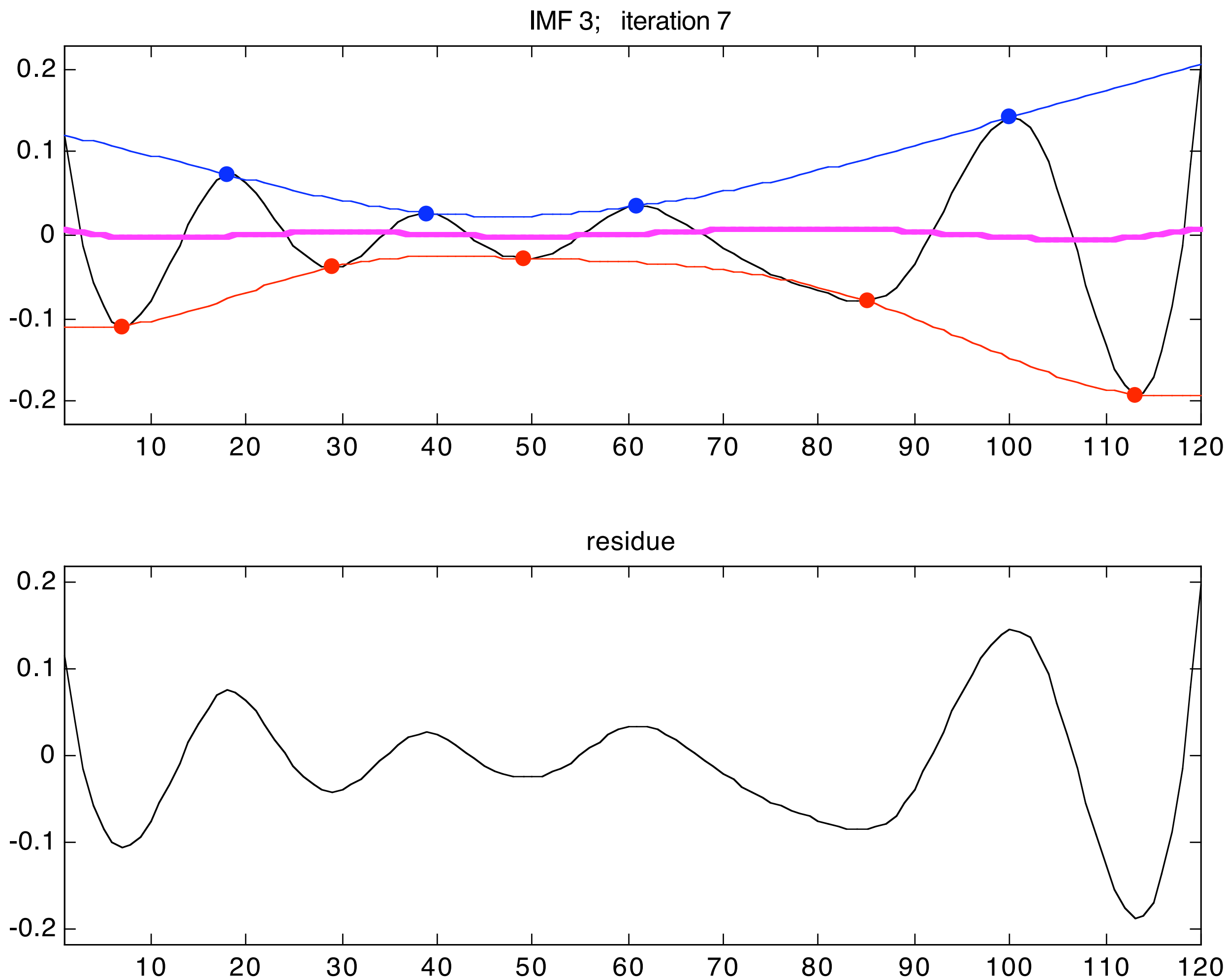

IMF 3;   iteration 5
0.2
0.1
0
-0.1
-0.2
10
20
30
40
50
60
70
80
90
100
110
120
residue
0.2
0.1
0
-0.1
-0.2
10
20
30
40
50
60
70
80
90
100
110
120

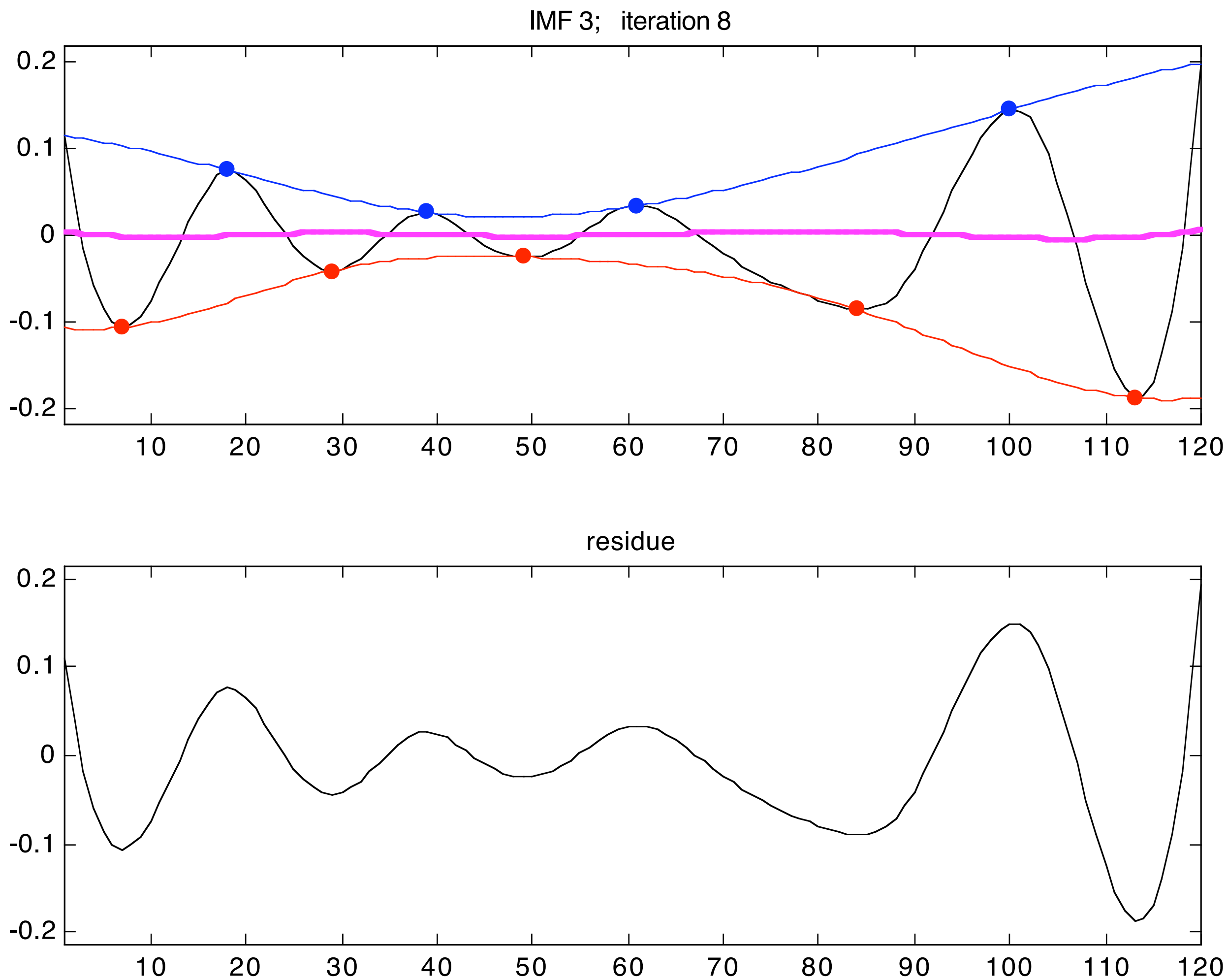


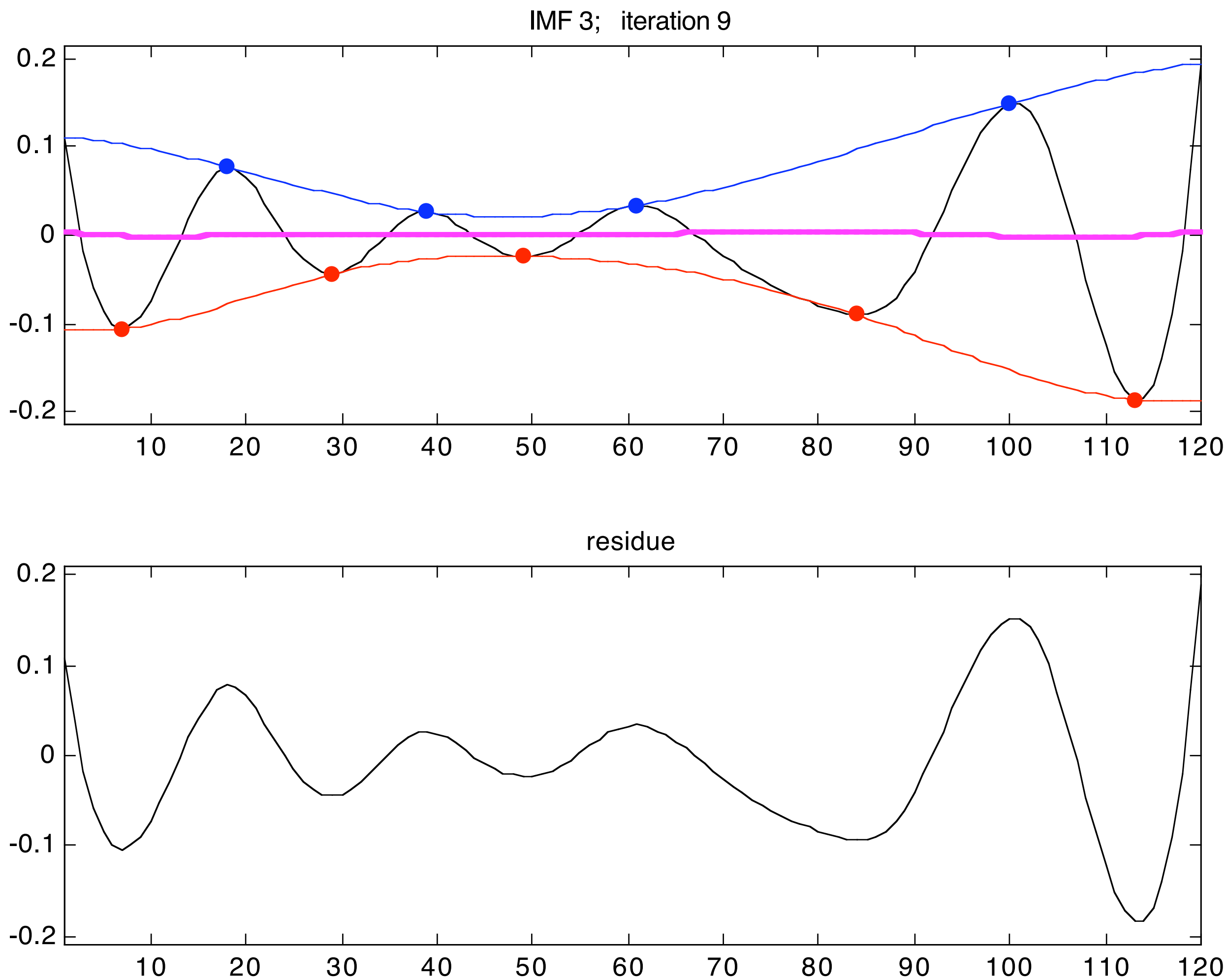
IMF 3;   iteration 6
0.2
0.1
0
-0.1
-0.2
10
20
30
40
50
60
70
80
90
100
110
120

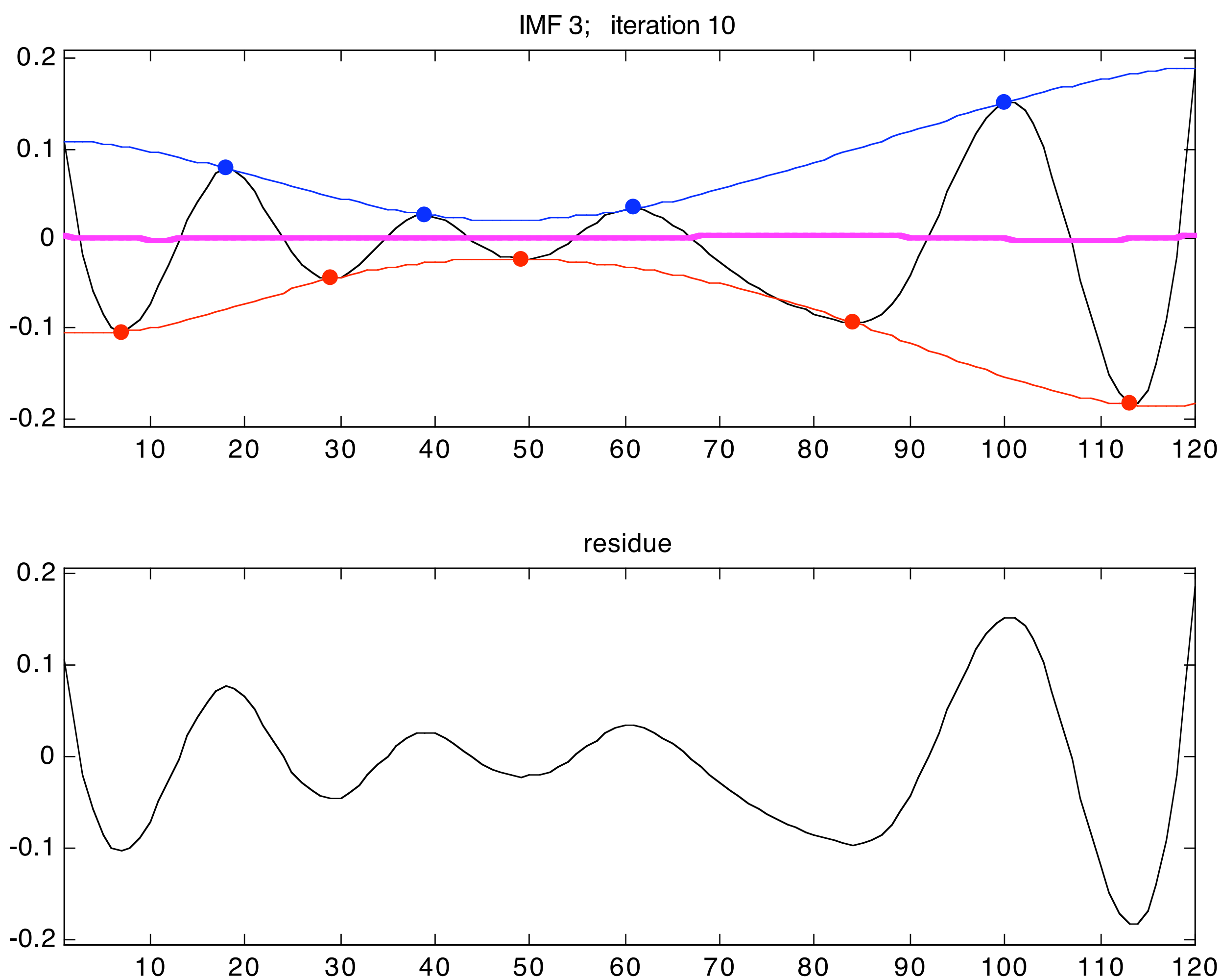


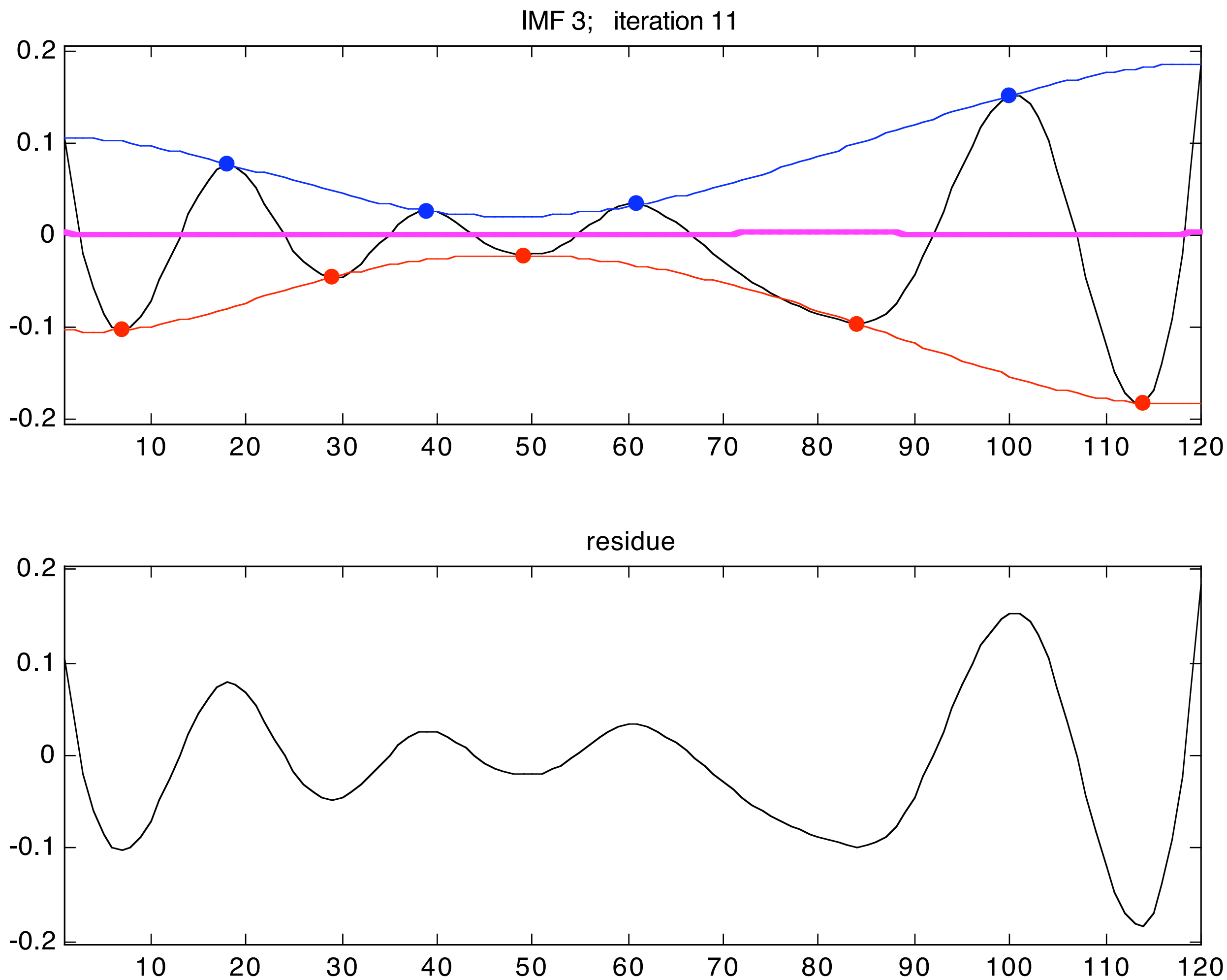
residue
0.2
0.1
0
-0.1
-0.2
10
20
30
40
50
60
70
80
90
100
110
120

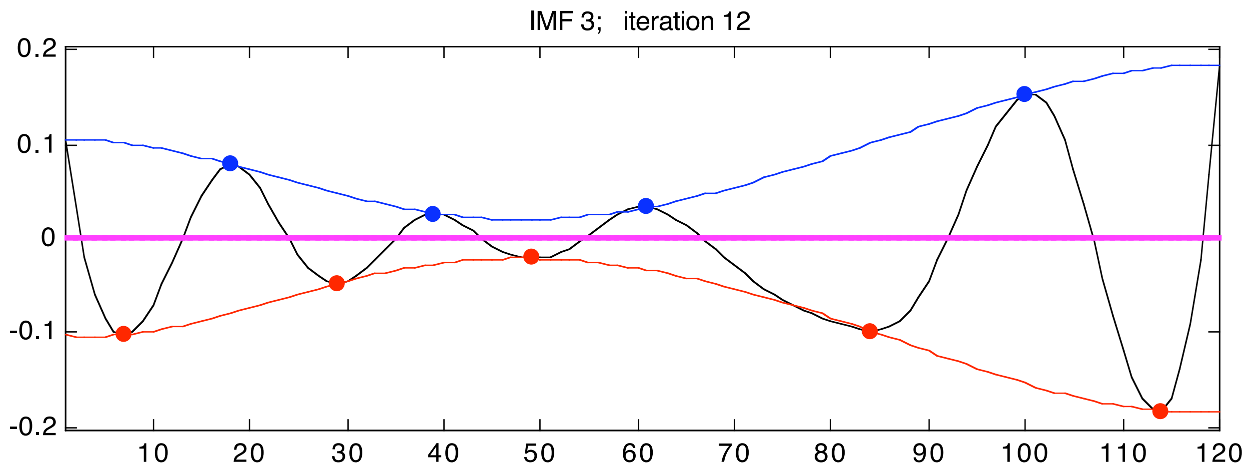

IMF 3;   iteration 7
0.2
0.1
0
-0.1
-0.2
10 20 30 40 50 60 70 80 90 100 110 120
residue
0.2
0.1
0
-0.1
-0.2
10 20 30 40 50 60 70 80 90 100 110 120

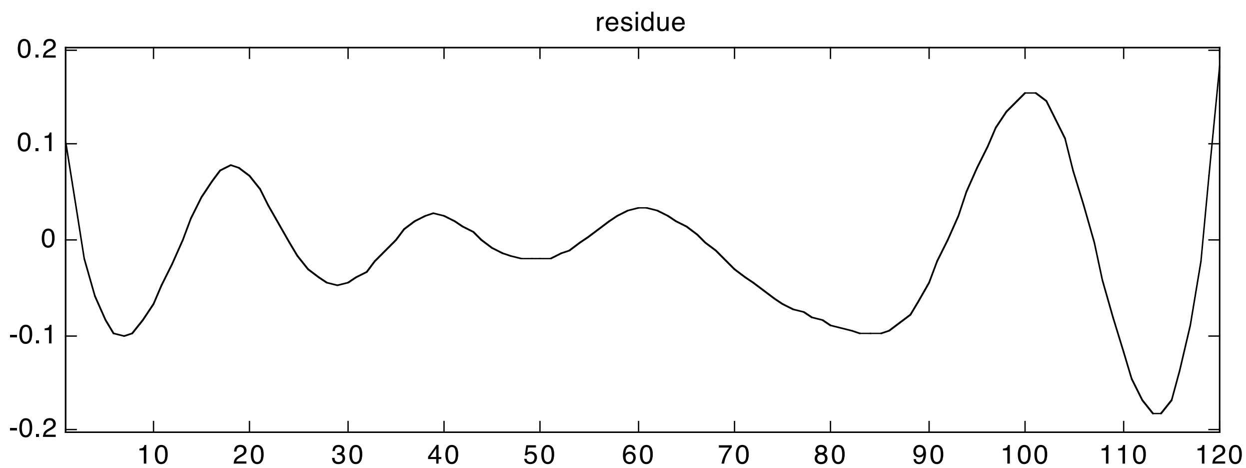

IMF 3; iteration 8
0.2
0.1
0
-0.1
-0.2
10
20
30
40
50
60
70
80
90
100
110
120
residue
0.2
0.1
0
-0.1
-0.2
10
20
30
40
50
60
70
80
90
100
110
120

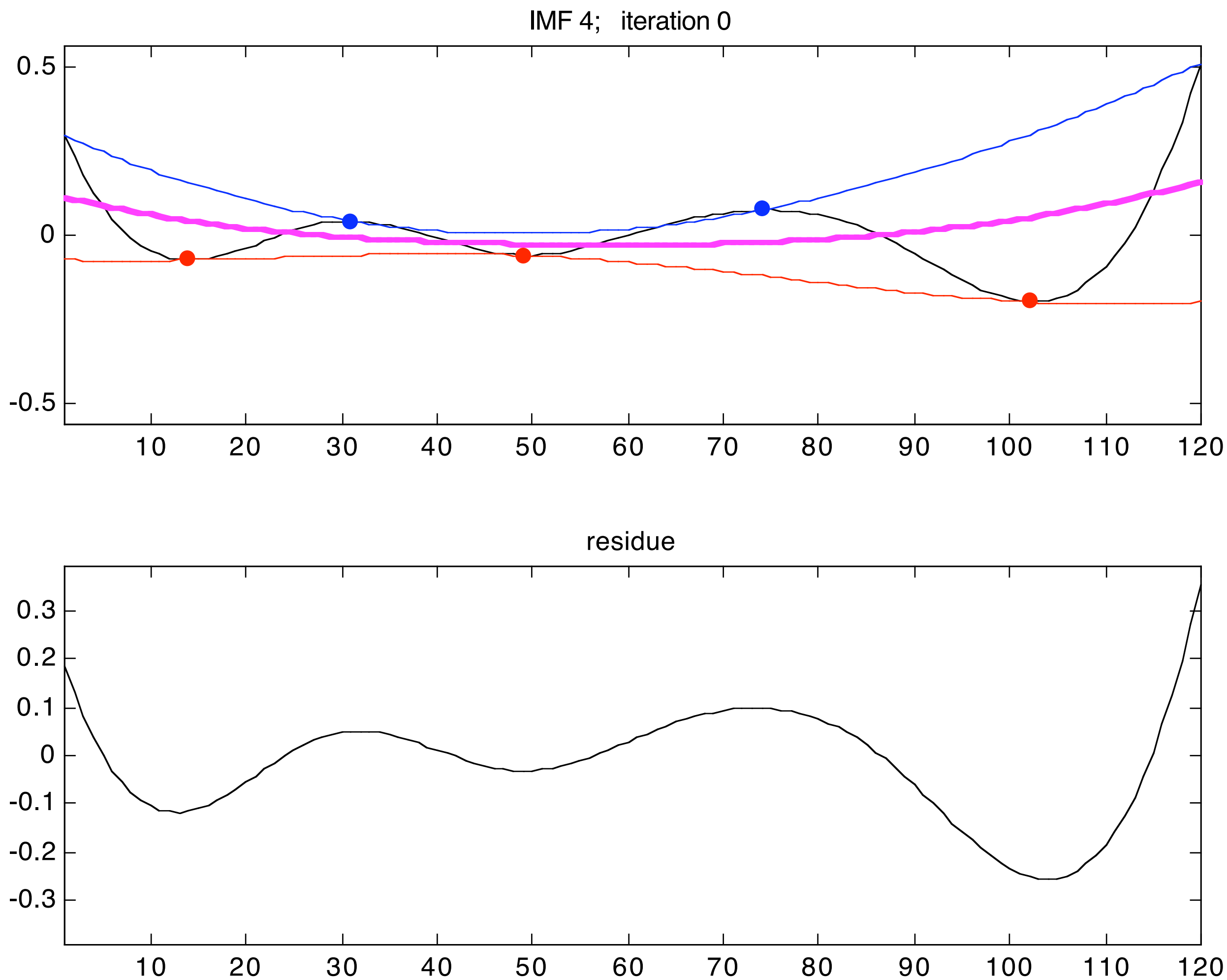

IMF 3;   iteration 9
0.2
0.1
0
-0.1
-0.2
10
20
30
40
50
60
70
80
90
100
110
120
residue
0.2
0.1
0
-0.1
-0.2
10
20
30
40
50
60
70
80
90
100
110
120

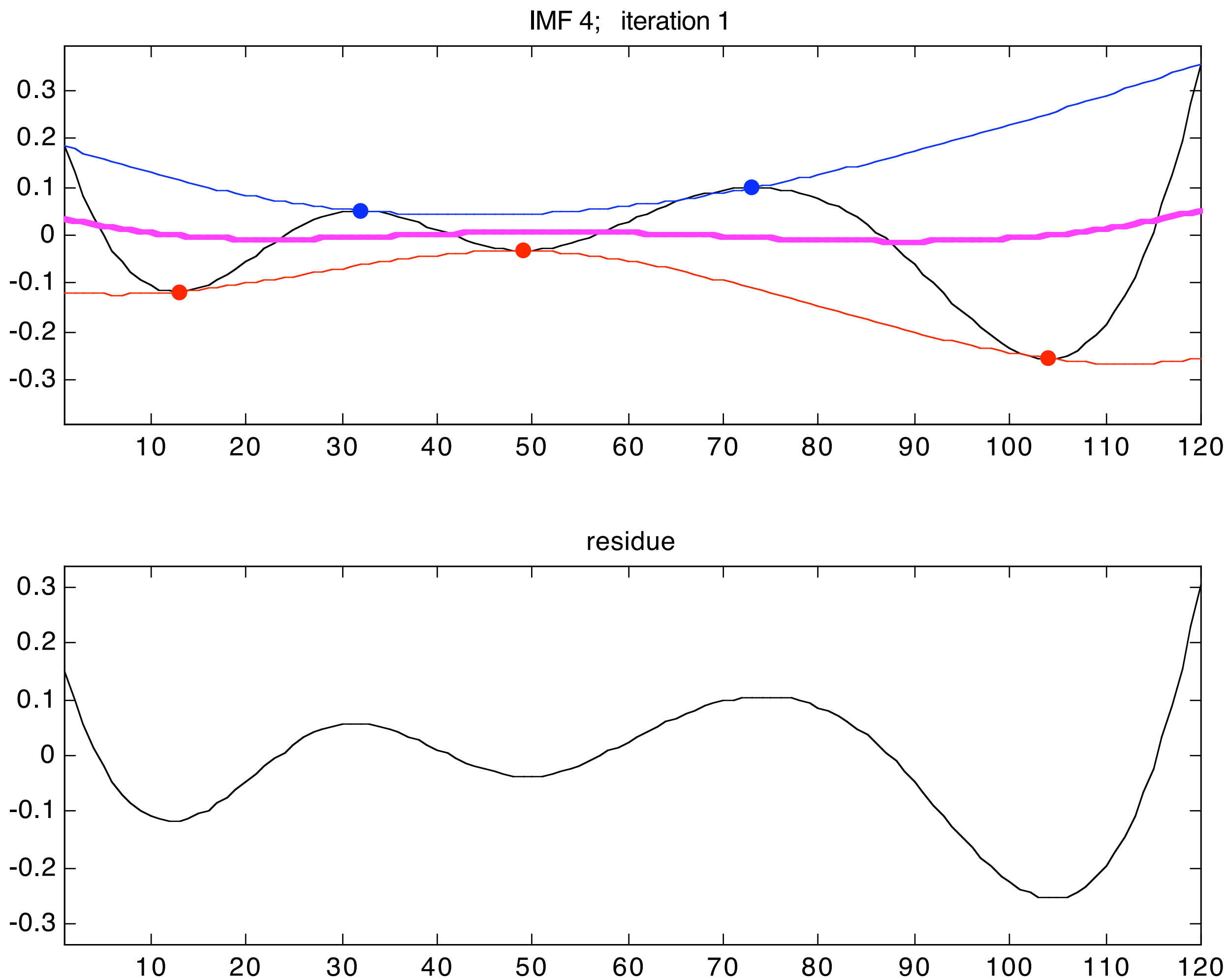

IMF 3;   iteration 10
0.2
0.1
0
-0.1
-0.2
10 20 30 40 50 60 70 80 90 100 110 120
residue
0.2
0.1
0
-0.1
-0.2
10 20 30 40 50 60 70 80 90 100 110 120

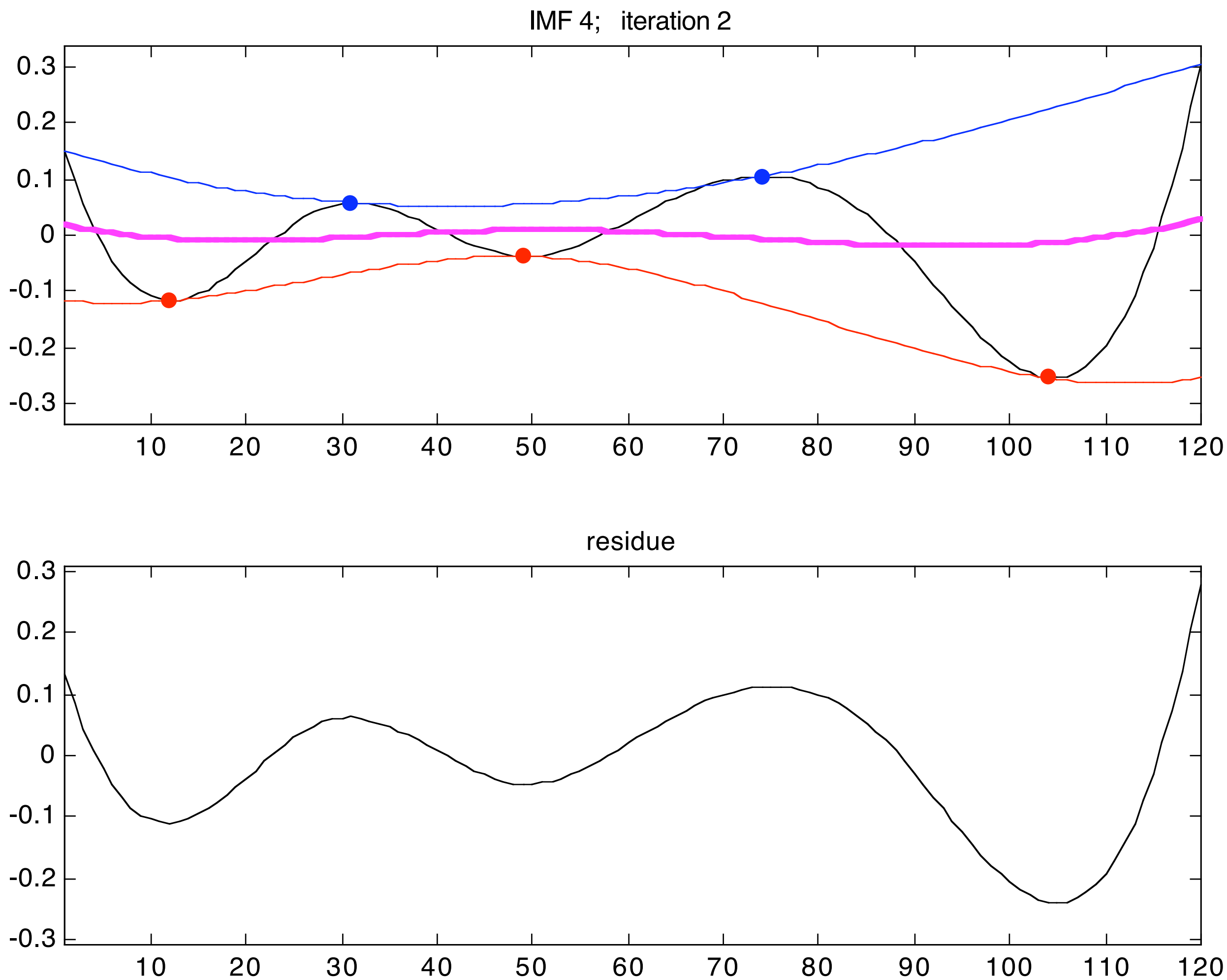

IMF 3; iteration 11
0.2
0.1
0
-0.1
-0.2
10
20
30
40
50
60
70
80
90
100
110
120
residue
0.2
0.1
0
-0.1
-0.2
10
20
30
40
50
60
70
80
90
100
110
120

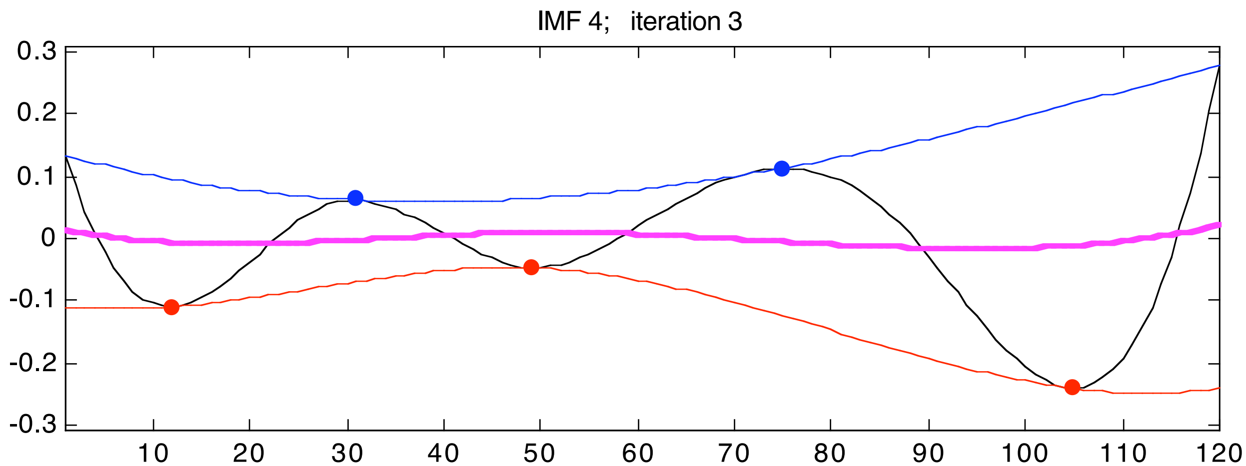

IMF 3;   iteration 12
0.2
0.1
0
-0.1
-0.2
10
20
30
40
50
60
70
80
90
100
110
120

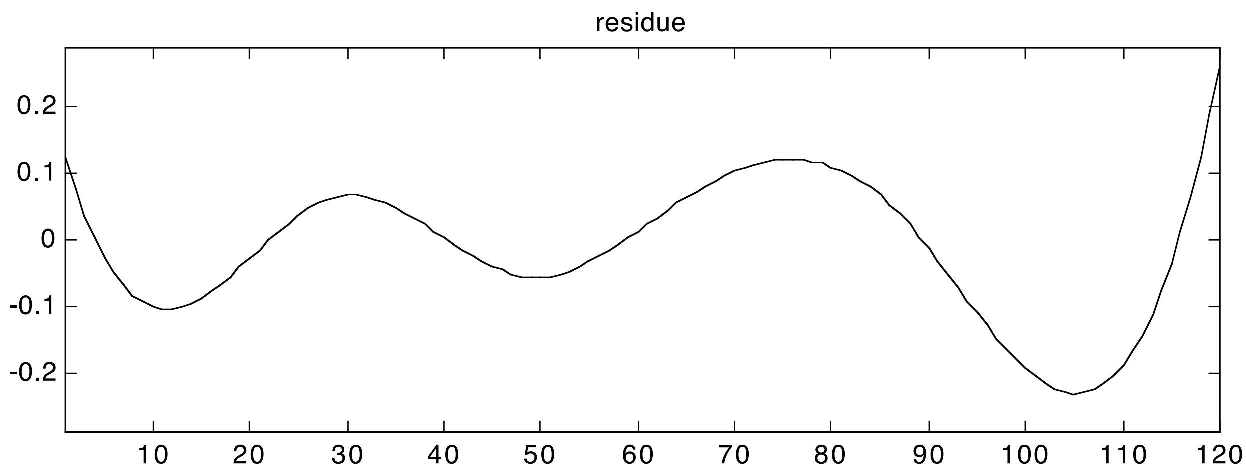

residue
0.2
0.1
0
-0.1
-0.2
10
20
30
40
50
60
70
80
90
100
110
120

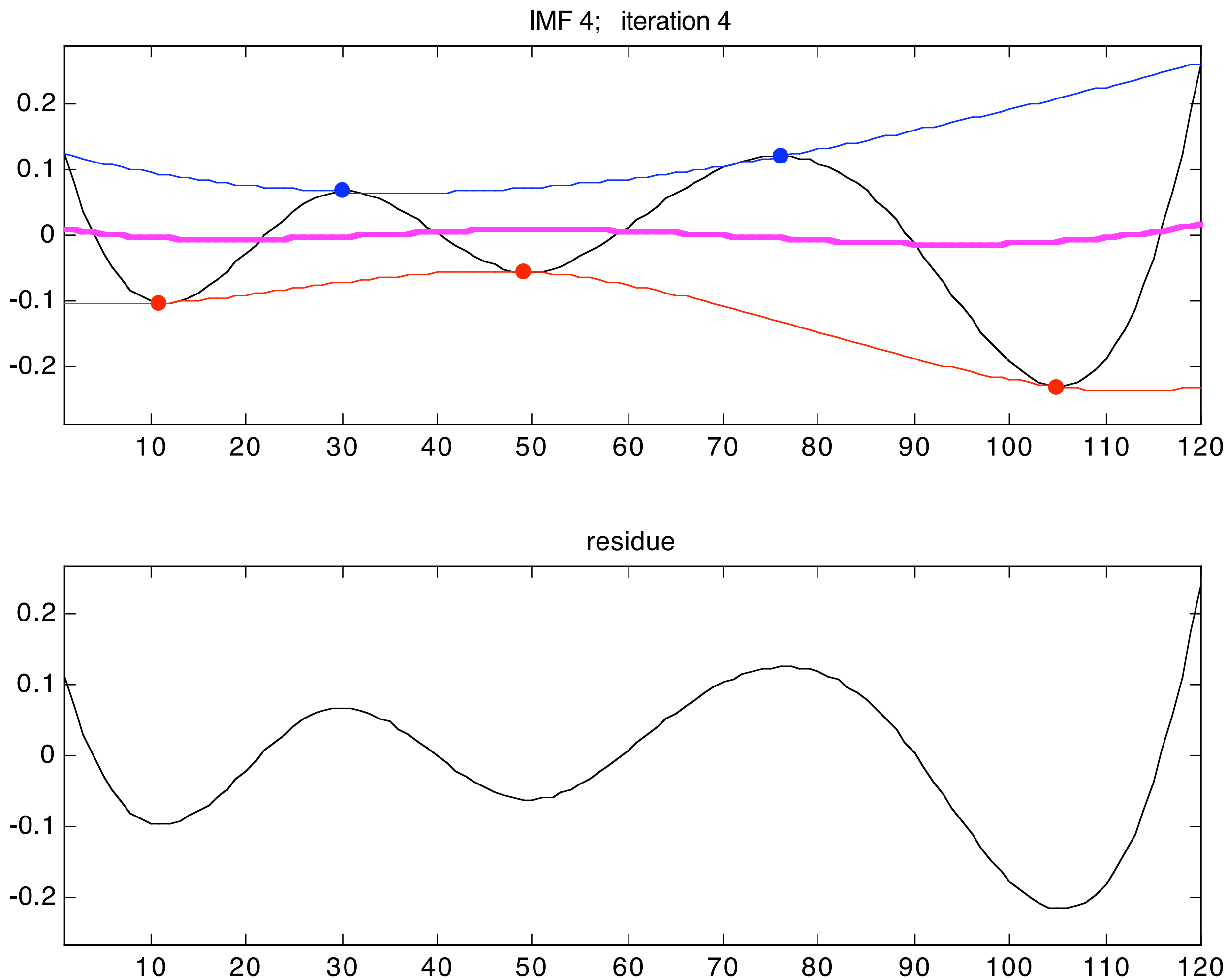

IMF 4;   iteration 0
0.5
0
-0.5
10
20
30
40
50
60
70
80
90
100
110
120
residue
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
10
20
30
40
50
60
70
80
90
100
110
120

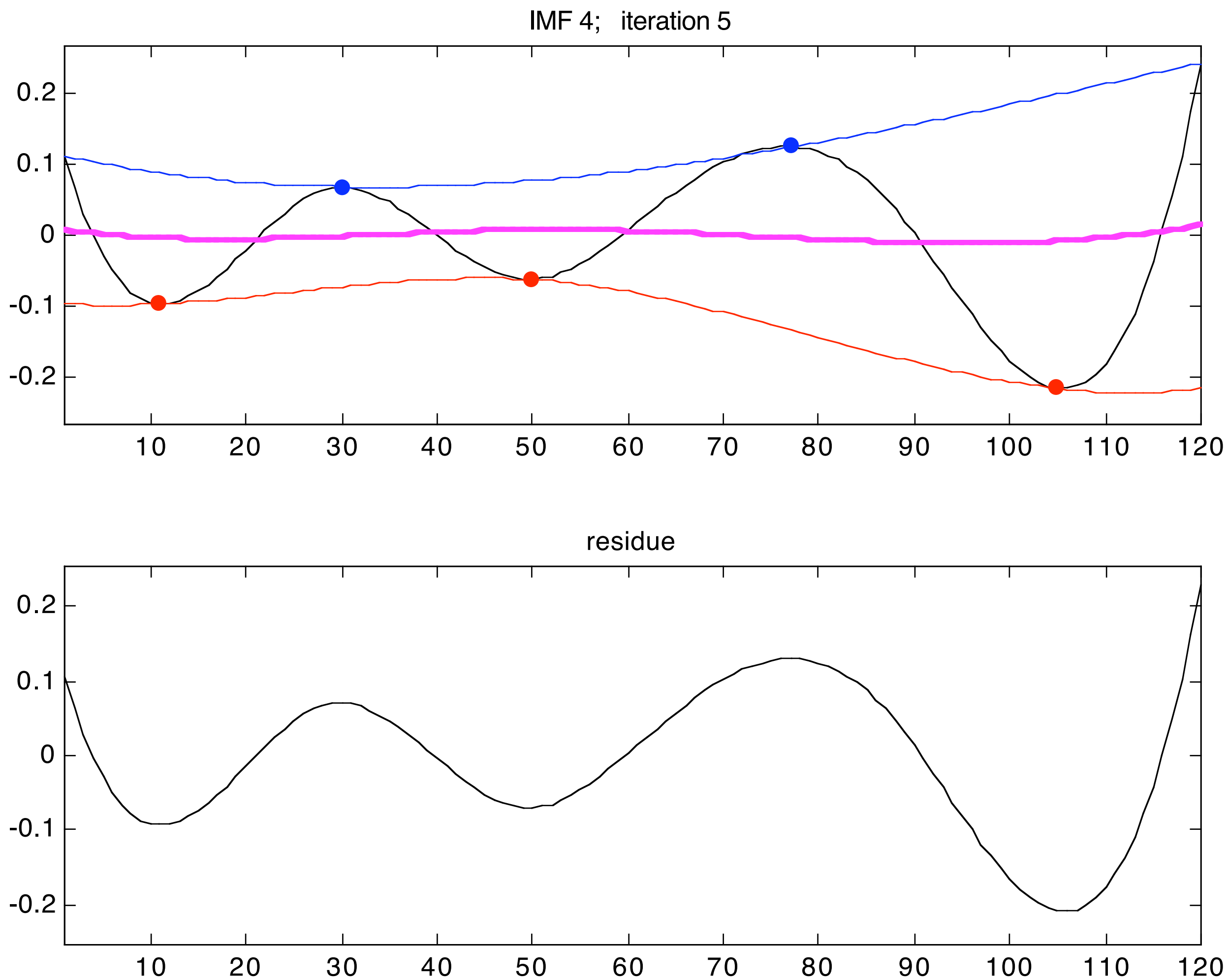

IMF 4;   iteration 1
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
10
20
30
40
50
60
70
80
90
100
110
120
residue

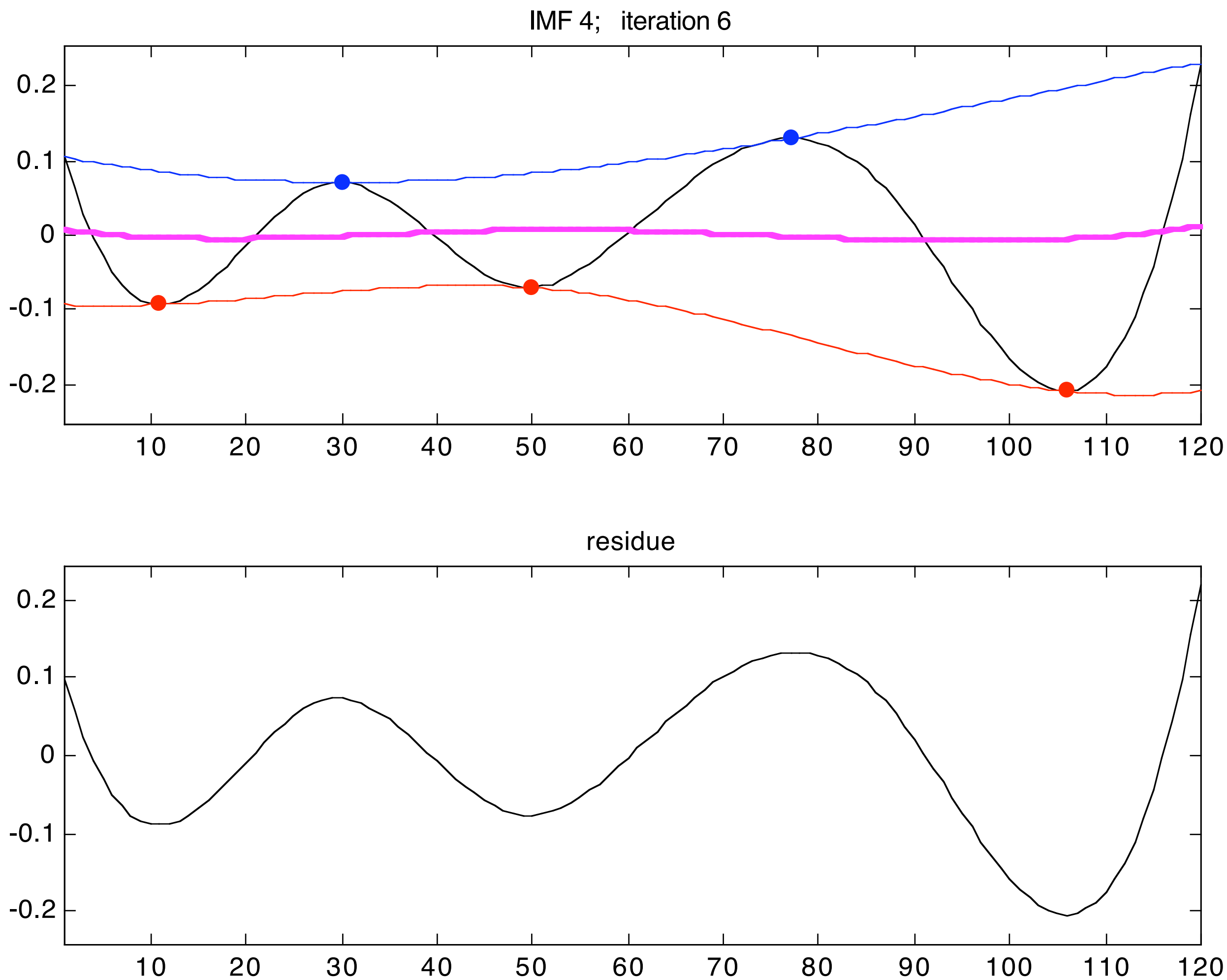

IMF 4;   iteration 2
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
10 20 30 40 50 60 70 80 90 100 110 120
residue
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
10 20 30 40 50 60 70 80 90 100 110 120

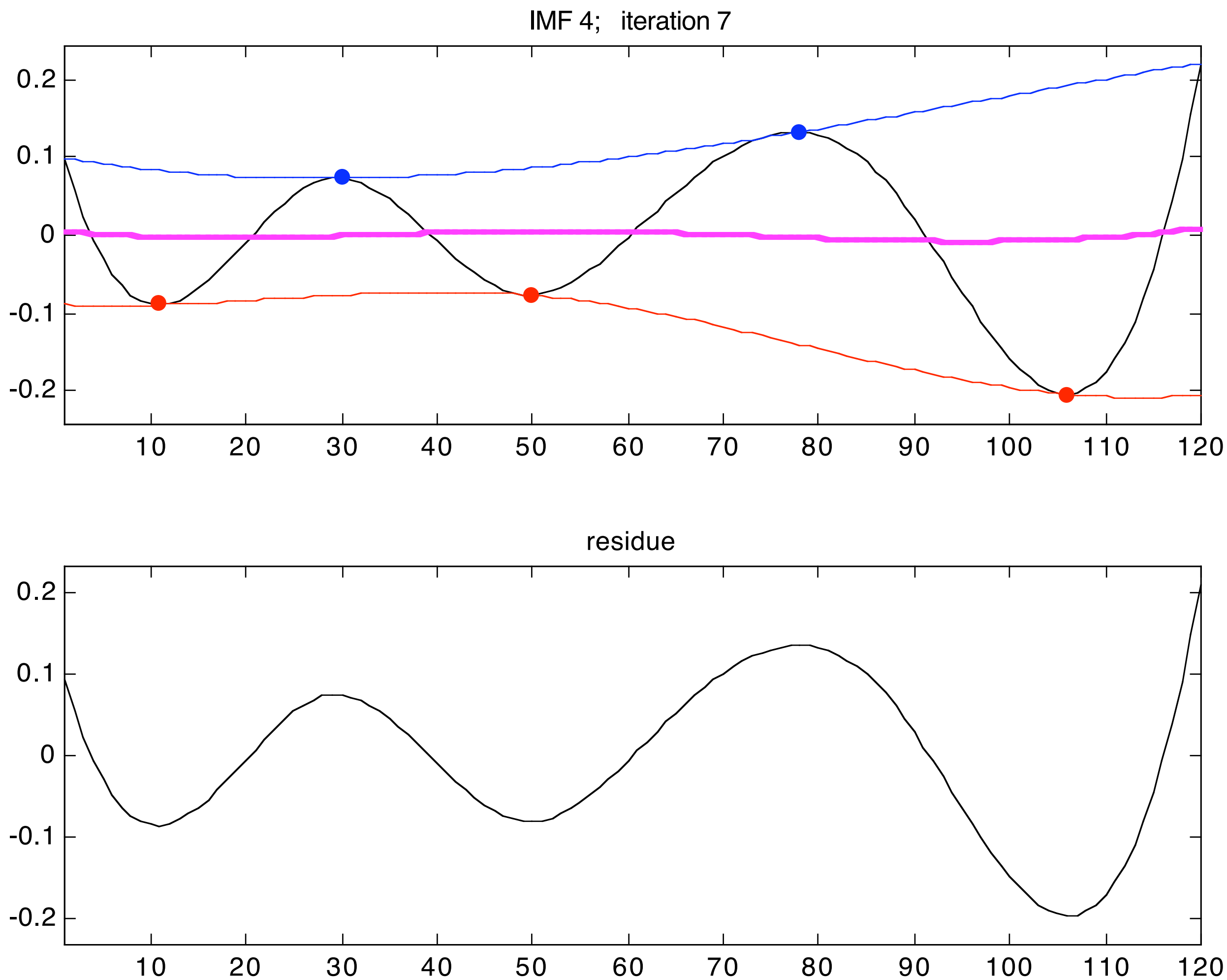

IMF 4;   iteration 3
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
10
20
30
40
50
60
70
80
90
100
110
120

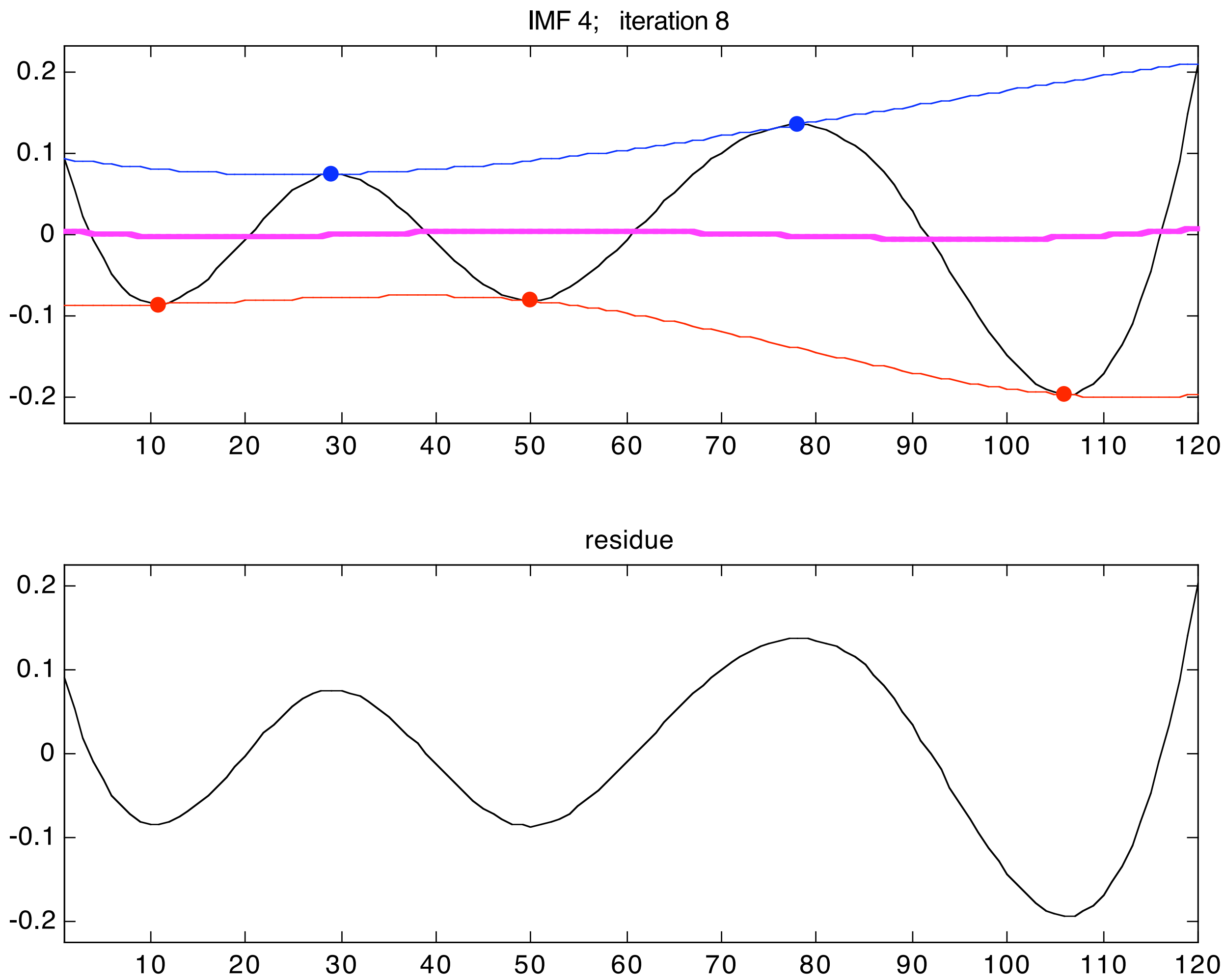

residue
0.2
0.1
0
-0.1
-0.2
10
20
30
40
50
60
70
80
90
100
110
120

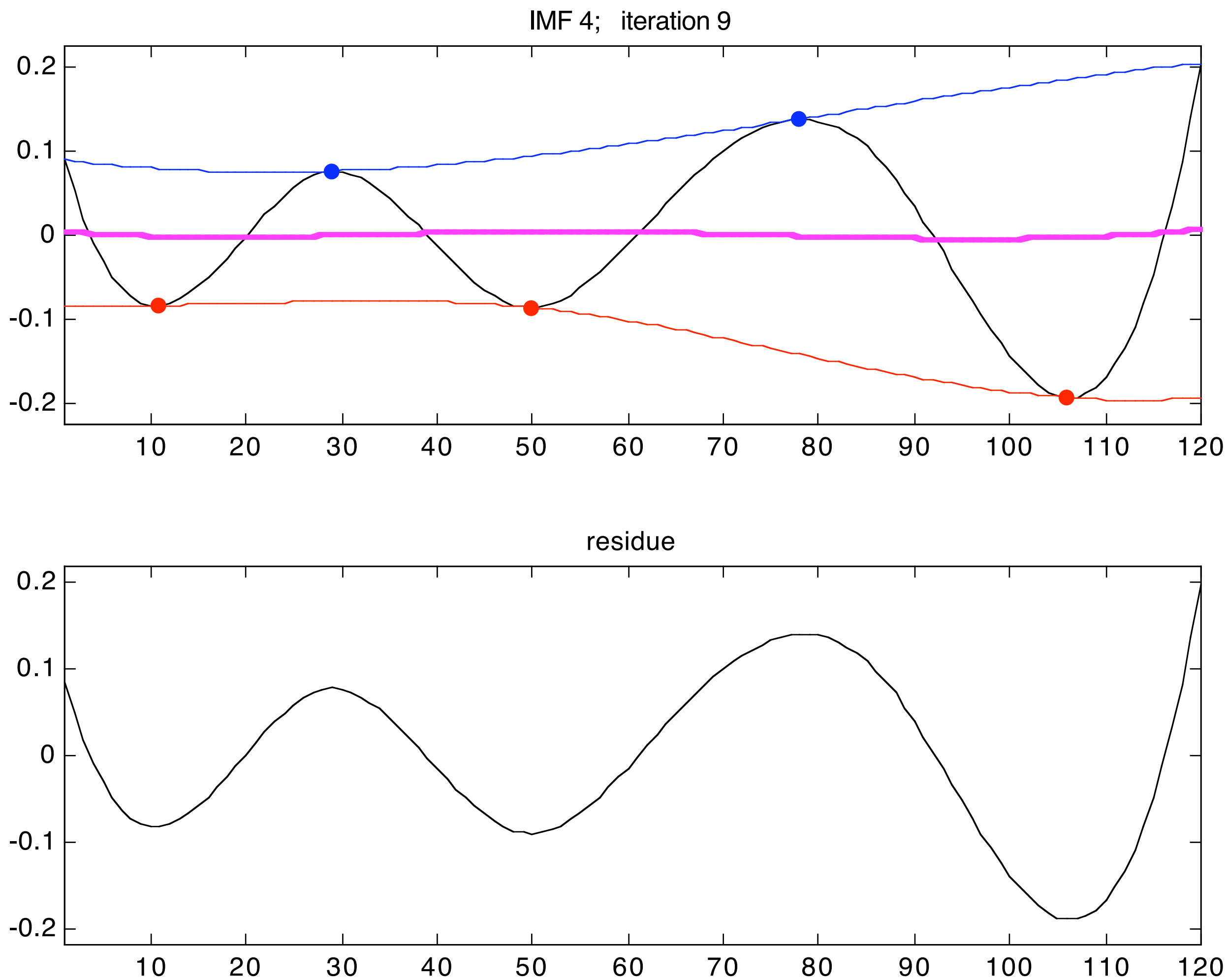

IMF 4;   iteration 4
0.2
0.1
0
-0.1
-0.2
10
20
30
40
50
60
70
80
90
100
110
120
residue
0.2
0.1
0
-0.1
-0.2
10
20
30
40
50
60
70
80
90
100
110
120

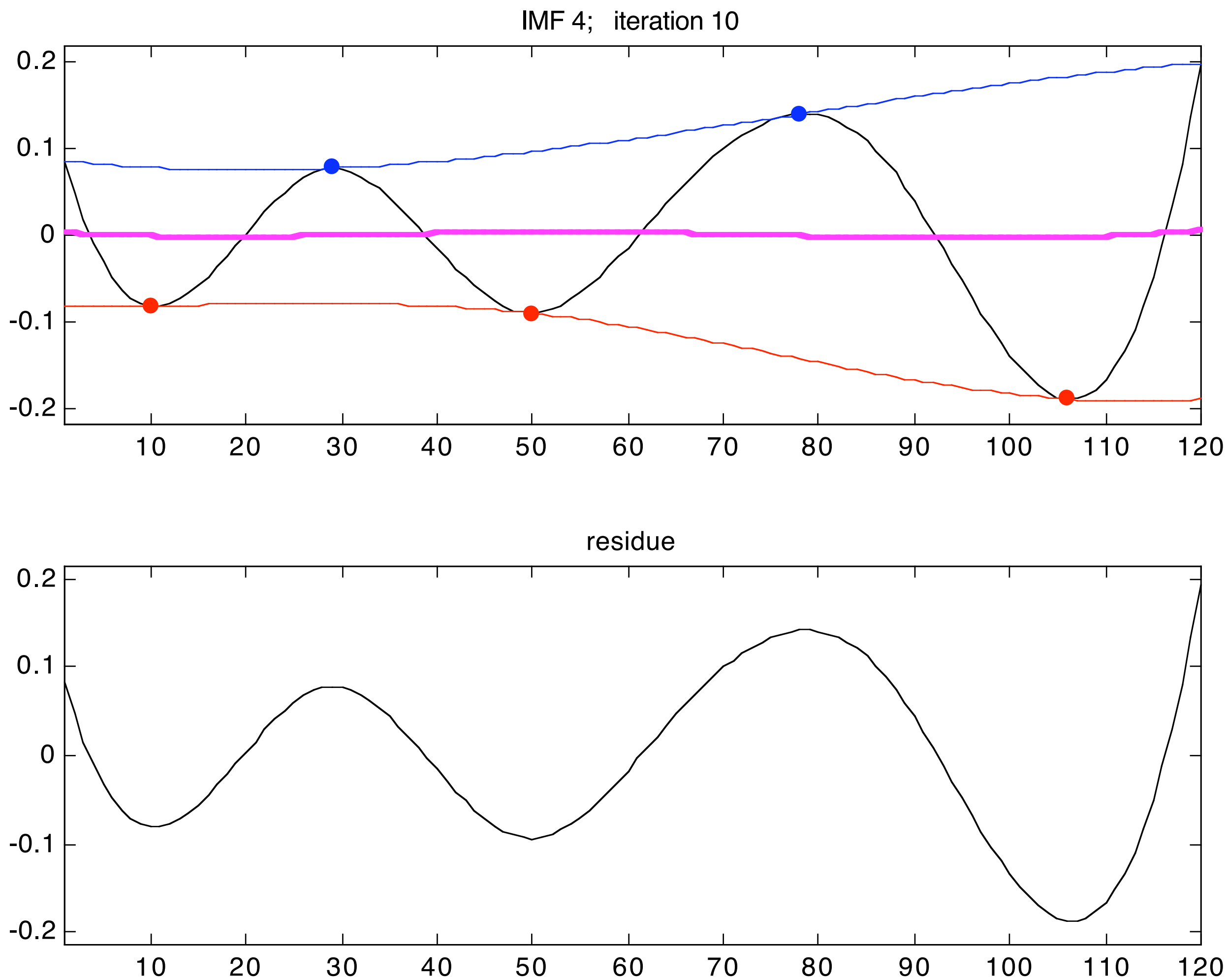

IMF 4;   iteration 5
0.2
0.1
0
-0.1
-0.2
10
20
30
40
50
60
70
80
90
100
110
120
residue
0.2
0.1
0
-0.1
-0.2
10
20
30
40
50
60
70
80
90
100
110
120


IMF 4;   iteration 6
0.2
0.1
0
-0.1
-0.2
10
20
30
40
50
60
70
80
90
100
110
120
residue
0.2
0.1
0
-0.1
-0.2
10
20
30
40
50
60
70
80
90
100
110
120

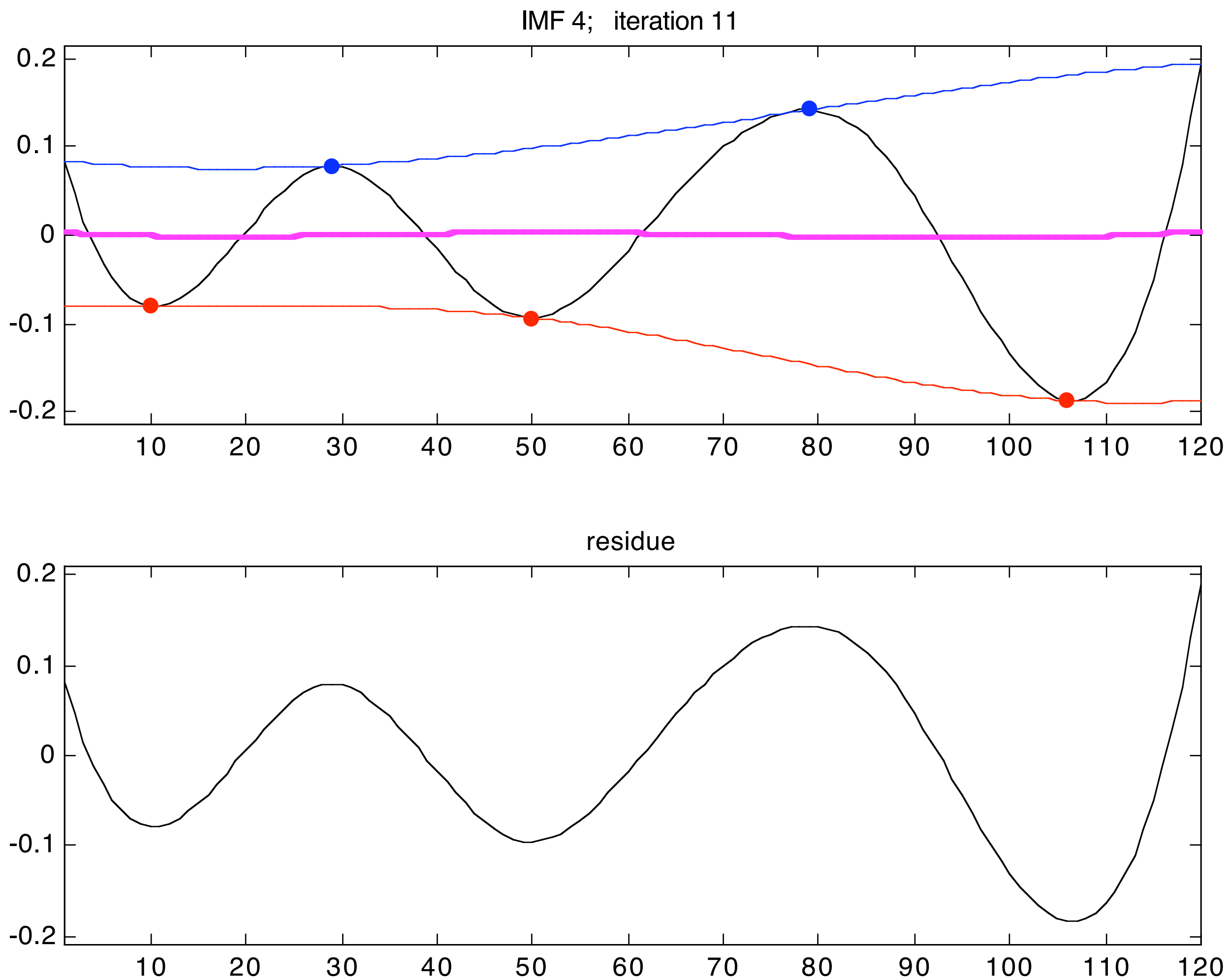

IMF 4; iteration 7
0.2
0.1
0
-0.1
-0.2
10
20
30
40
50
60
70
80
90
100
110
120
residue
0.2
0.1
0
-0.1
-0.2
10
20
30
40
50
60
70
80
90
100
110
120


IMF 4;   iteration 8
0.2
0.1
0
-0.1
-0.2
10
20
30
40
50
60
70
80
90
100
110
120
residue
0.2
0.1
0
-0.1
-0.2
10
20
30
40
50
60
70
80
90
100
110
120

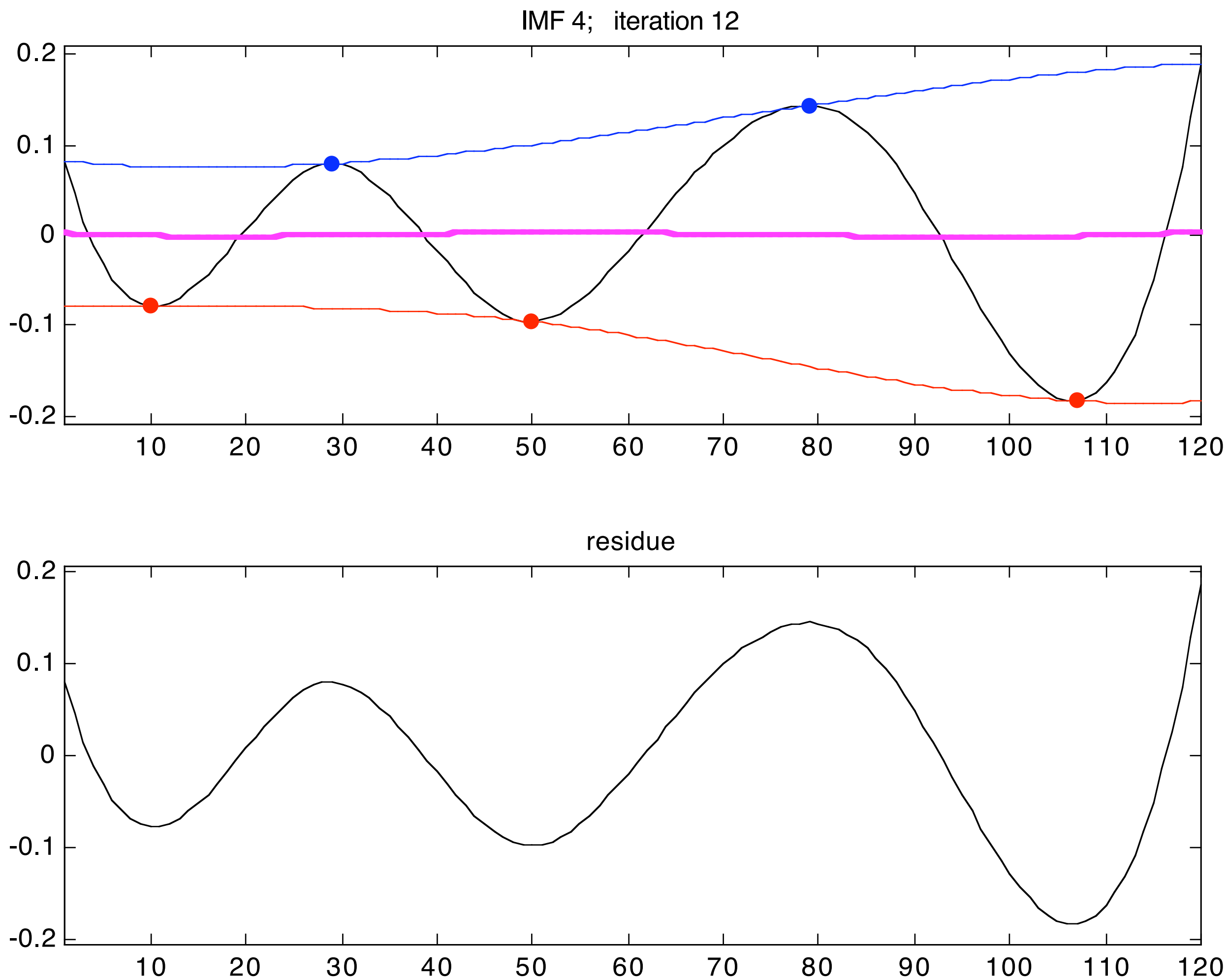

IMF 4; iteration 9
0.2
0.1
0
-0.1
-0.2
10
20
30
40
50
60
70
80
90
100
110
120
residue
0.2
0.1
0
-0.1
-0.2
10
20
30
40
50
60
70
80
90
100
110
120

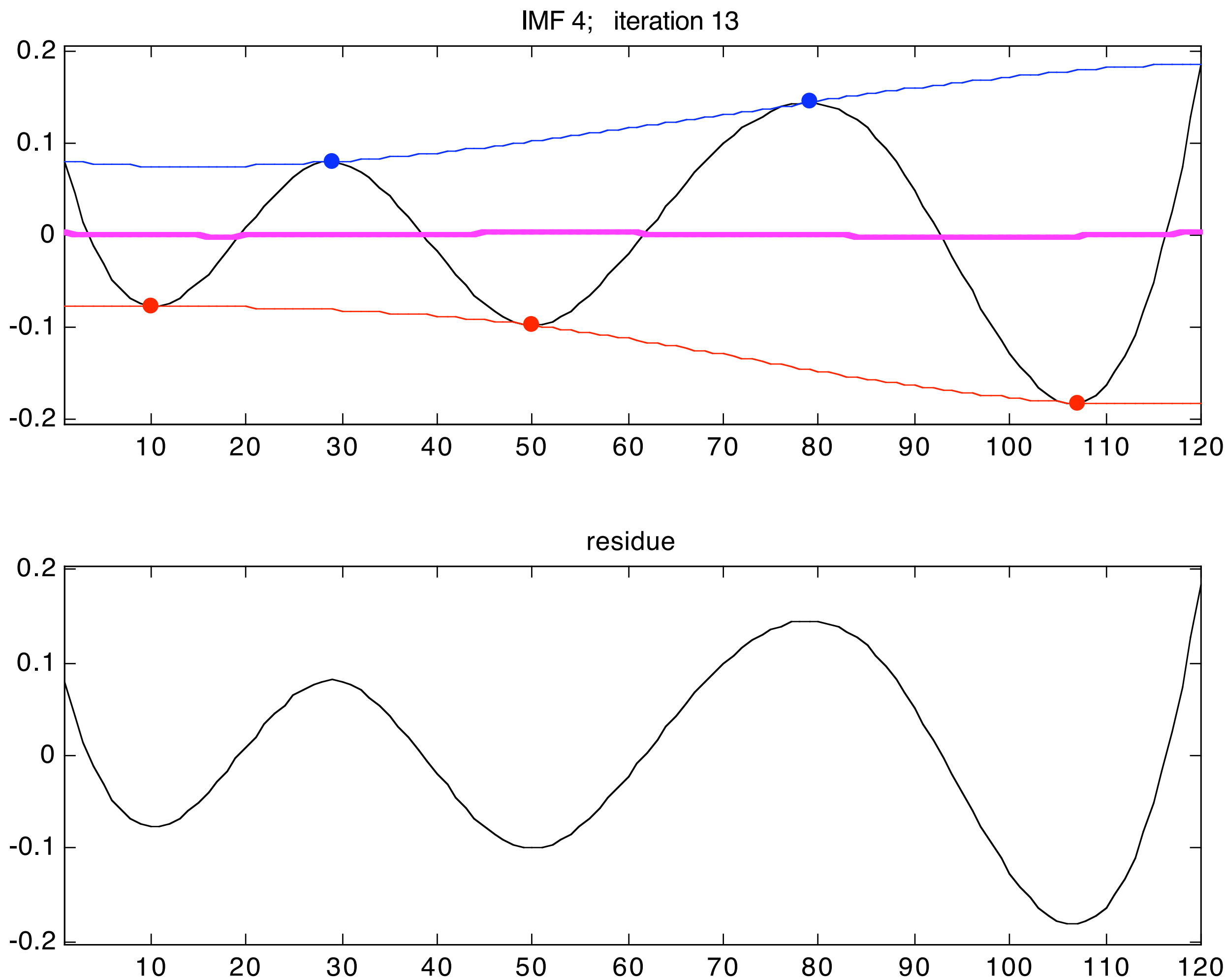


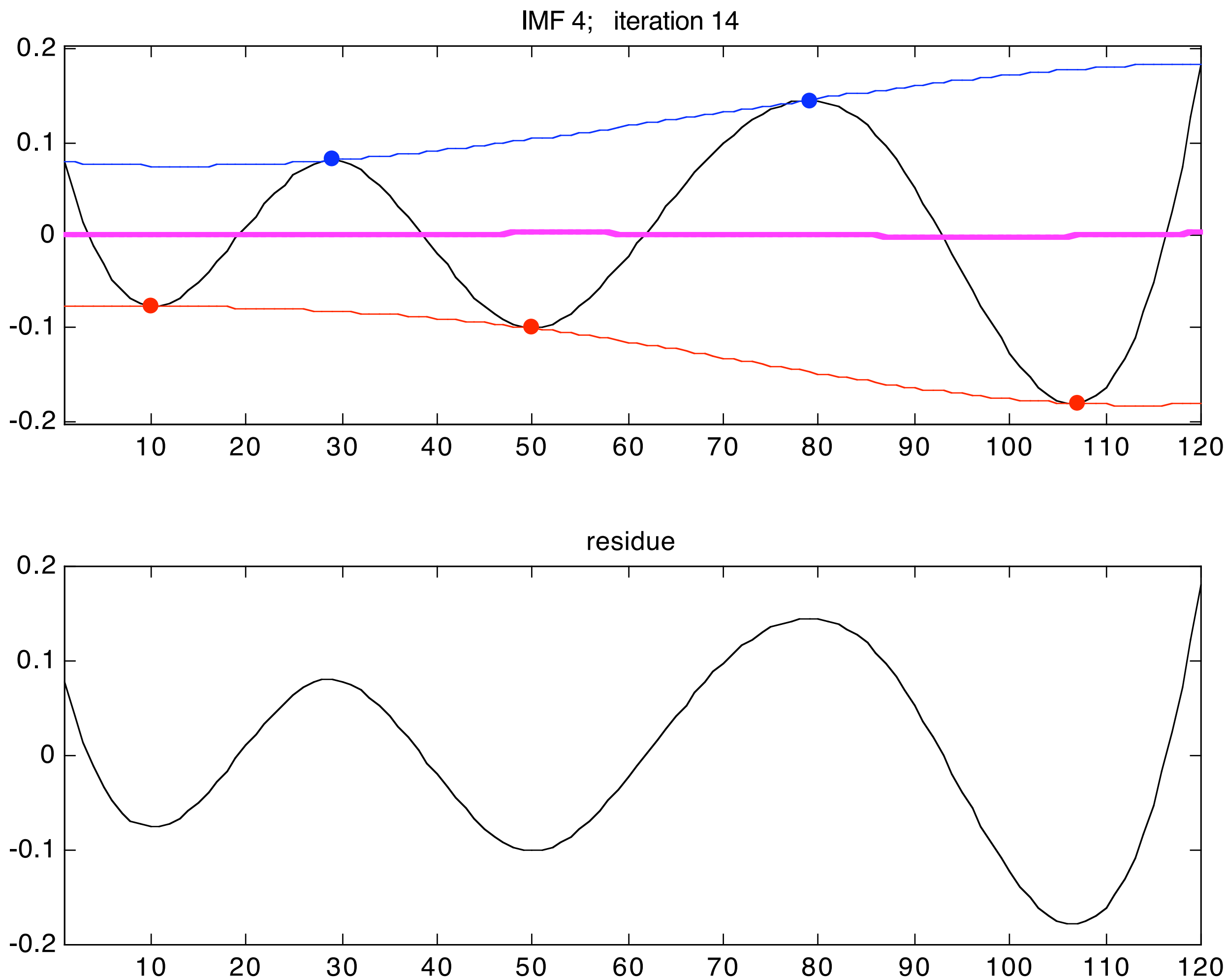
IMF 4;   iteration 10
0.2
0.1
0
-0.1
-0.2
10
20
30
40
50
60
70
80
90
100
110
120
residue
0.2
0.1
0
-0.1
-0.2
10
20
30
40
50
60
70
80
90
100
110
120



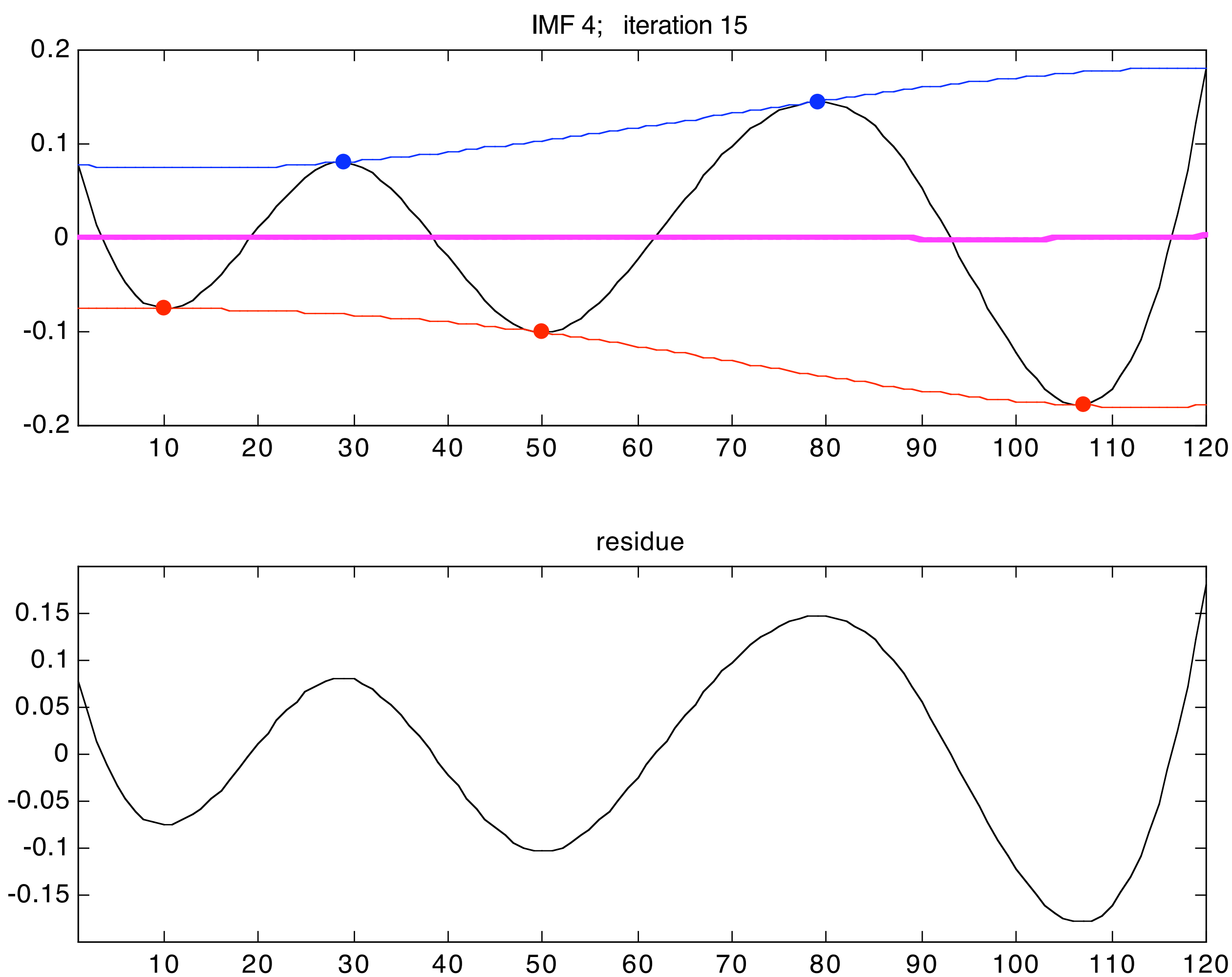
IMF 4;   iteration 11
0.2
0.1
0
-0.1
-0.2
10
20
30
40
50
60
70
80
90
100
110
120
residue


IMF 4; iteration 12
0.2
0.1
0
-0.1
-0.2
10 20 30 40 50 60 70 80 90 100 110 120
residue
0.2
0.1
0
-0.1
-0.2
10 20 30 40 50 60 70 80 90 100 110 120

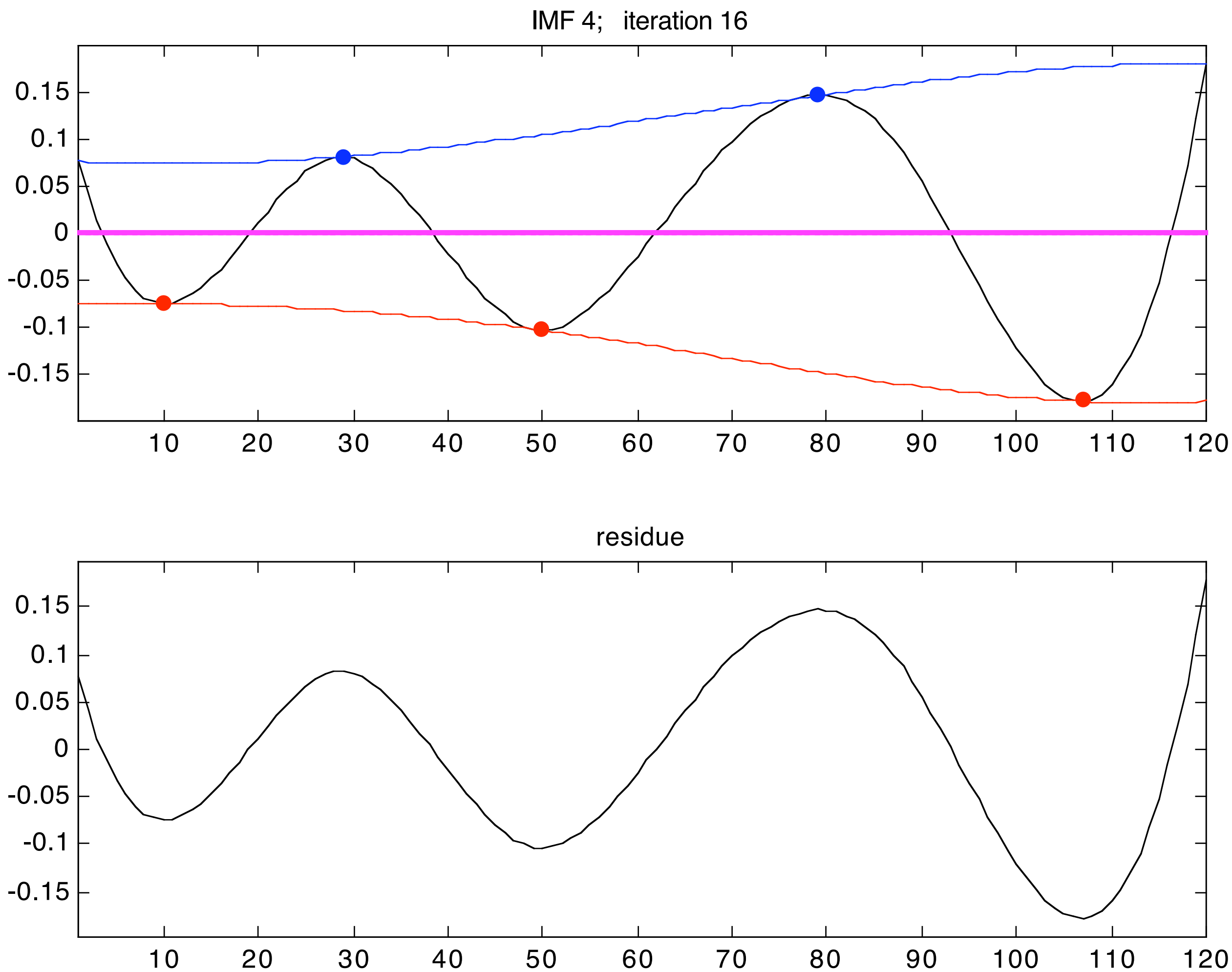

IMF 4;   iteration 13
0.2
0.1
0
-0.1
-0.2
10
20
30
40
50
60
70
80
90
100
110
120
residue
0.2
0.1
0
-0.1
-0.2
10
20
30
40
50
60
70
80
90
100
110
120

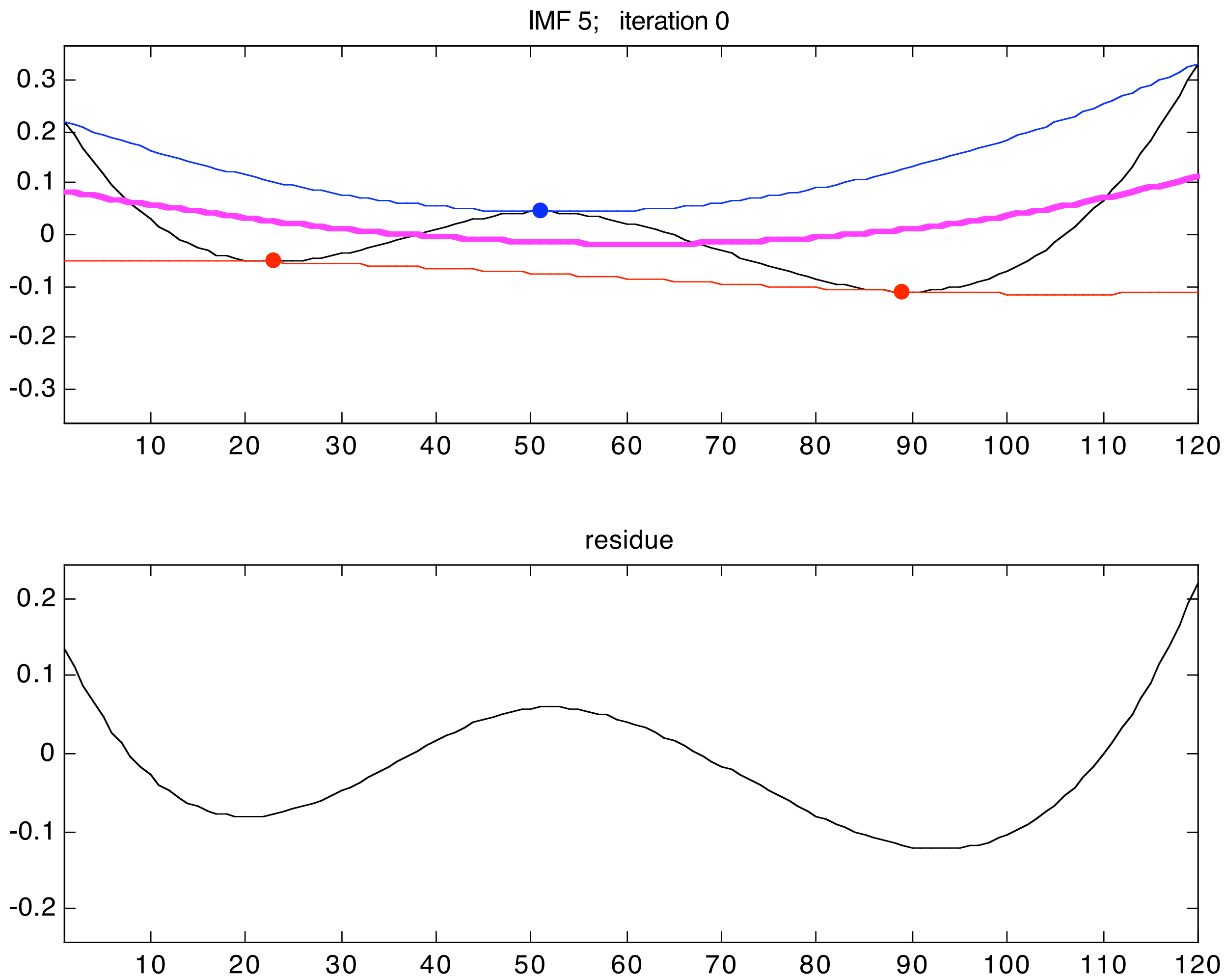


IMF 4; iteration 14
0.2
0.1
0
-0.1
-0.2
10
20
30
40
50
60
70
80
90
100
110
120
residue
0.2
0.1
0
-0.1
-0.2
10
20
30
40
50
60
70
80
90
100
110
120

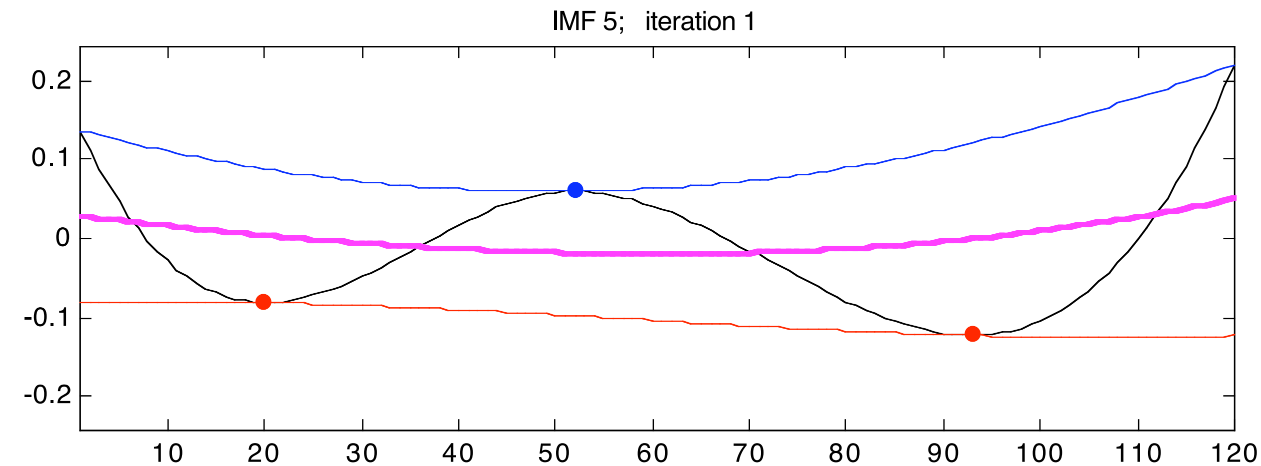


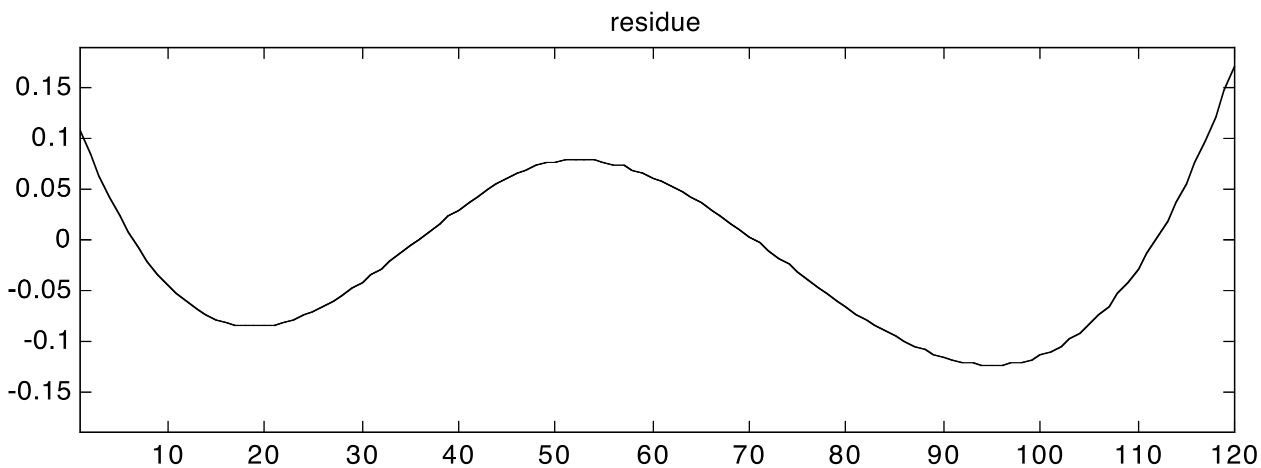
IMF 4; iteration 15
0.2
0.1
0
-0.1
-0.2
10
20
30
40
50
60
70
80
90
100
110
120
residue
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
10
20
30
40
50
60
70
80
90
100
110
120

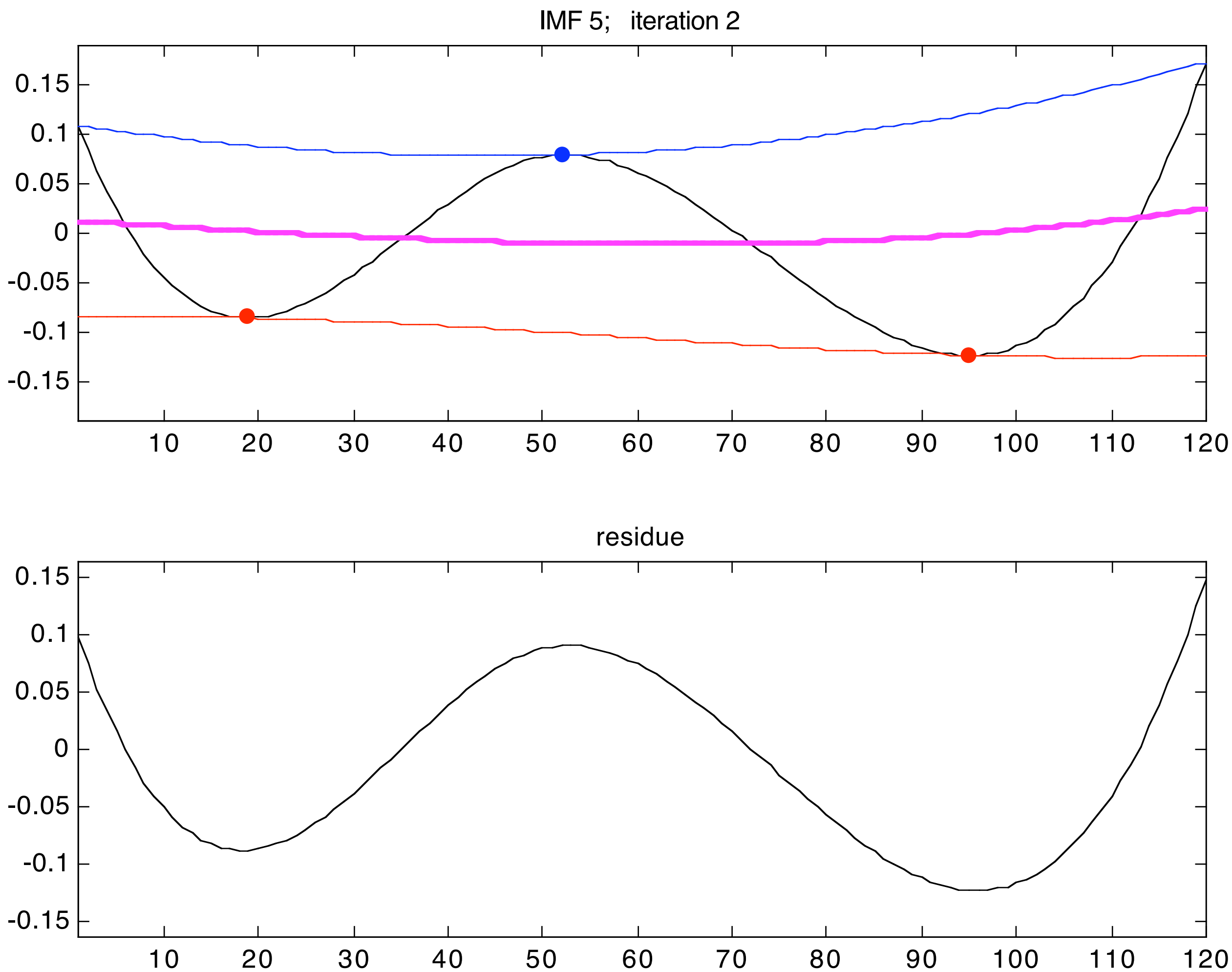

IMF 4;   iteration 16
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
10
20
30
40
50
60
70
80
90
100
110
120
residue
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
10
20
30
40
50
60
70
80
90
100
110
120

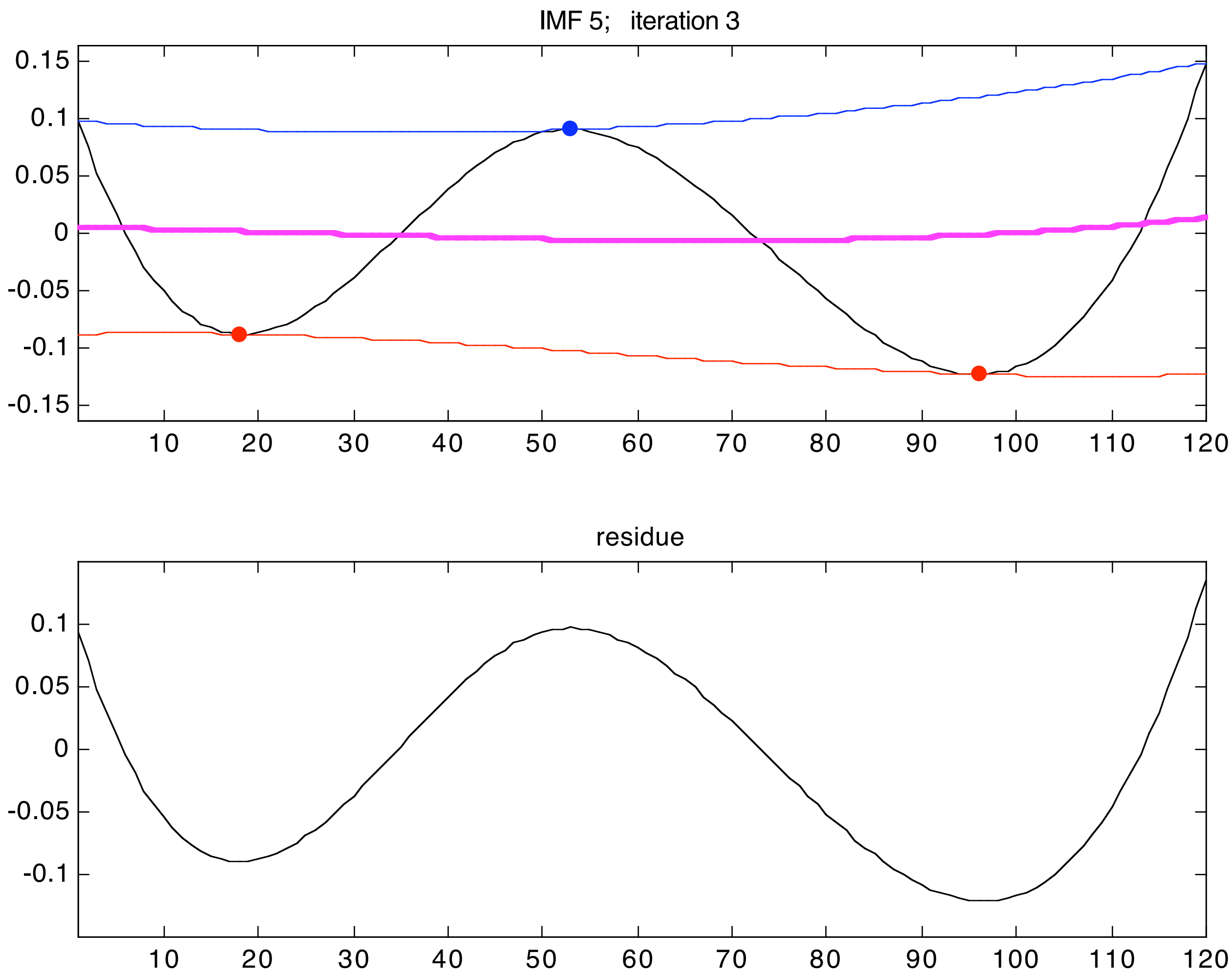


IMF 5;   iteration 0
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
10
20
30
40
50
60
70
80
90
100
110
120
residue
0.2
0.1
0
-0.1
-0.2
10
20
30
40
50
60
70
80
90
100
110
120

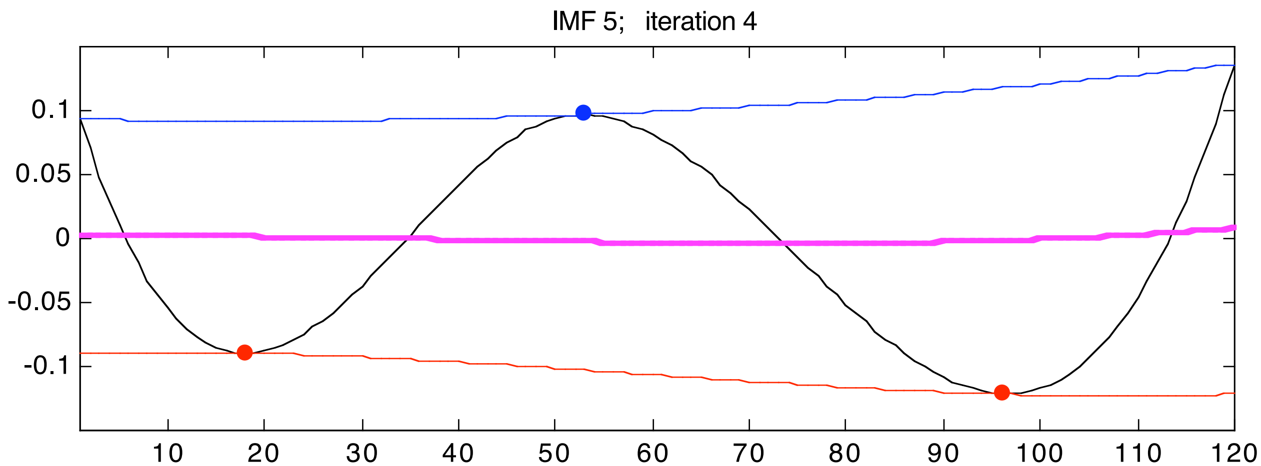

IMF 5;   iteration 1
0.2
0.1
0
-0.1
-0.2
10 20 30 40 50 60 70 80 90 100 110 120


residue
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
10 20 30 40 50 60 70 80 90 100 110 120


IMF 5;   iteration 2
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
10
20
30
40
50
60
70
80
90
100
110
120
residue



IMF 5; iteration 3
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
10
20
30
40
50
60
70
80
90
100
110
120
residue
0.1
0.05
0
-0.05
-0.1
10
20
30
40
50
60
70
80
90
100
110
120



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10
20
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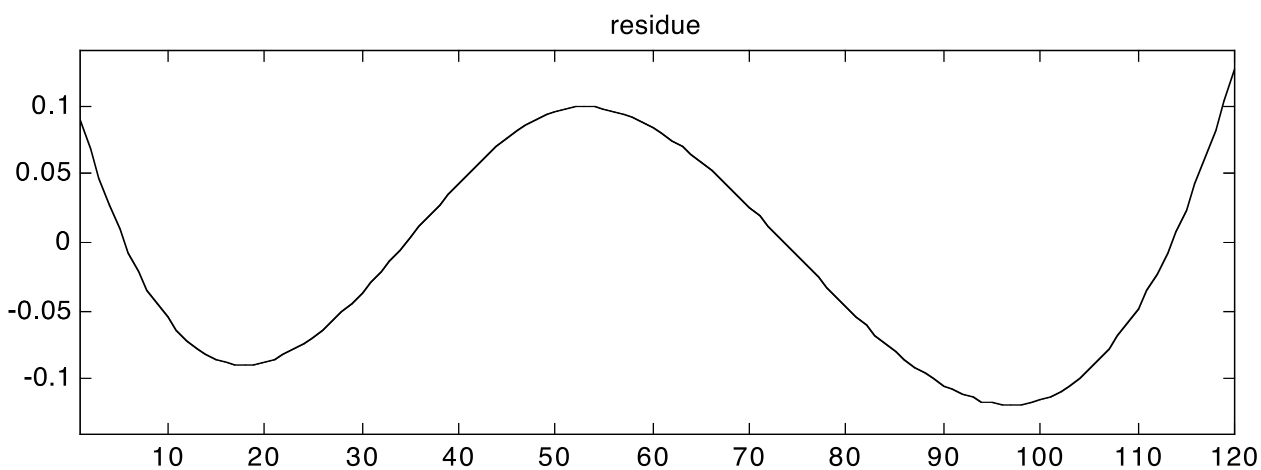
90
100
110
120



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0
-0.05
-0.1
10
20
30
40
50
60
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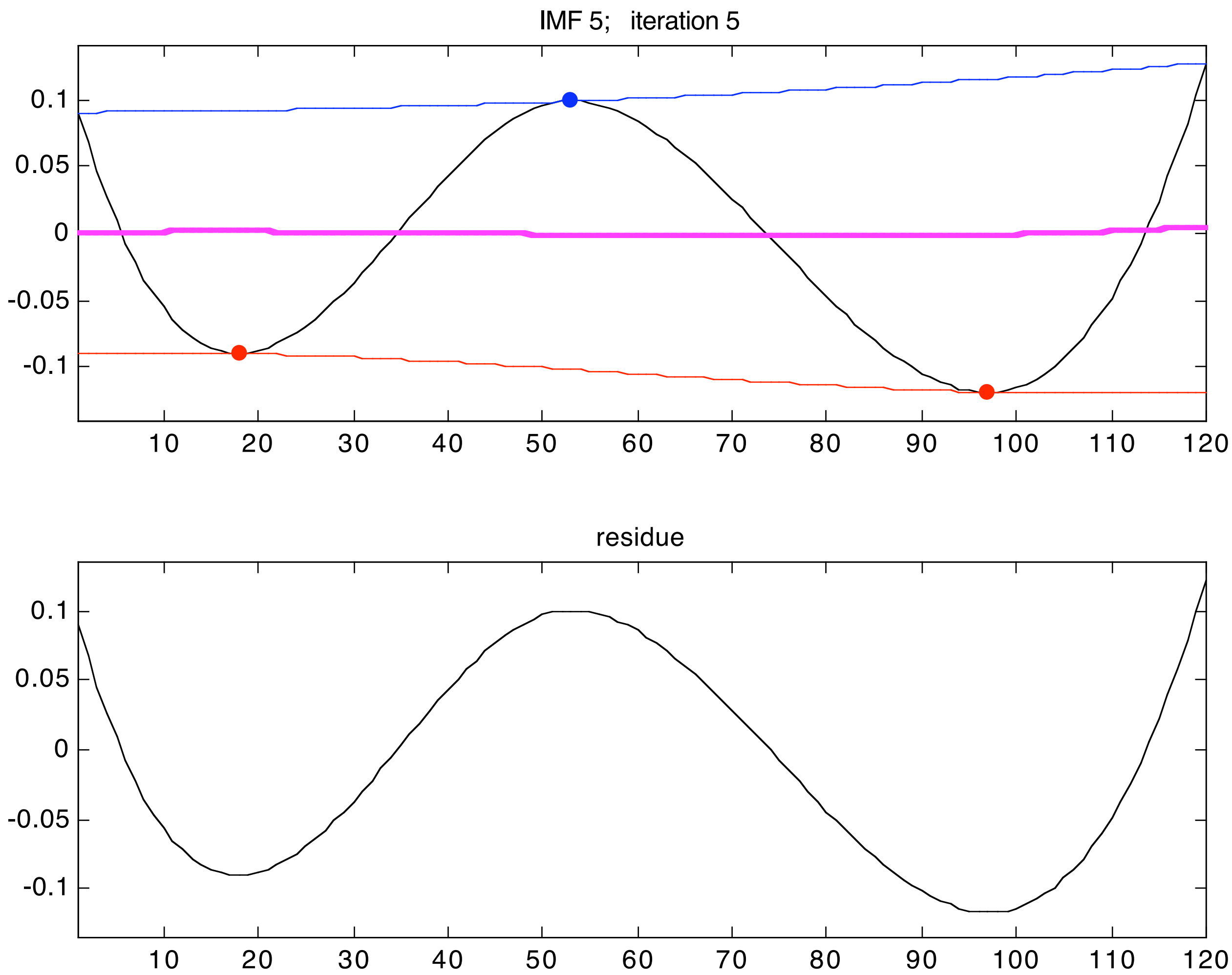
80
90
100
110
120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-0.1
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20
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80
90
100
110
120
residue
0.1
0.05
0
-0.05
-0.1
10
20
30
40
50
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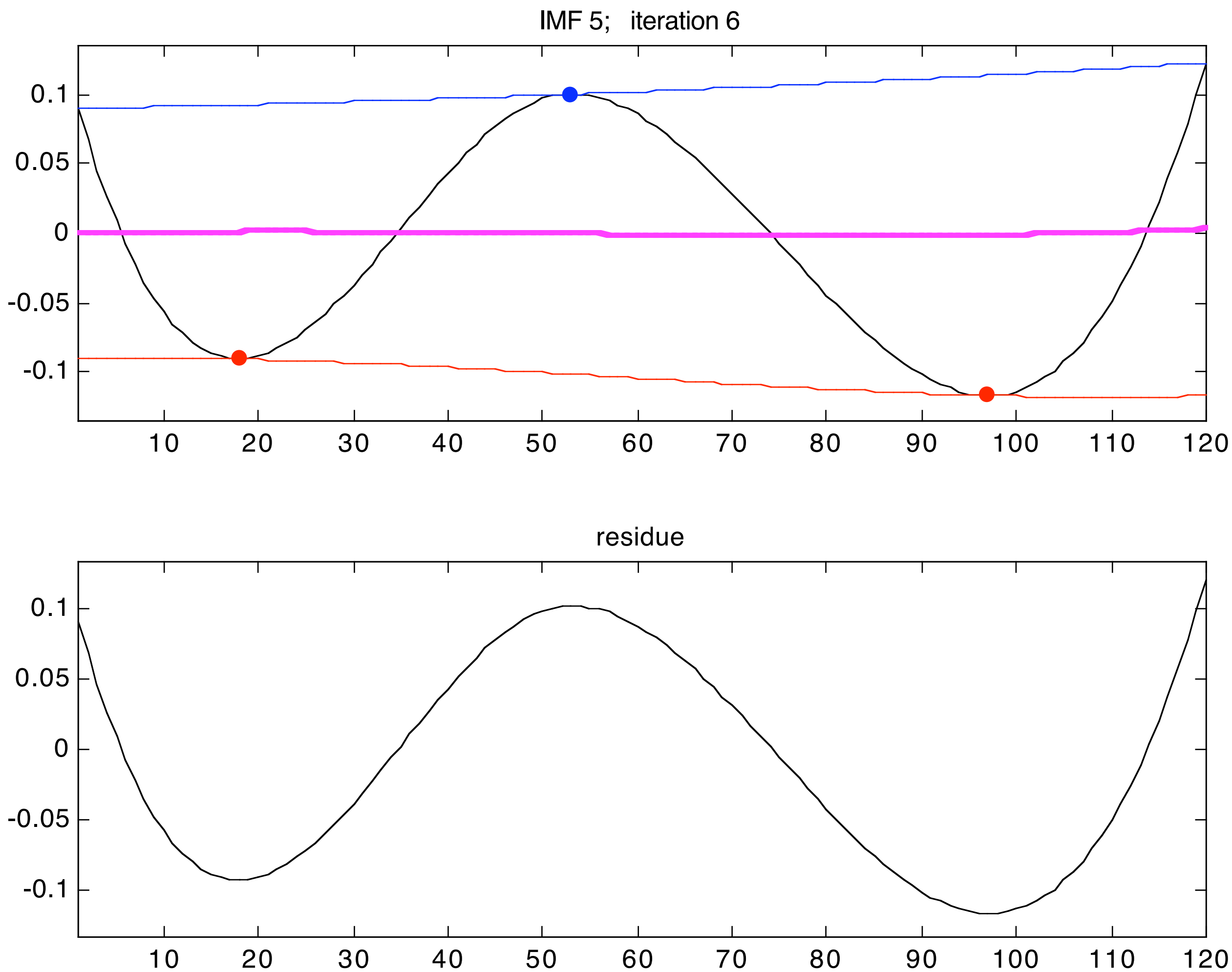
120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-0.1
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100
110
120
residue
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0.05
0
-0.05
-0.1
10
20
30
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50
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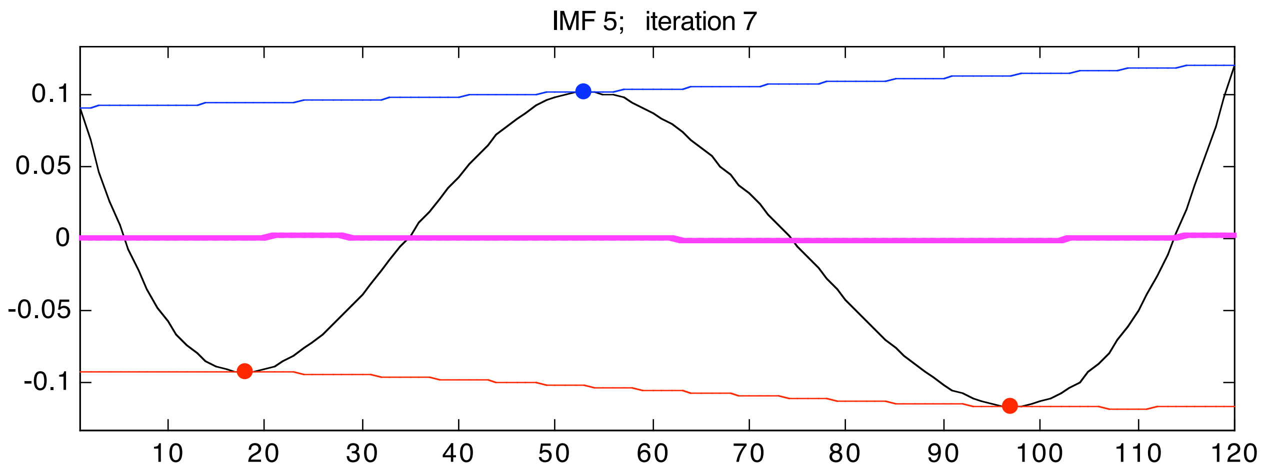
90
100
110
120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110
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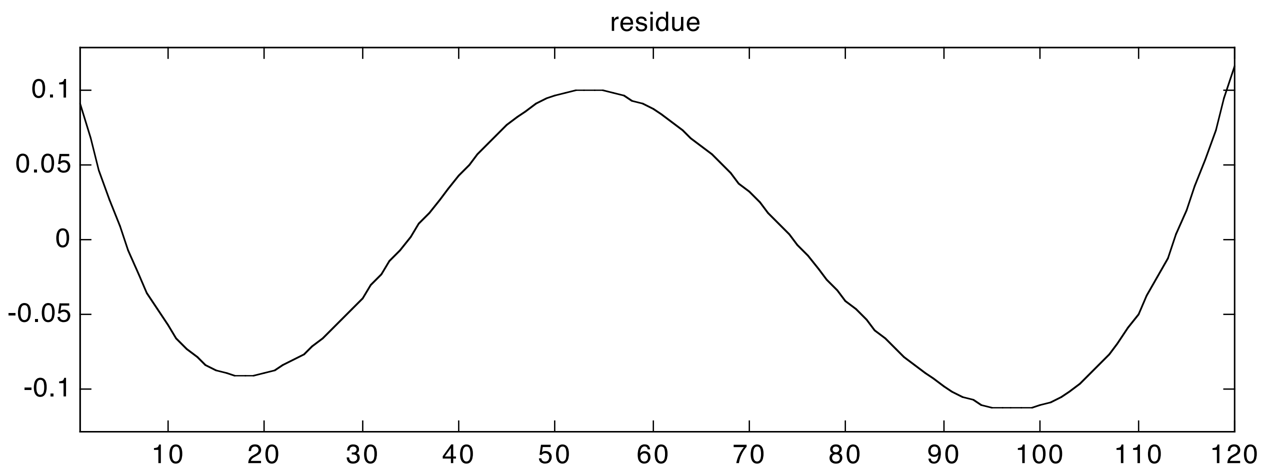
residue
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-0.05
-0.1
10
20
30
40
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60
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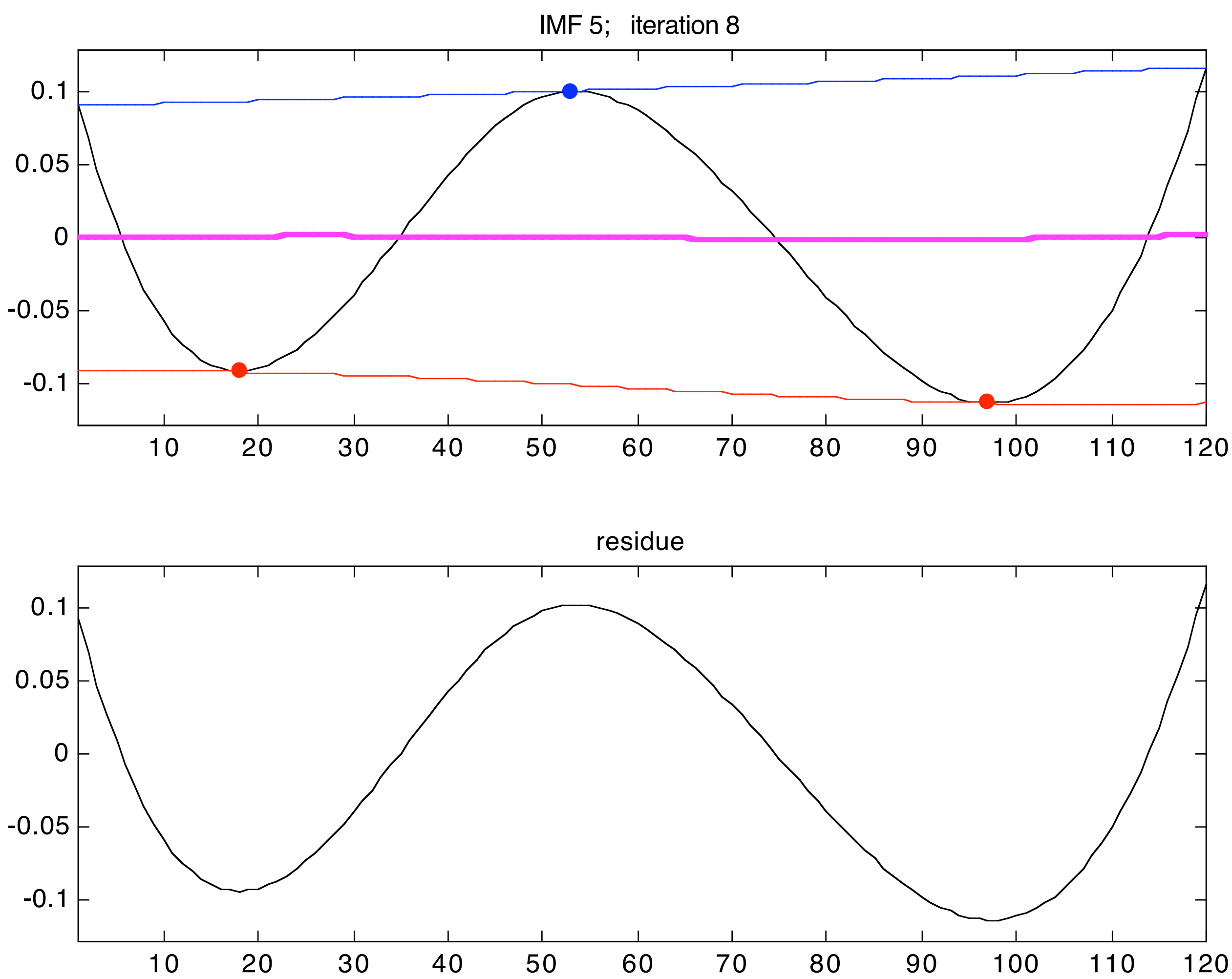
100
110
120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0
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-0.1
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20
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40
50
60
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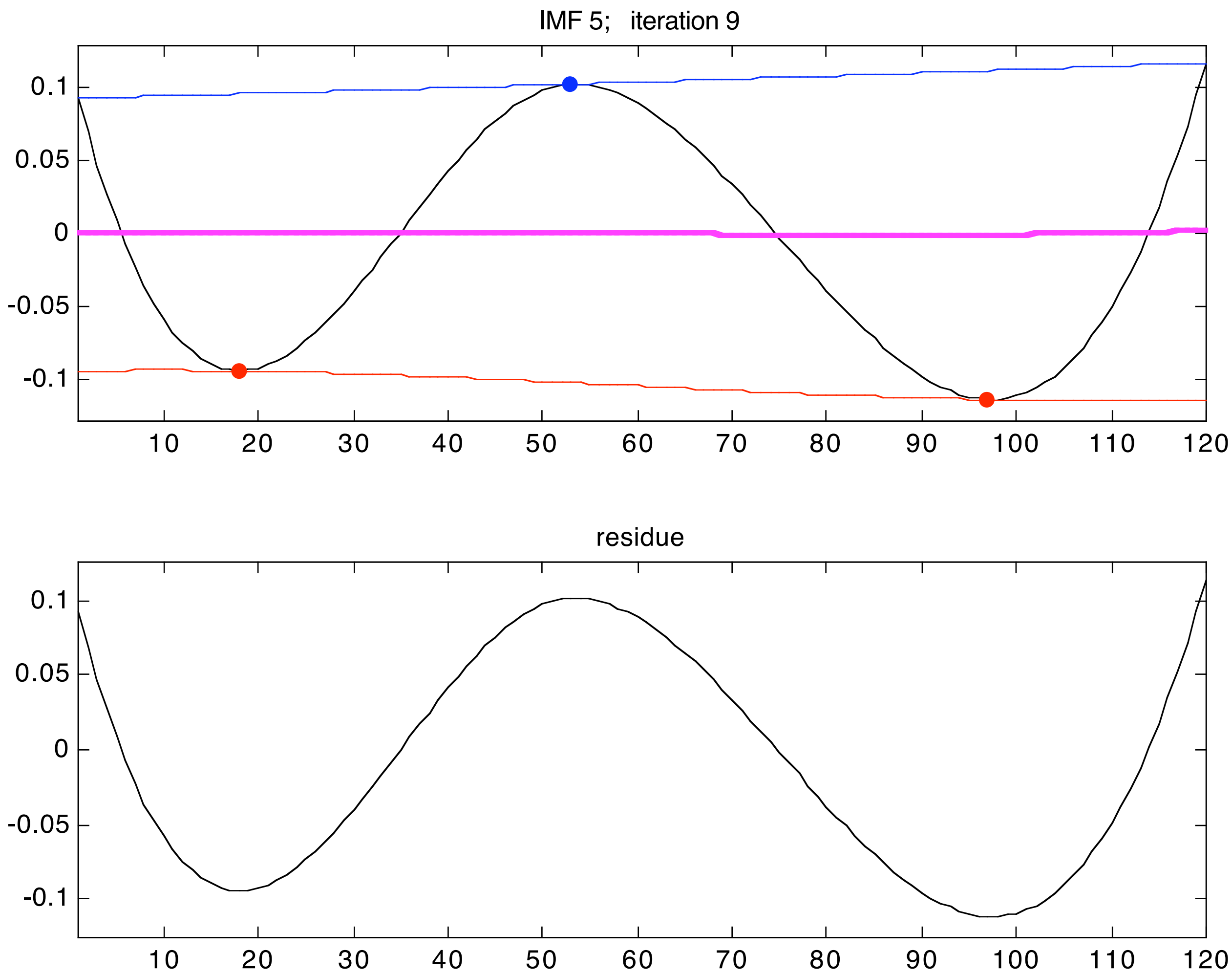
80
90
100
110
120
residue
0.1
0.05
0
-0.05
-0.1
10
20
30
40
50
60
70
80
90
100
110
120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0
-0.05
-0.1
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40
50
60
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80
90
100
110
120
residue
0.1
0.05
0
-0.05
-0.1
10
20
30
40
50
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80
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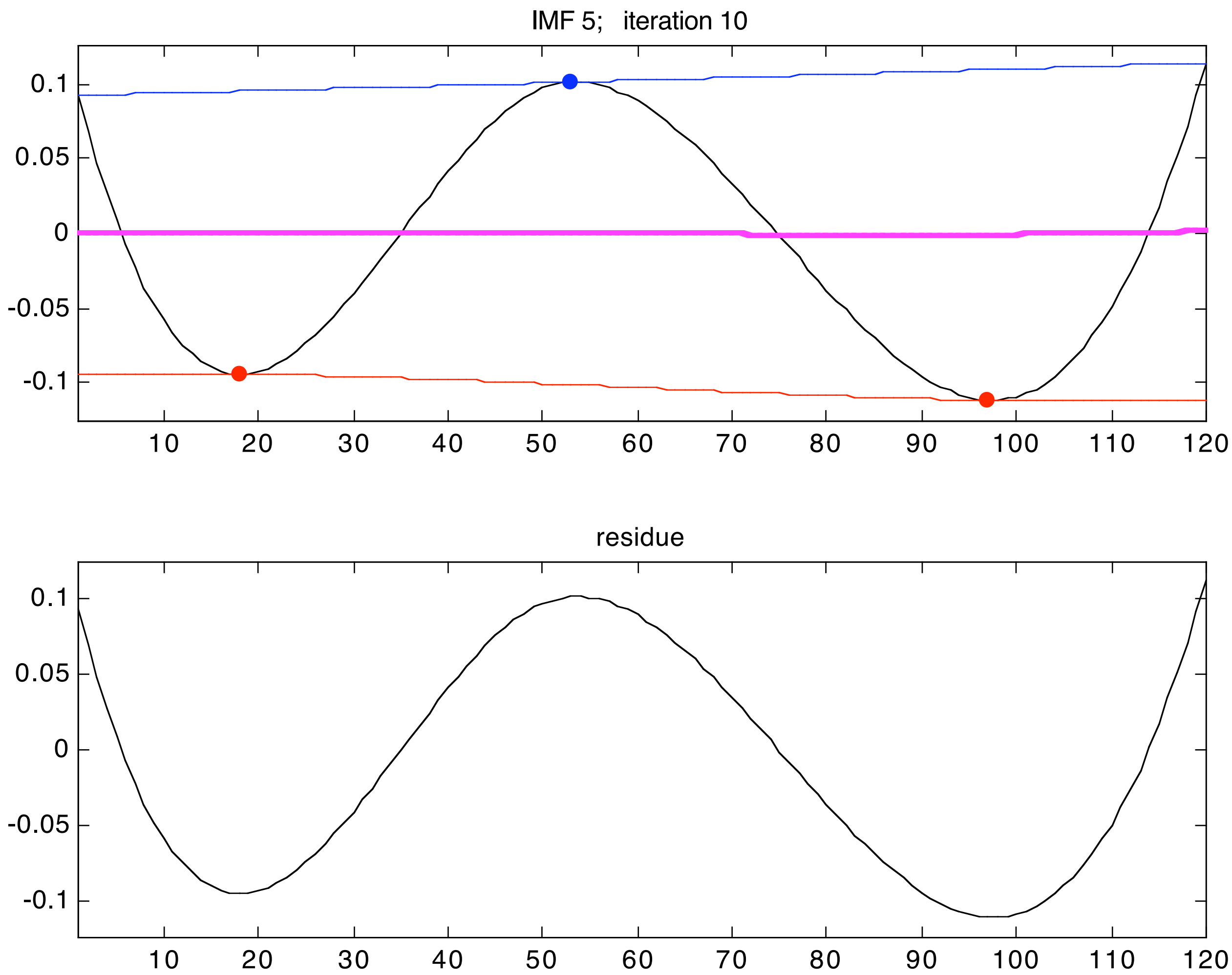
100
110
120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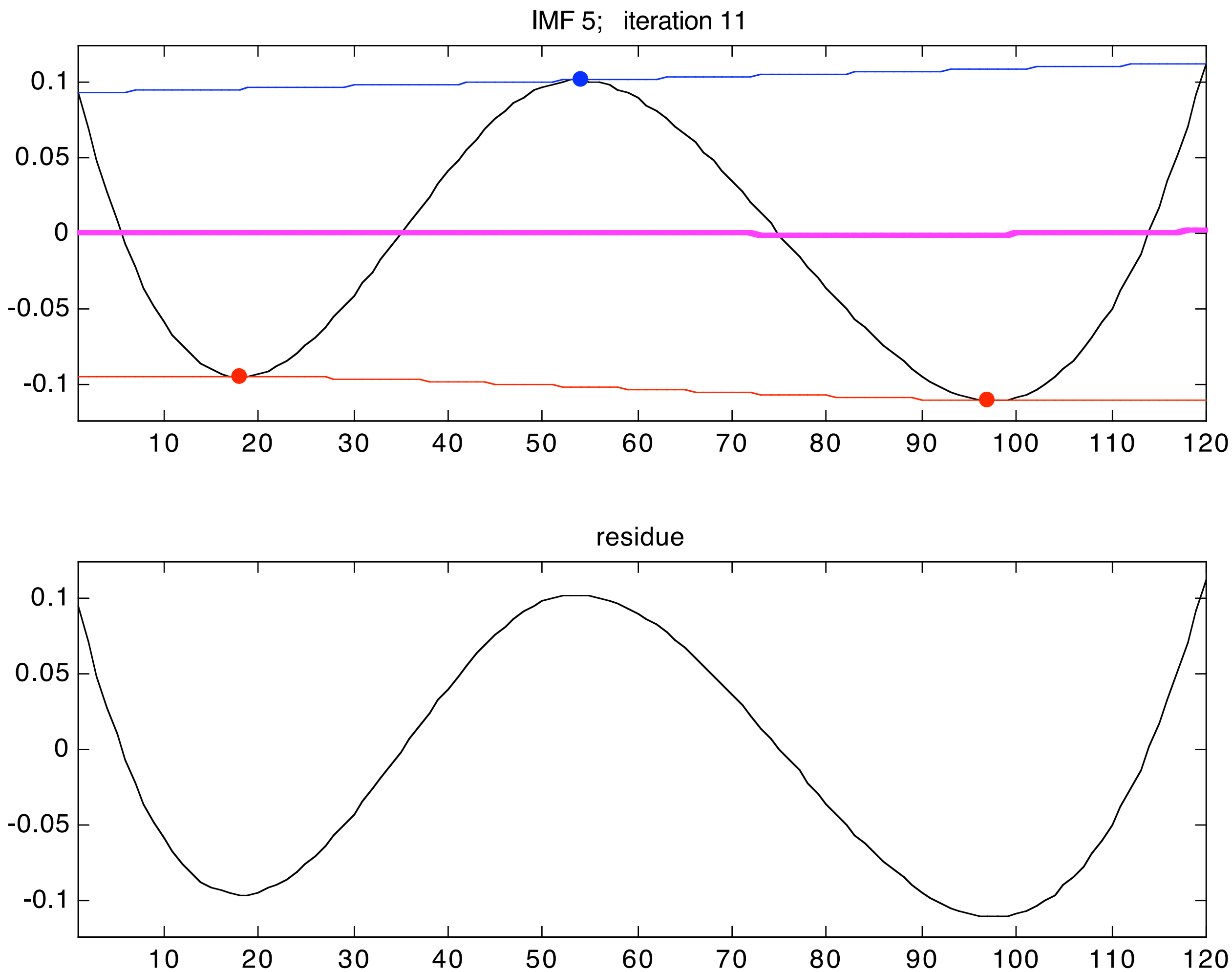
-0.05
-0.1
10
20
30
40
50
60
70
80
90
100
110
120
residue
0.1
0.05
0
-0.05
-0.1
10
20
30
40
50
60
70
80
90
100
110
120



IMF 5;   iteration 11
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0.05
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-0.05
-0.1
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20
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40
50
60
70
80
90
100
110
120
residue
0.1
0.05
0
-0.05
-0.1
10
20
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50
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80
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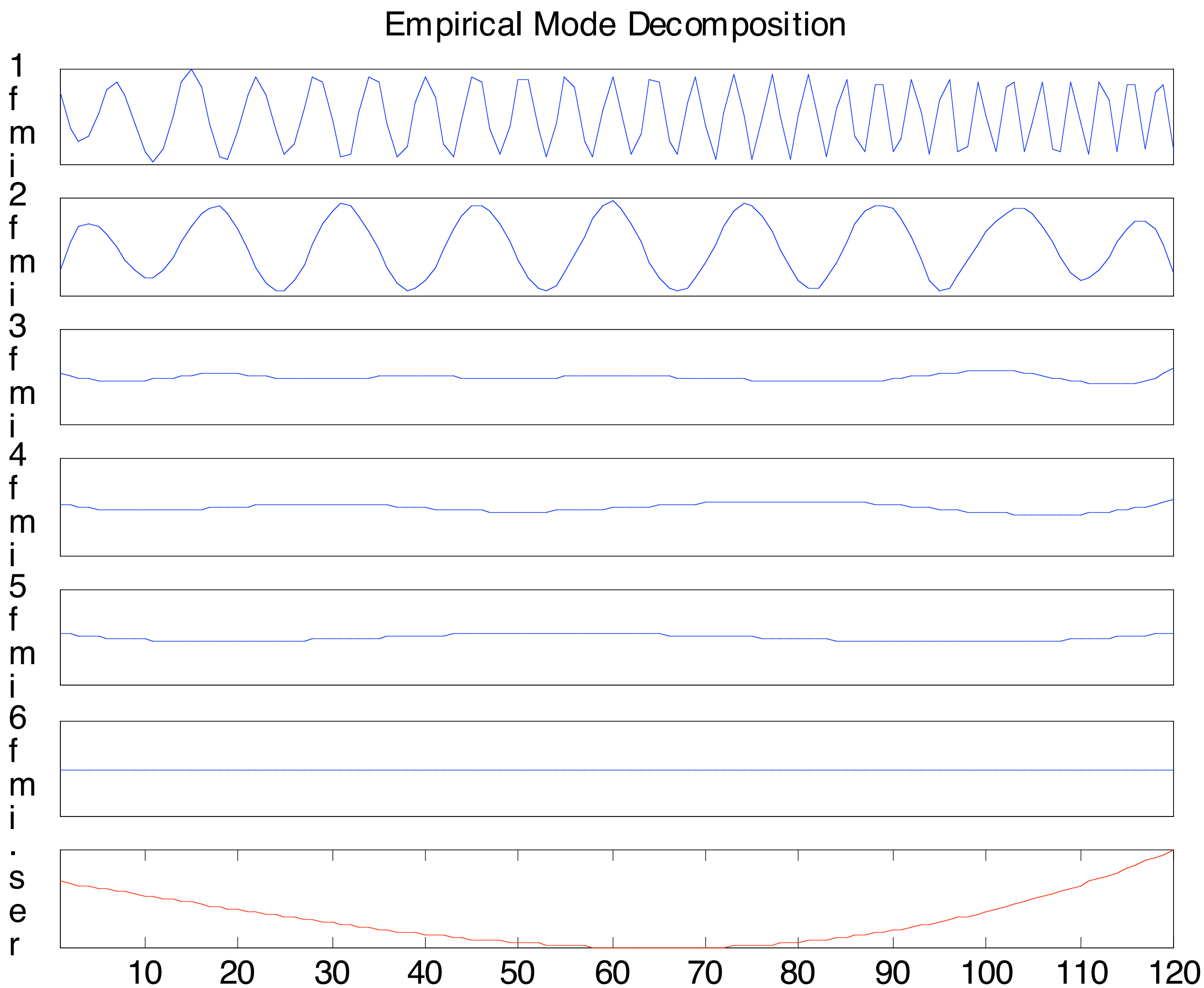
100
110
120


Empirical Mode Decomposition
imf1
imf2
imf3
imf4
imf5
imf6
res.
10
20
30
40
50
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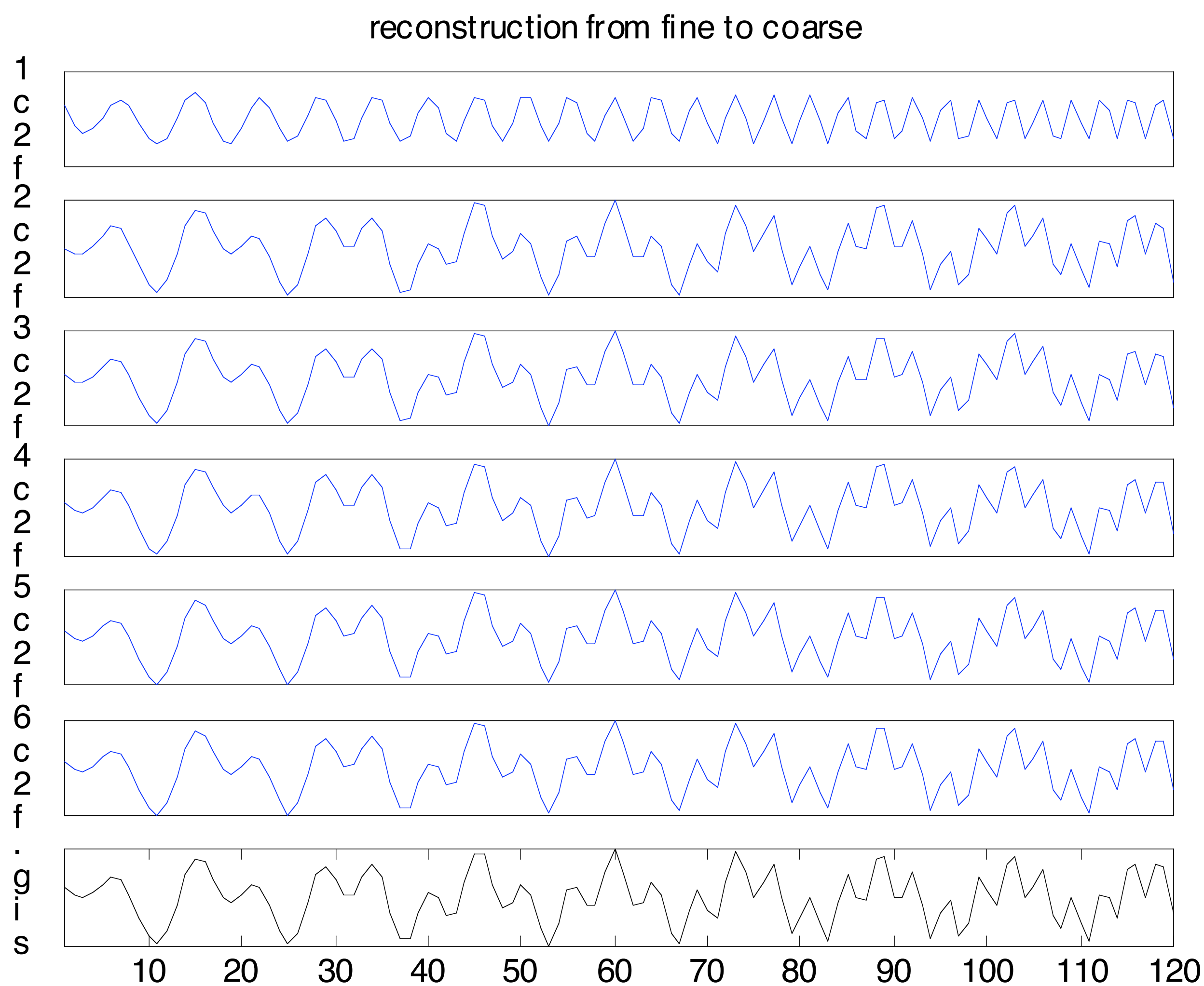
80
90
100
110
120


reconstruction from fine to coarse
f2c1
f2c2
f2c3
f2c4
f2c5
f2c6
sig.
10
20
30
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50
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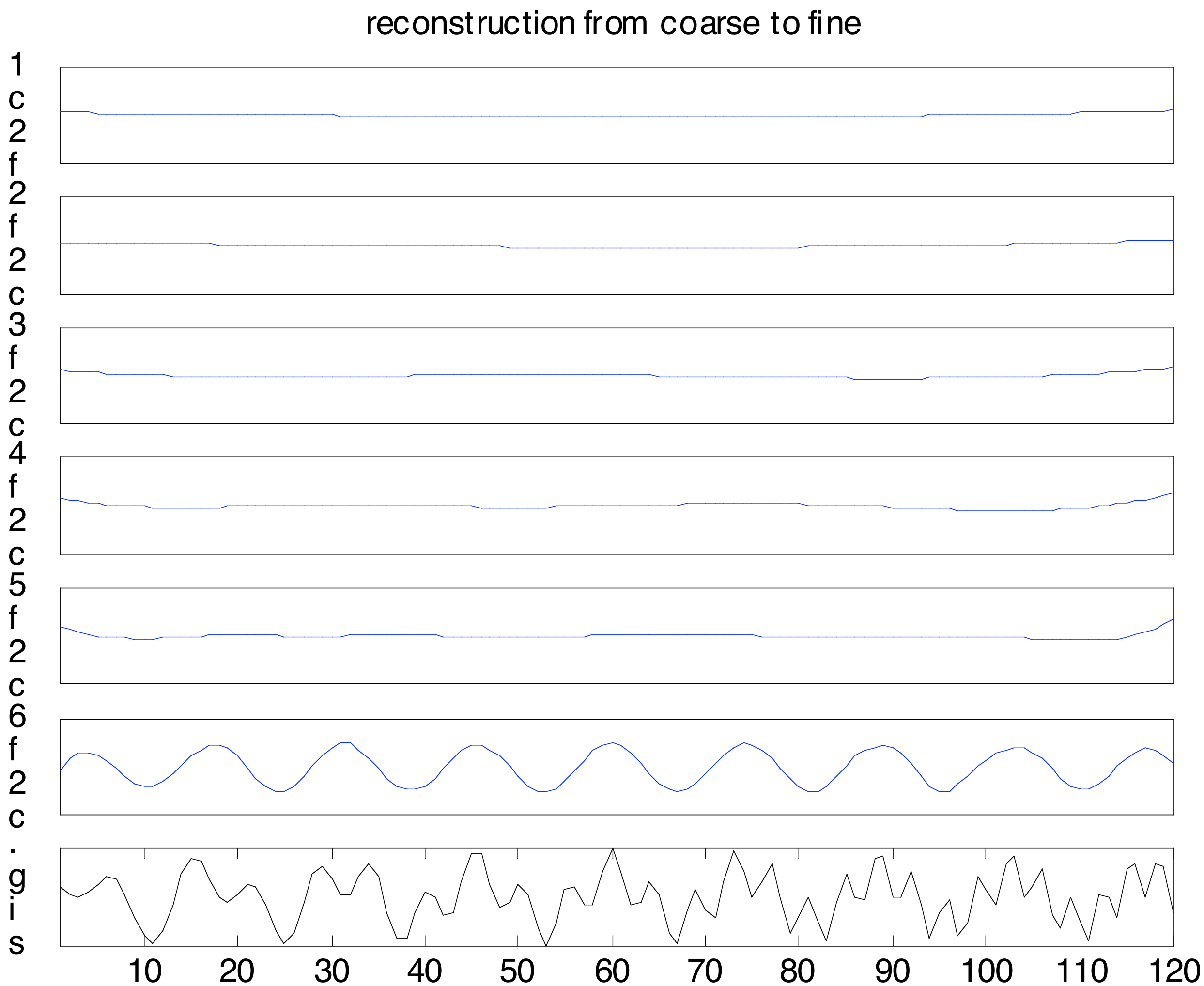
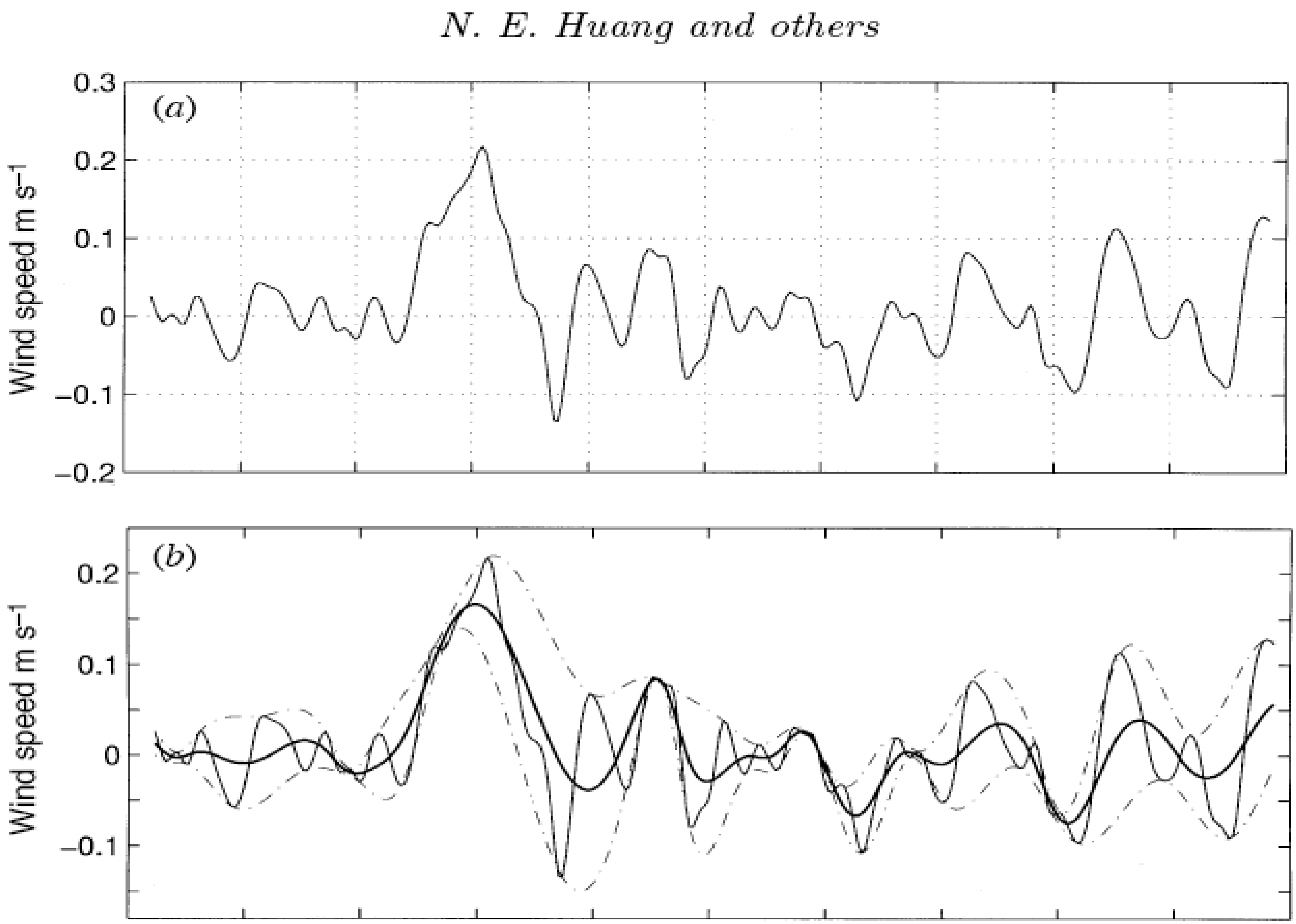
80
90
100
110
120



reconstruction from coarse to fine
c2f1
c2f2
c2f3
c2f4
c2f5
c2f6
sig.
10
20
30
40
50
60
70
80
90
100
110
120

Nontrivial Example



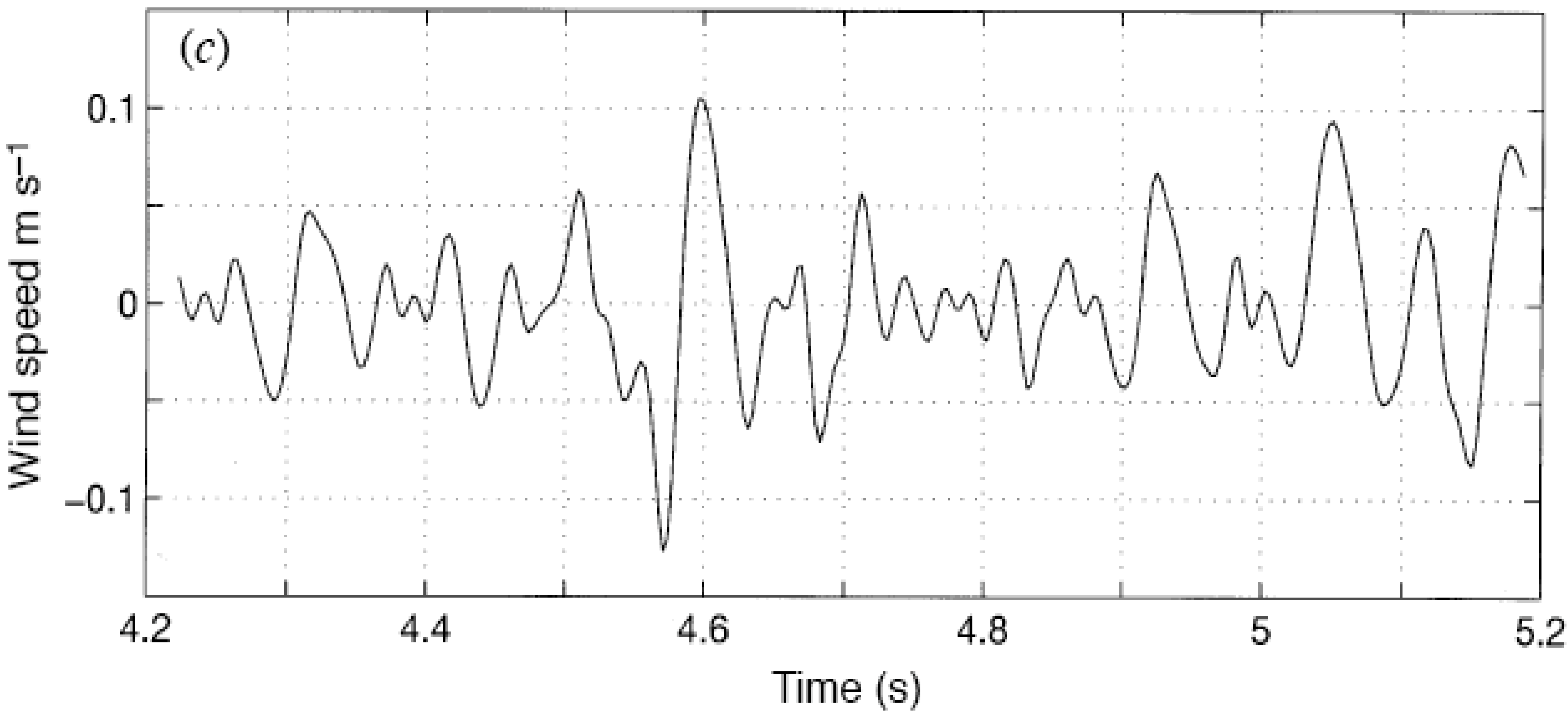


Figure 3. Illustration of the sifting processes: (a) the original data; (b) the data in thin solid line, with the upper and lower envelopes in dot-dashed lines and the mean in thick solid line; (c) the difference between the data and m_1. This is still not an IMF, for there are negative local maxima and positive minima suggesting riding waves.

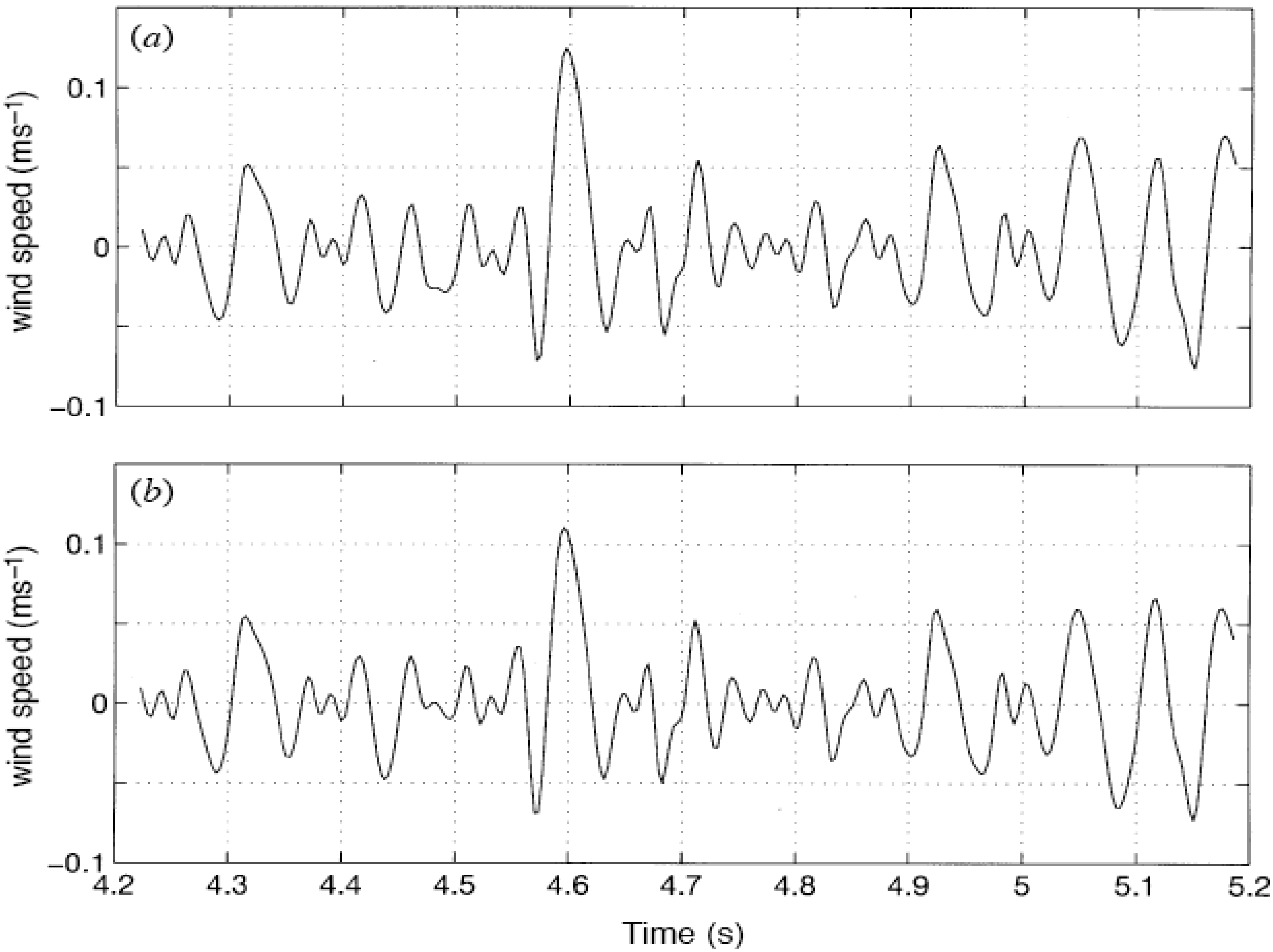


Figure 4. Illustration of the effects of repeated sifting process: (a) after one more sifting of the result in figure 3c, the result is still asymmetric and still not a IMF; (b) after three siftings, the result is much improved, but more sifting needed to eliminate the asymmetry. The final IMF is shown in figure 2 after nine siftings.

Converged h_1 after 9 iterations

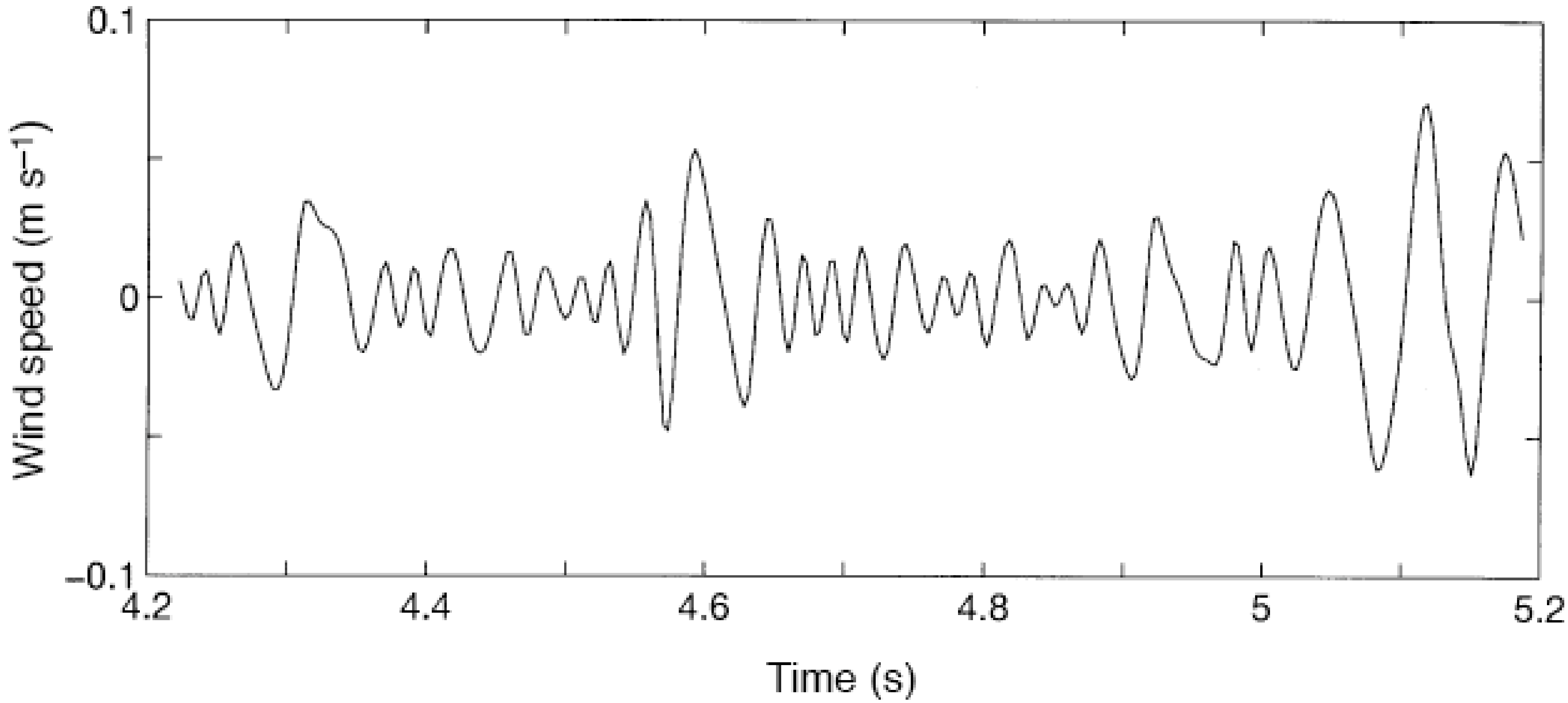


Figure 2. A typical intrinsic mode function with the same numbers of zero crossings and extrema, and symmetry of the upper and lower envelopes with respect to zero.

Intrinsic Filtering Capability

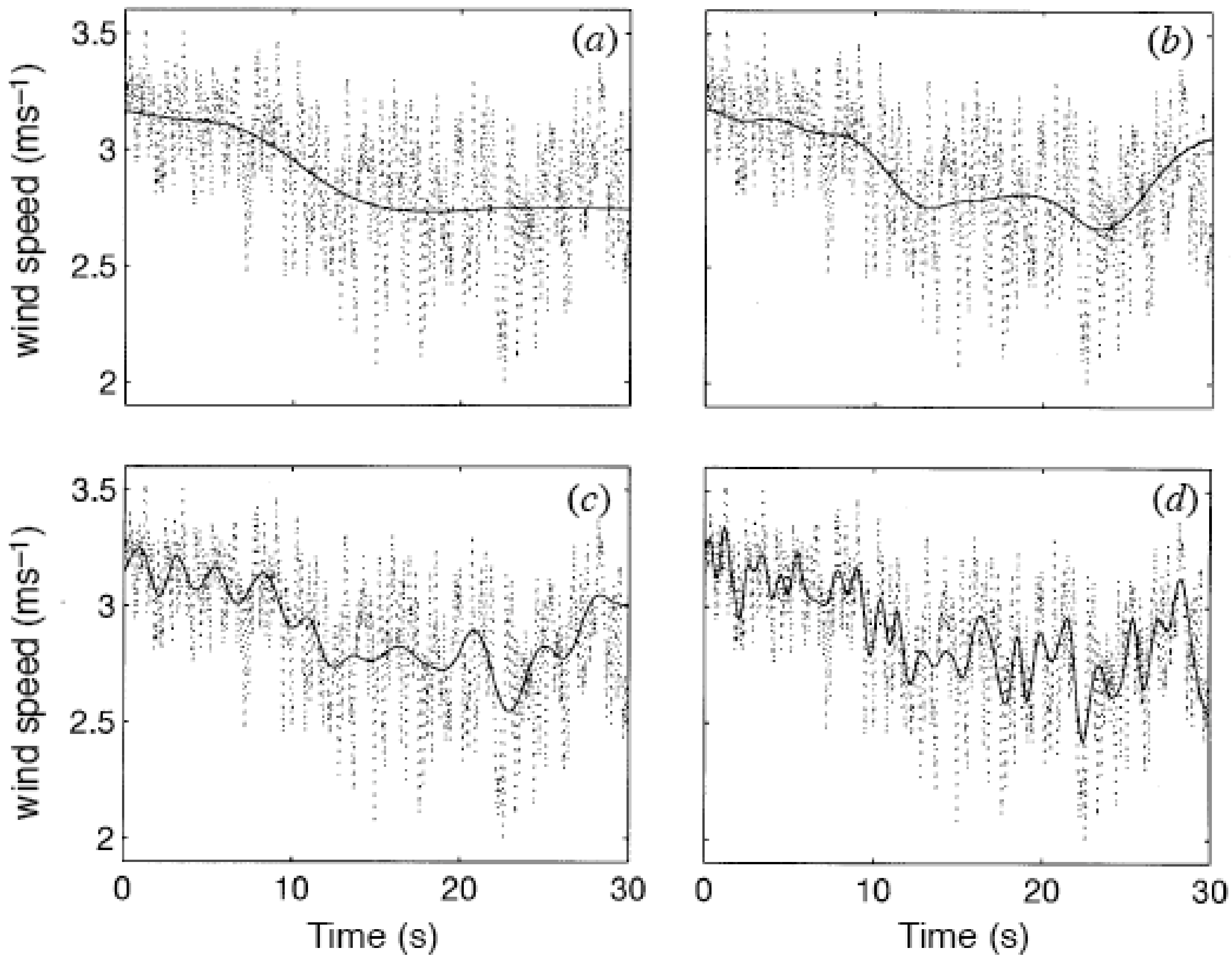
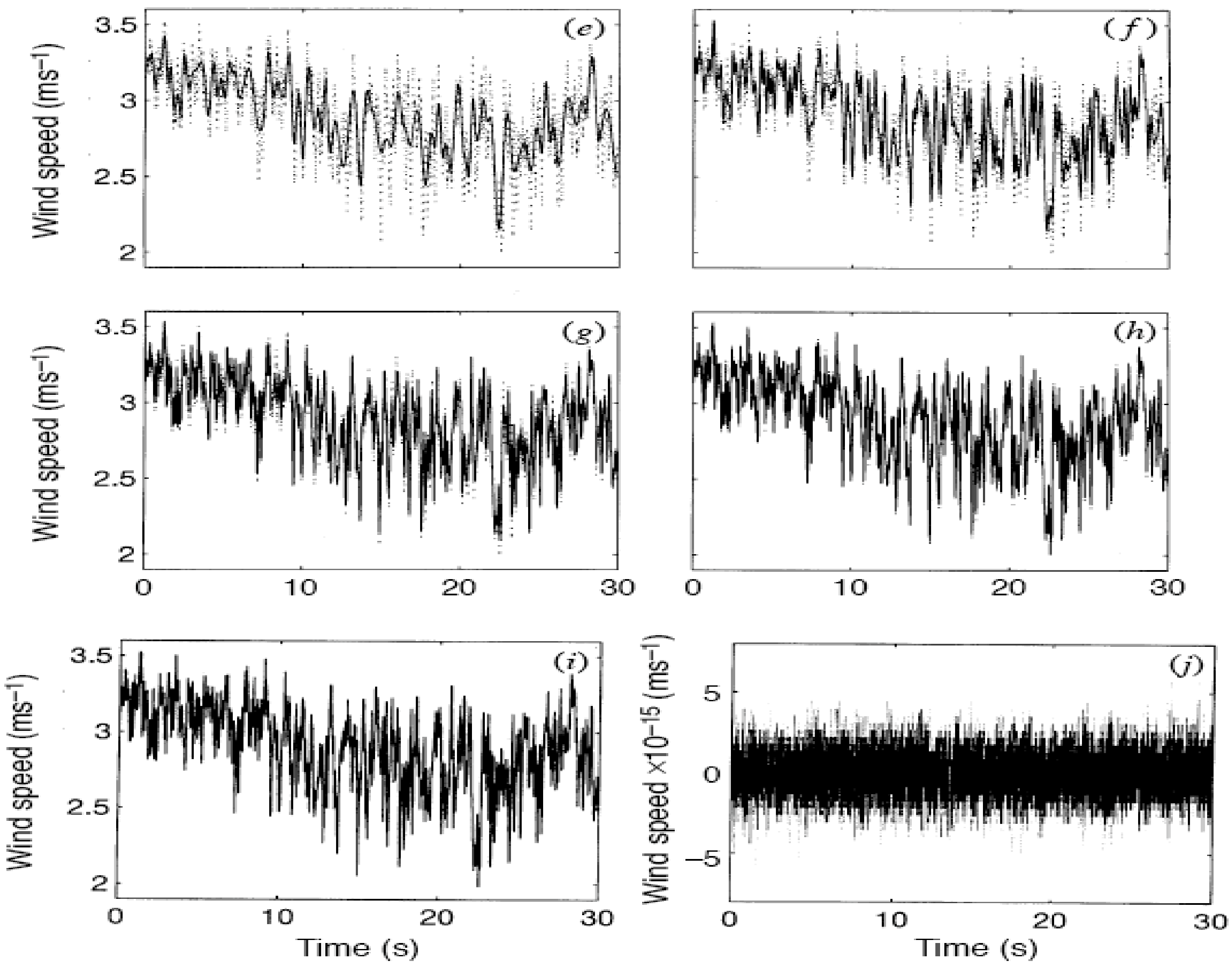


Figure 7. Numerical proof of the completeness of the EMD through reconstruction of the original data from the IMF components. (*a*) Data (in the dotted line) and the c_9 component (in the solid line). c_9 serves as a running mean, but it is not obtained from either mean or filtering. (*b*) Data (in the dotted line) and the sum of c_9–c_8 components (in the solid line). (*c*) Data (in the dotted line) and the sum of c_9–c_7 components (in the solid line). (*d*) Data (in the dotted line) and the sum of c_9–c_6 components (in the solid line).

Completeness Demonstration



gure 7. *Cont.* (*e*) Data (in the dotted line) and the sum of c_9–c_5 components (in the solid line).
) Data (in the dotted line) and the sum of c_9–c_4 components (in the solid line). (*g*) Data (in
e dotted line) and the sum of c_9–c_3 components (in the solid line). By now, we seem to have
covered all the energy containing eddies. (*h*) Data (in the dotted line) and the sum of c_9–c_2
nponents (in the solid line). (*i*) Data (in the dotted line) and the sum of c_9–c_1 components (in
e solid line). This is the final reconstruction of the data from the IMFs. It appears no different
m the original data. (*j*) The difference between the original data and the reconstructed one;
e difference is the limit of the computational precision of the personal computer (PC) used.

Why the Hilbert Transform?

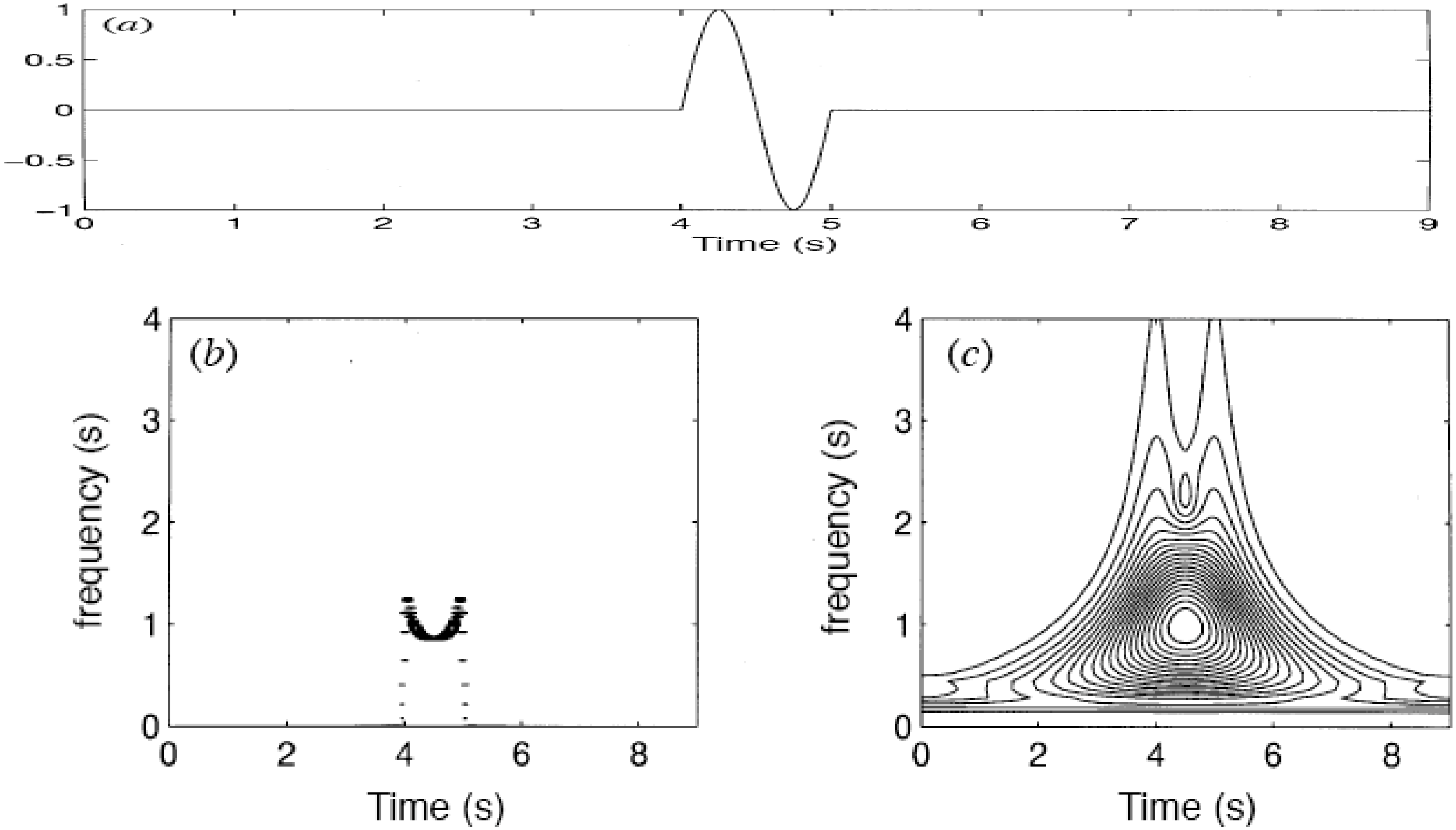


Figure 14. A calibration of time localization of the Hilbert spectrum analysis. (*a*) The calibration data, a single sine wave. (*b*) The Hilbert spectrum for the calibration signal: the energy is highly localized in time and frequency, though there are some end effects. (*c*) The Morlet wavelet spectrum for the calibration signal: the calibration signal is localized by the high-frequency components, yet the energy distribution in the frequency space spreads widely in comparison with the Hilbert spectrum.

Back to HHT mathematics:
The Hilbert Transform and Instantaneous Frequency

For an arbitrary time series, x(t), its Hilbert transform, y(t), is defined as

$$y(t) = \frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{x(\tau)}{t-\tau} d\tau$$

where PV indicates the Cauchy principal value.

Observe that the Hilbert transform, when viewed from a windowed sampling, the weight $1/(t-\tau)$ can be interpreted as a windowing width as the Hilbert transform is nothing but a convolution integral of $x(t)$ with $1/t$. Thus, the Hilbert transform emphasizes the local properties of the signal, $x(t)$.

The Hilbert Spectrum

For each of IMFs $(c_j, j = 1, ..., k)$ and their corresponding instantaneous frequencies $(\omega_j, j = 1, ..., k)$, let their Hilbert complexification be

$$z_j(t) = c_j(t) + i\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{c_j(t')}{t-t'}dt' = a_j(t)e^{i\theta(t)},$$

$$\theta_j(t) = \int \omega_j(t)dt, \quad j = 1, ..., k.$$

$$X(t) = \sum_{j=1}^{k} a_j(t)\ e^{i\omega_j t}$$

The Hilbert spectrum: a three-dimensional plot of the amplitude $a_j(t)$ vs time (t) and the instantaneous frequency $\omega_j(t)$, designated as $H(\omega,t)$,either for each amplitude or the sum of all the amplitudes.

Utilizations of the Hilbert Spectrum

Marginal Spectrum: $h(\omega) = \int_0^T H(\omega, t)dt$

Instantaneous Energy density Level: $IE(t) = \int_\omega H^2(\omega, t)d\omega$

Mean Marginal Spectrum: $n(\omega) = \frac{1}{T}h(\omega)$

Degree of Stationarity: $DS(\omega) = \frac{1}{T}\int_0^T (1 - \frac{H(\omega,t)}{n(\omega)})^2 dt$

Degree of Statistic Stationarity: $DSS(\omega, \triangle T) = \frac{1}{T}\int_0^T (1 - \frac{\overline{H(\omega,t)}}{n(\omega)})^2 dt$

Sampling Guidlines:

$$N = \frac{1/n\triangle t)}{(1/T)} = \frac{T}{n\triangle t}$$

N is the maximum number of frequency cells

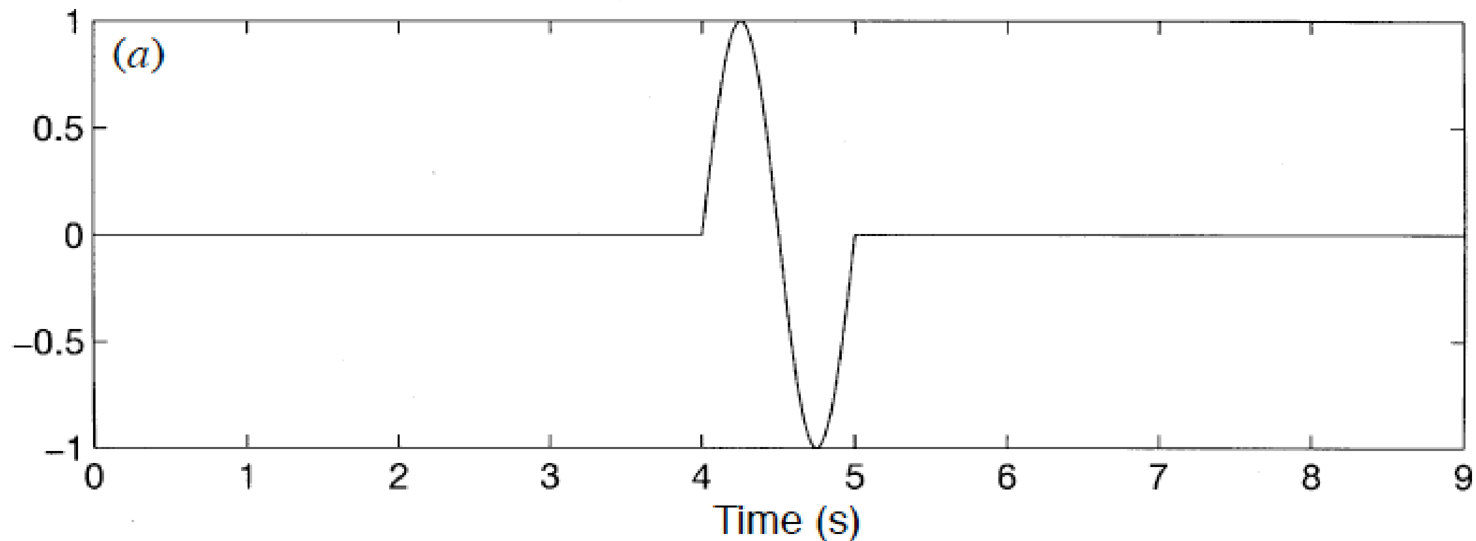
T is the total data length

$\triangle t$ is the sampling rate

n is the minimum number of $\triangle t$ needed to define the frequency accurtely

Use an averaged values over three cell values!

Comparison of Hilbert Spectrum



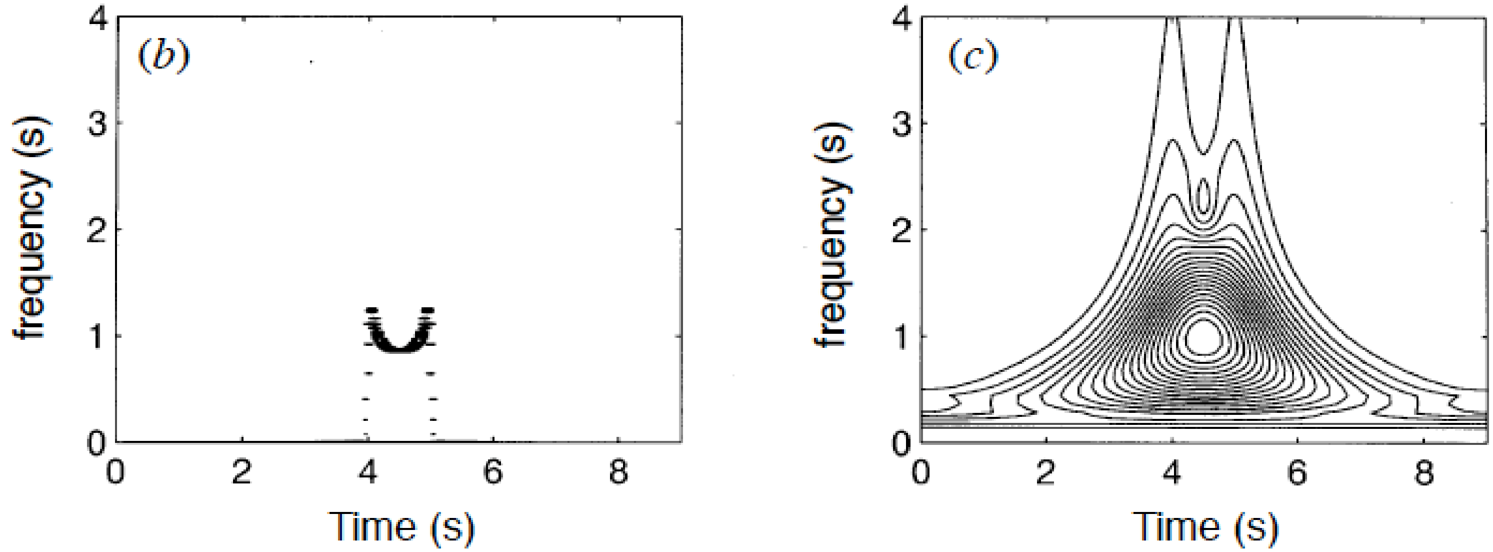


Figure 14. A calibration of time localization of the Hilbert spectrum analysis. (*a*) The calibration data, a single sine wave. (*b*) The Hilbert spectrum for the calibration signal: the energy is highly localized in time and frequency, though there are some end effects. (*c*) The Morlet wavelet spectrum for the calibration signal: the calibration signal is localized by the high-frequency components, yet the energy distribution in the frequency space spreads widely in comparison with the Hilbert spectrum.

How good is the instantaneous frequency?

Duffing Equation: $$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + x(1 + \epsilon x^2) = \gamma \cos \omega t$$

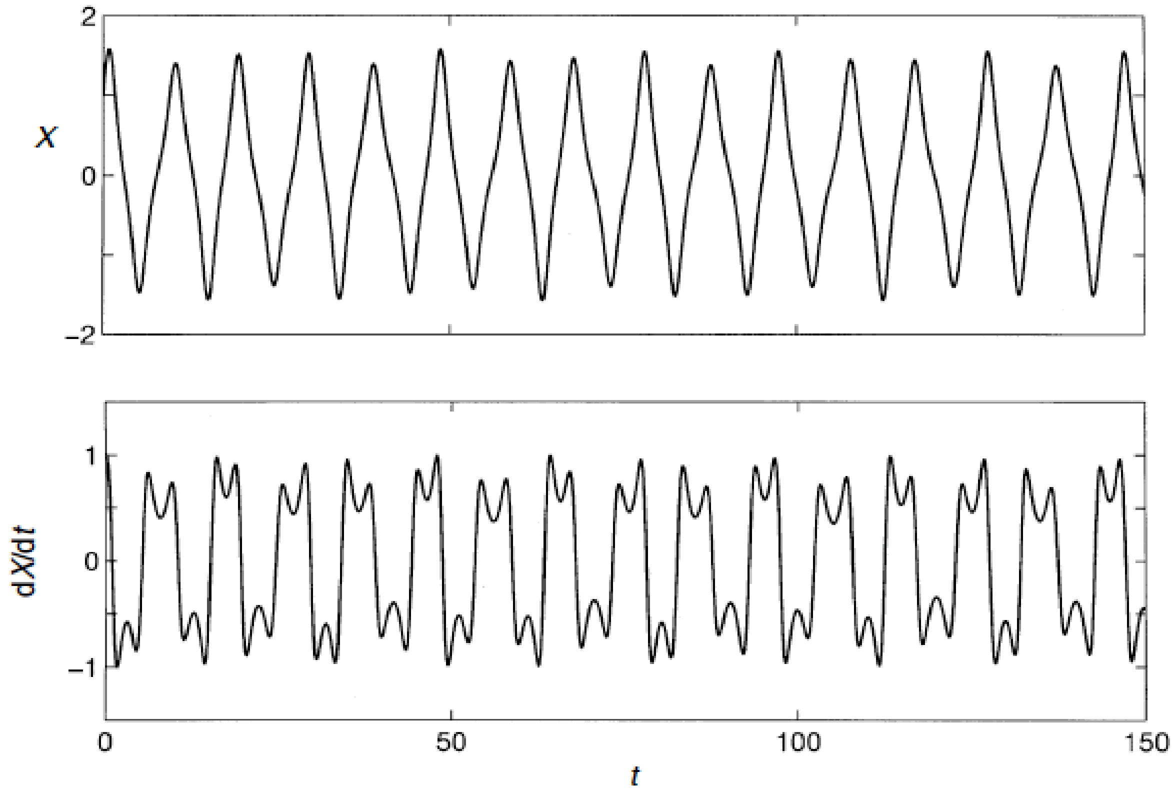


Figure 30. Numerical solution for the Duffing equation. The oscillation is highly nonlinear as indicated by the deformed wave-profile with pointed crests and troughs.

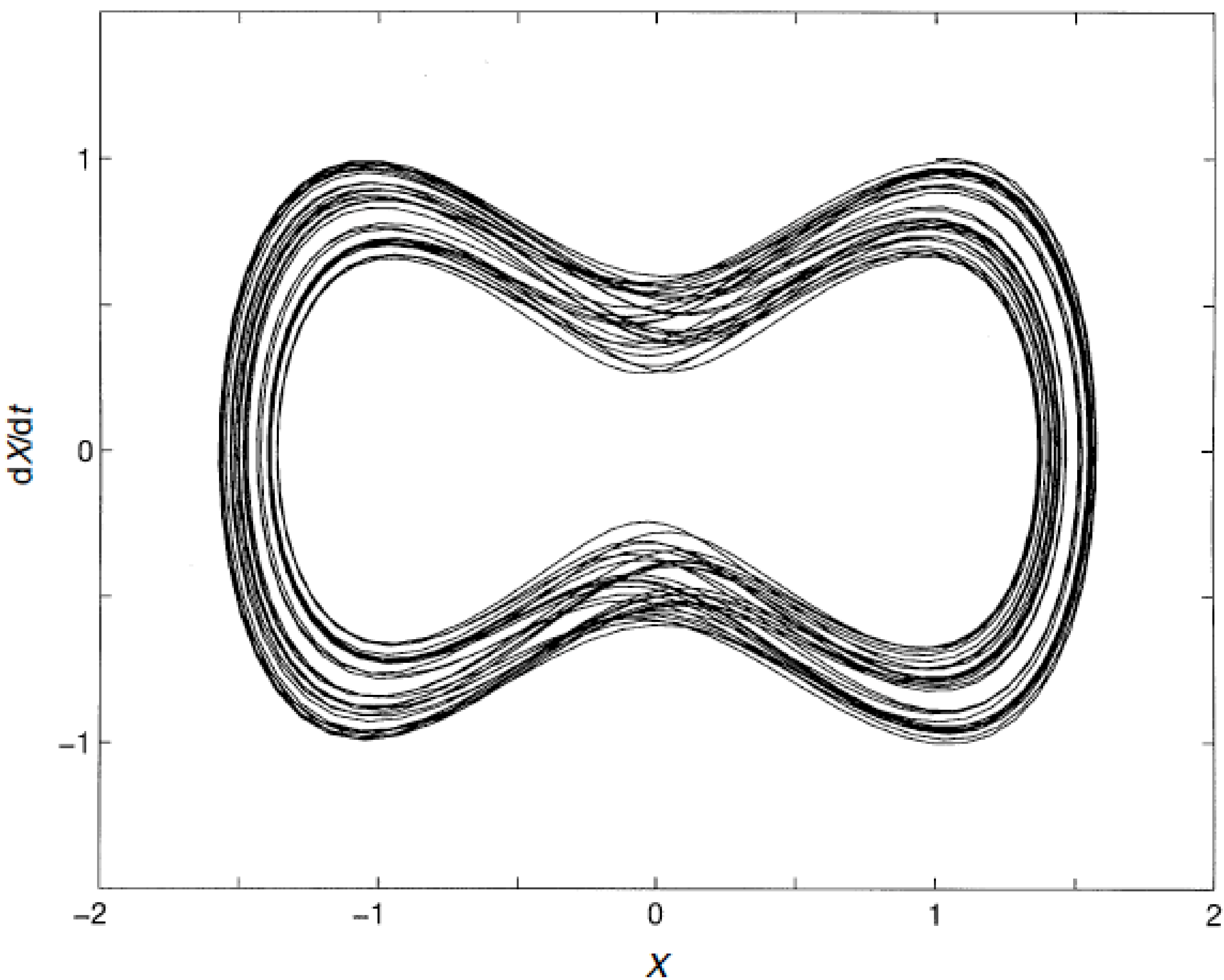


Figure 31. The phase diagram of the Duffing equation solutions with continuous winding indicating no fixed period of the oscillations. The strongly deformed shape from a simple circle gives the pointed crests and troughs of the time series data.

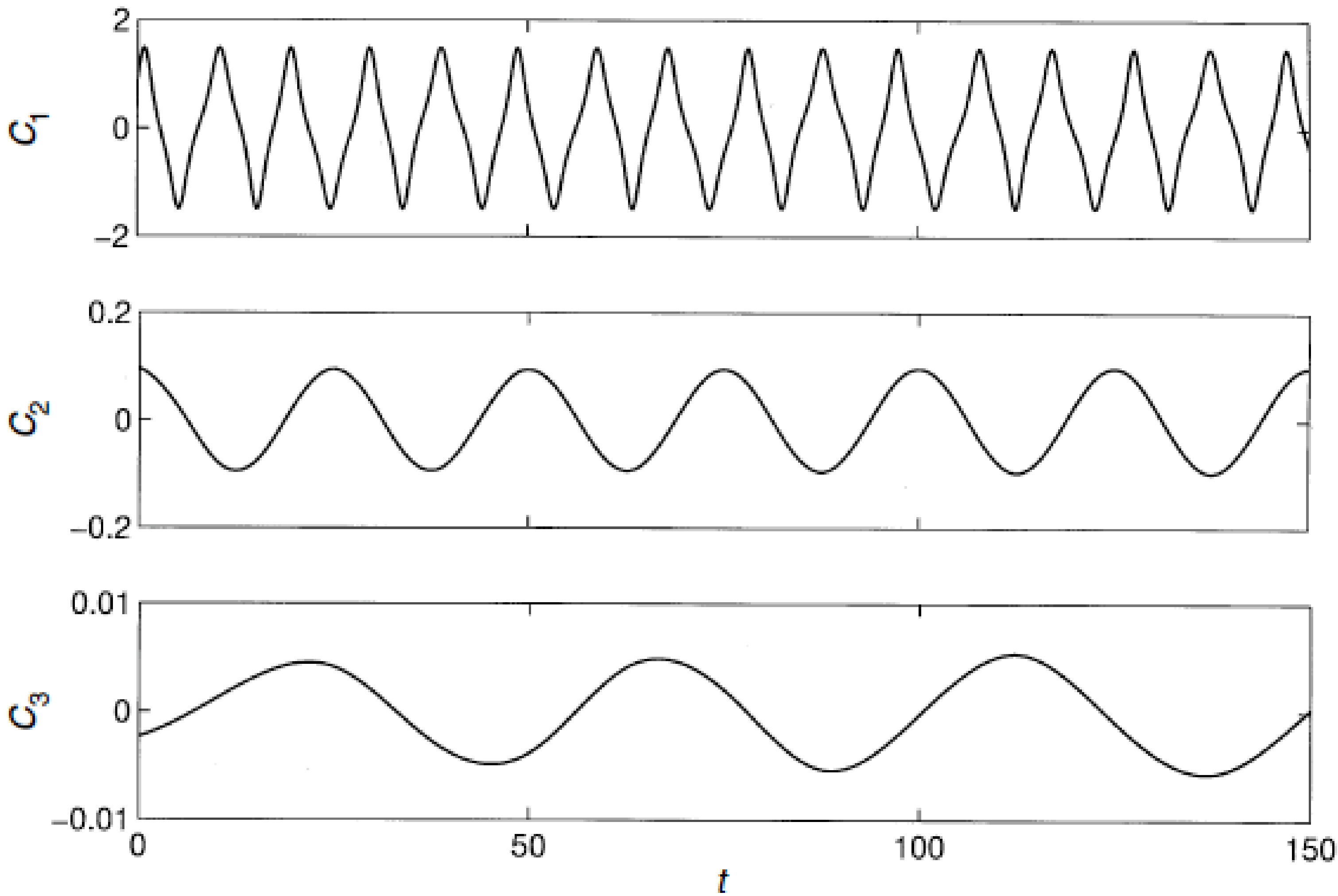


Figure 32. The IMF components of the Duffing equation from EMD: the first component is nonlinear, but it is a physical mode. This example shows that EMD is a nonlinear decomposition method.

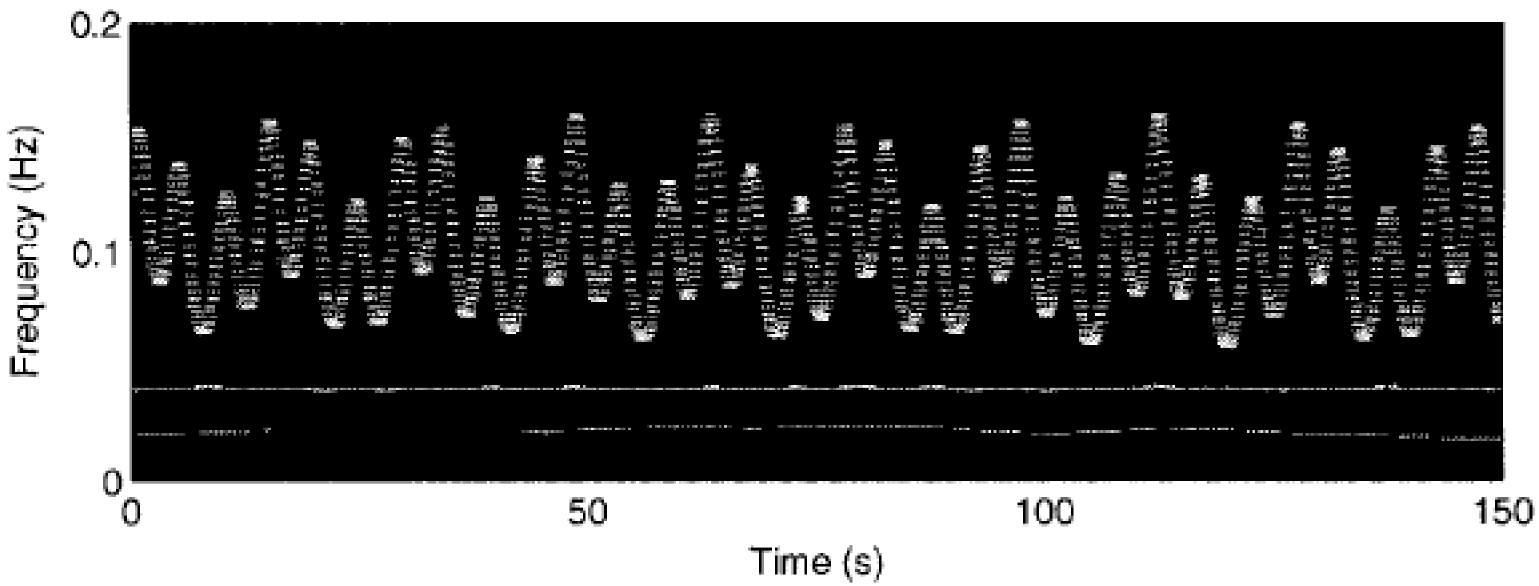


Figure 33. The Hilbert spectrum for the Duffing equation solution showing a strongly intrawave frequency modulated oscillation and two low-frequency oscillations. The intermediate frequency component is the forcing function.

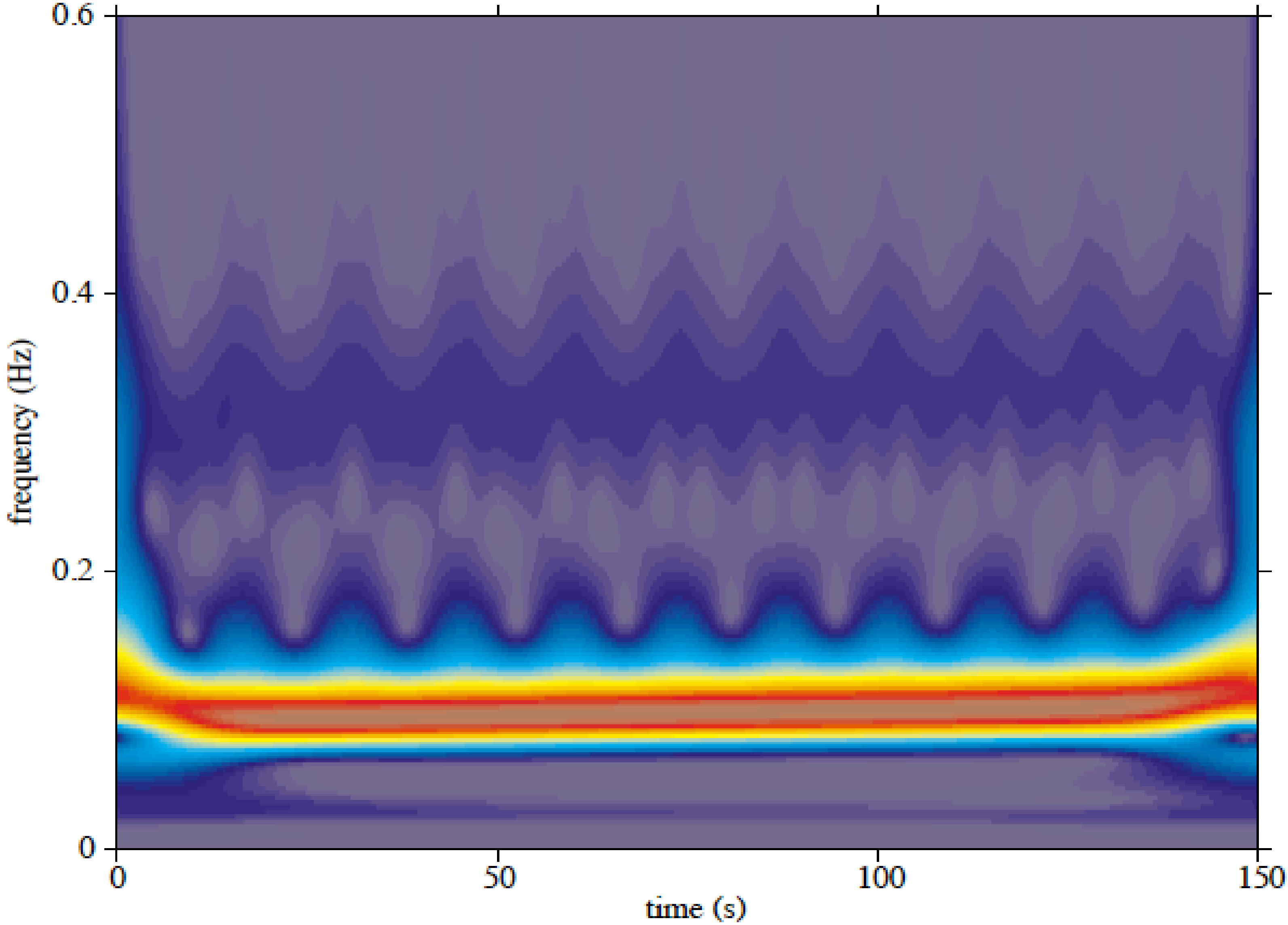


Figure 34. The Morlet wavelet spectrum for the Duffing equation solution showing a rich distribution of harmonics but no details of the intra- or interwave frequency modulations. Notice the scale of this figure. If the Hilbert spectrum is plotted on this scale, all the information is confined to the region between 0.05 and 0.15 Hz.

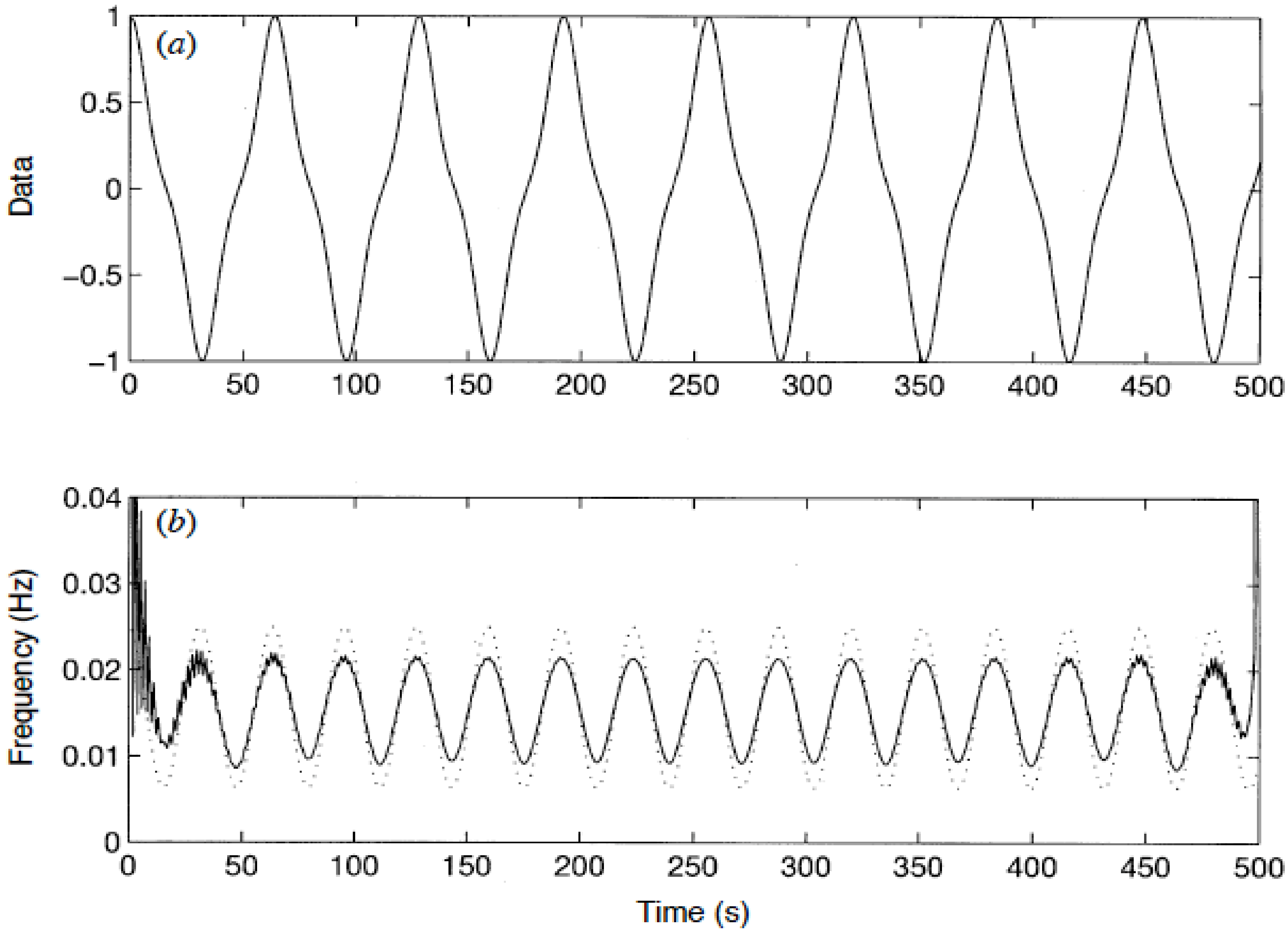


Figure 35. An intrawave frequency modulated Duffing wave: (a) the wave-profile for the model based on equation (9.7) showing the distinct Duffing wave form; (b) the instantaneous frequency of the Duffing wave model based on Hilbert transform (solid line) and based on the classical wave theory and equation (3.4) (dotted line).

Example 2: Period doubling of *Rössler equation*

$$\dot{x} = -(y+z), \quad \dot{y} = x + \tfrac{1}{5}y, \quad \dot{z} = \tfrac{1}{5} + z(x-\mu) \quad \mu = 3.5$$

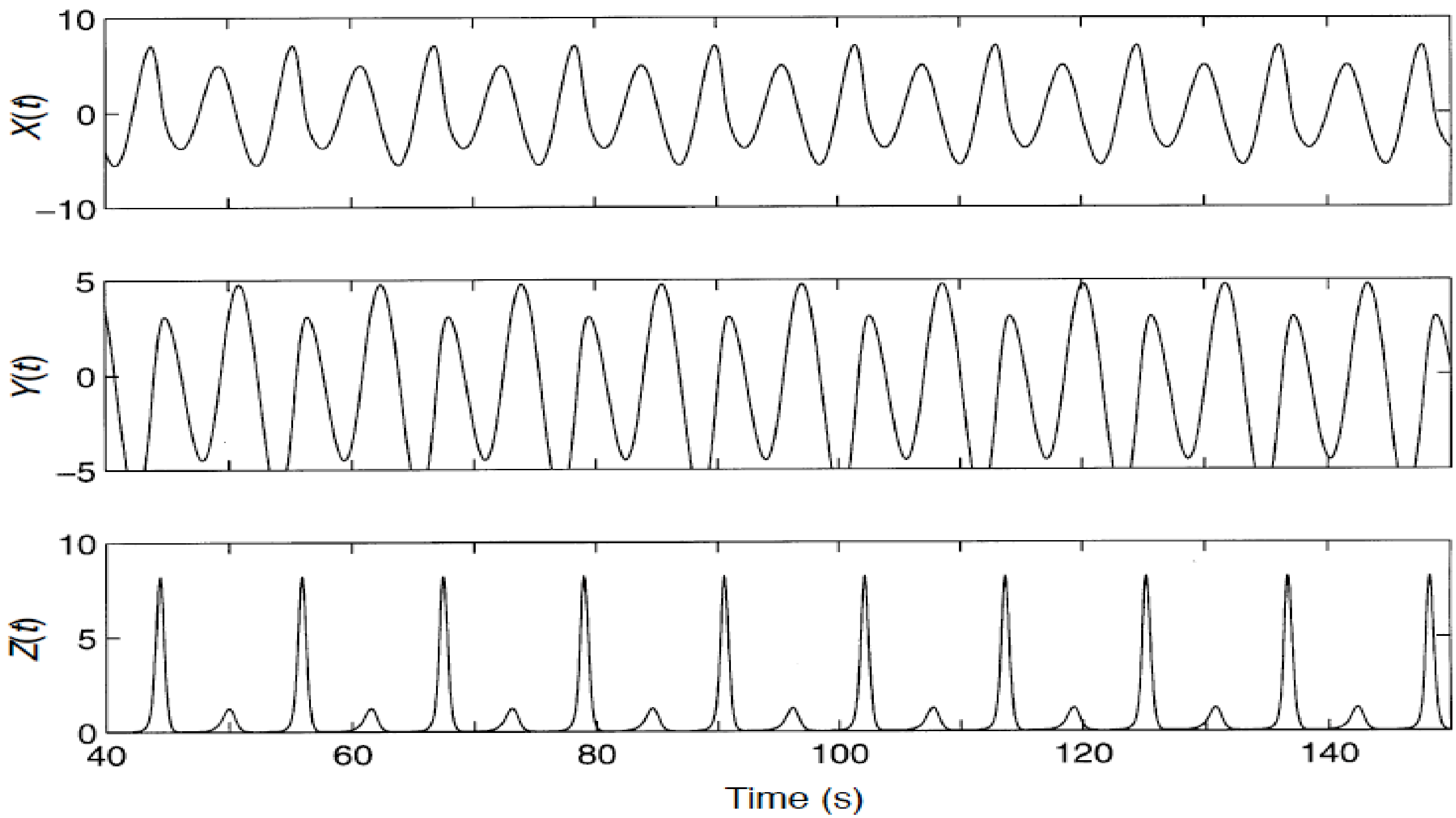


Figure 44. The time series for the numerical solution of the Rössler equation given in equation (9.11).

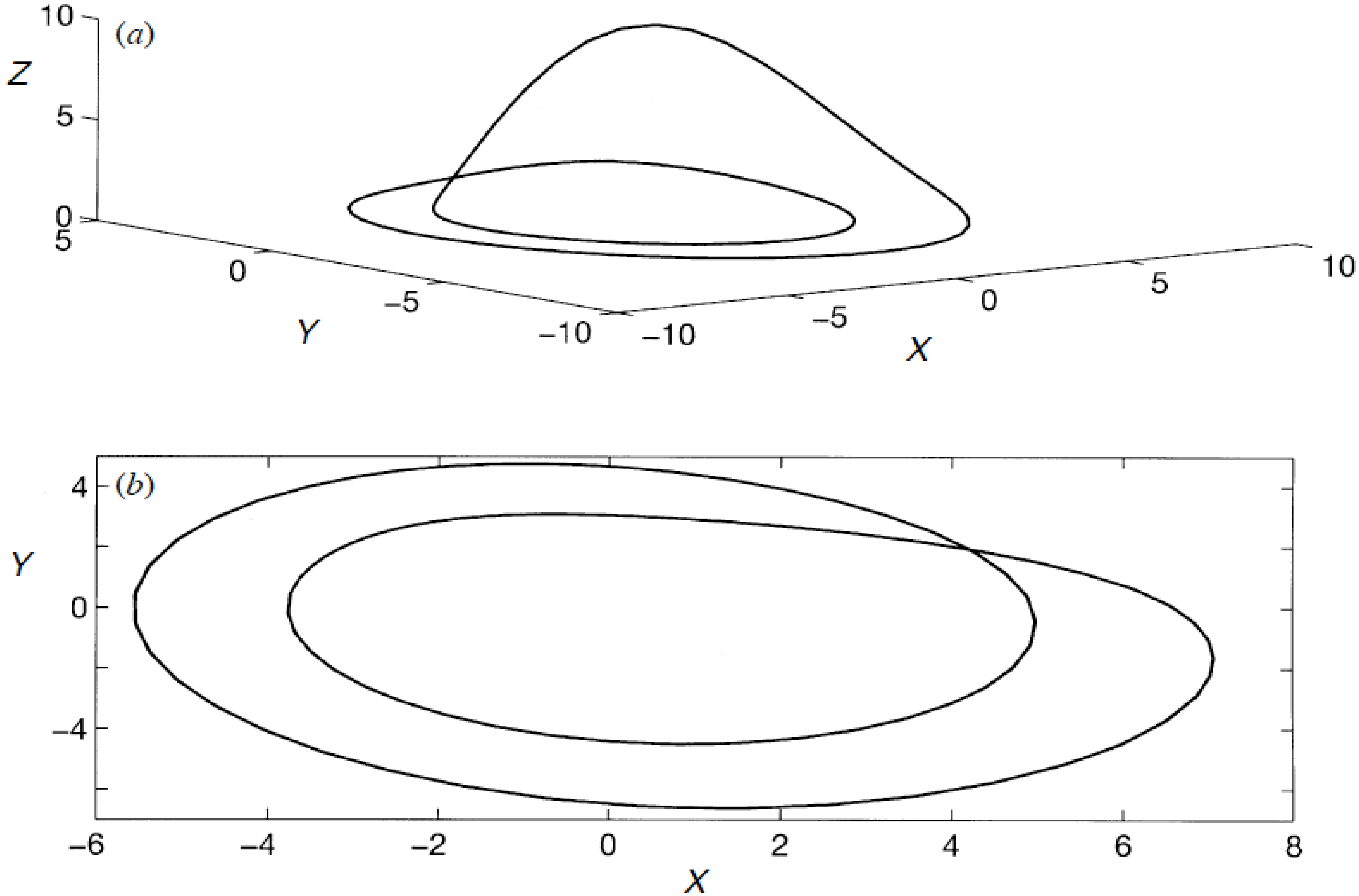


Figure 43. The numerical solution of the Rössler equation given in equation (9.11), for the period-doubling case: (*a*) the three-dimensional representation; (*b*) the two-dimensional representation.

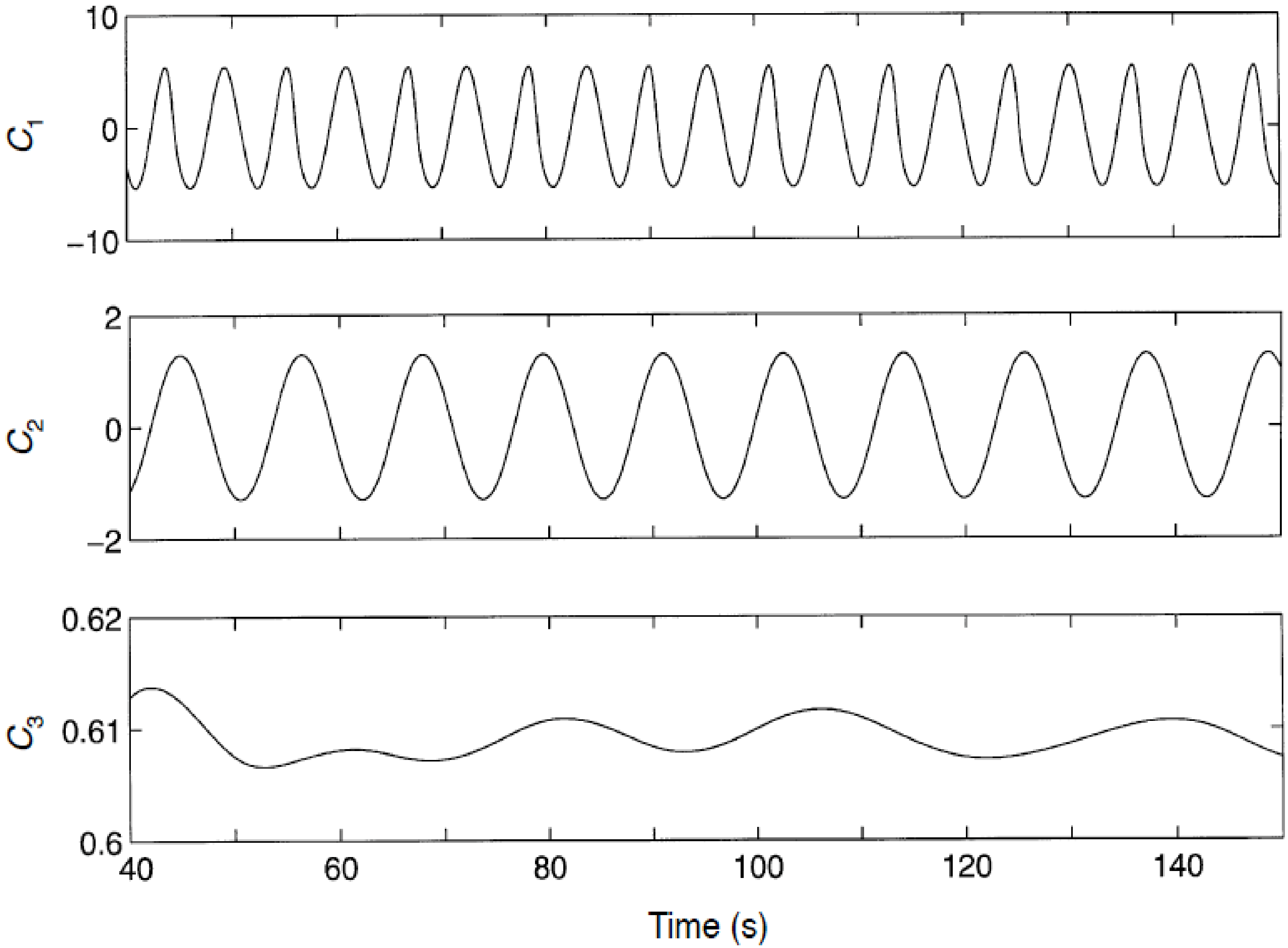


Figure 45. The IMF components of the x-components of the Rössler equation given in equation (9.11): component c_1 is non-uniform but the most energetic; c_2 is uniform; and the last component contains negligible energy.

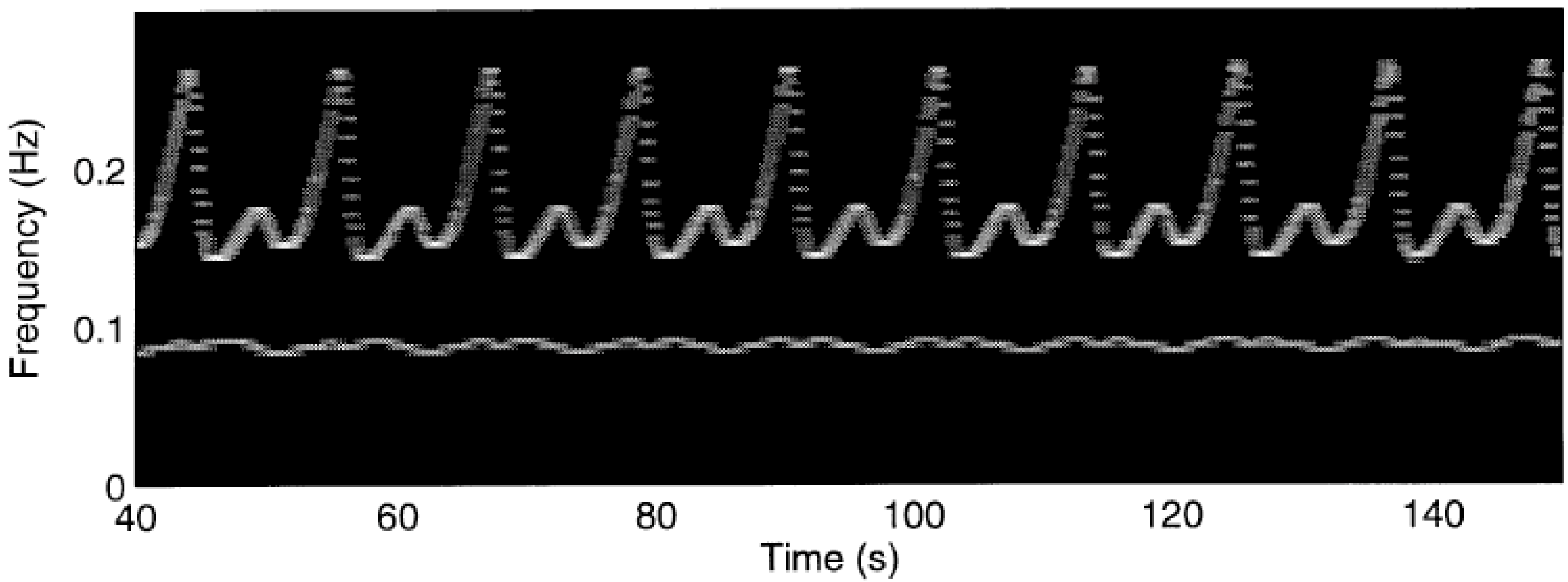


Figure 46. The Hilbert spectrum for the x-component. The most energetic component shows regular intra- and interwave frequency modulations. The frequency variations bear one-to-one correspondence with the time series data or the IMF component.

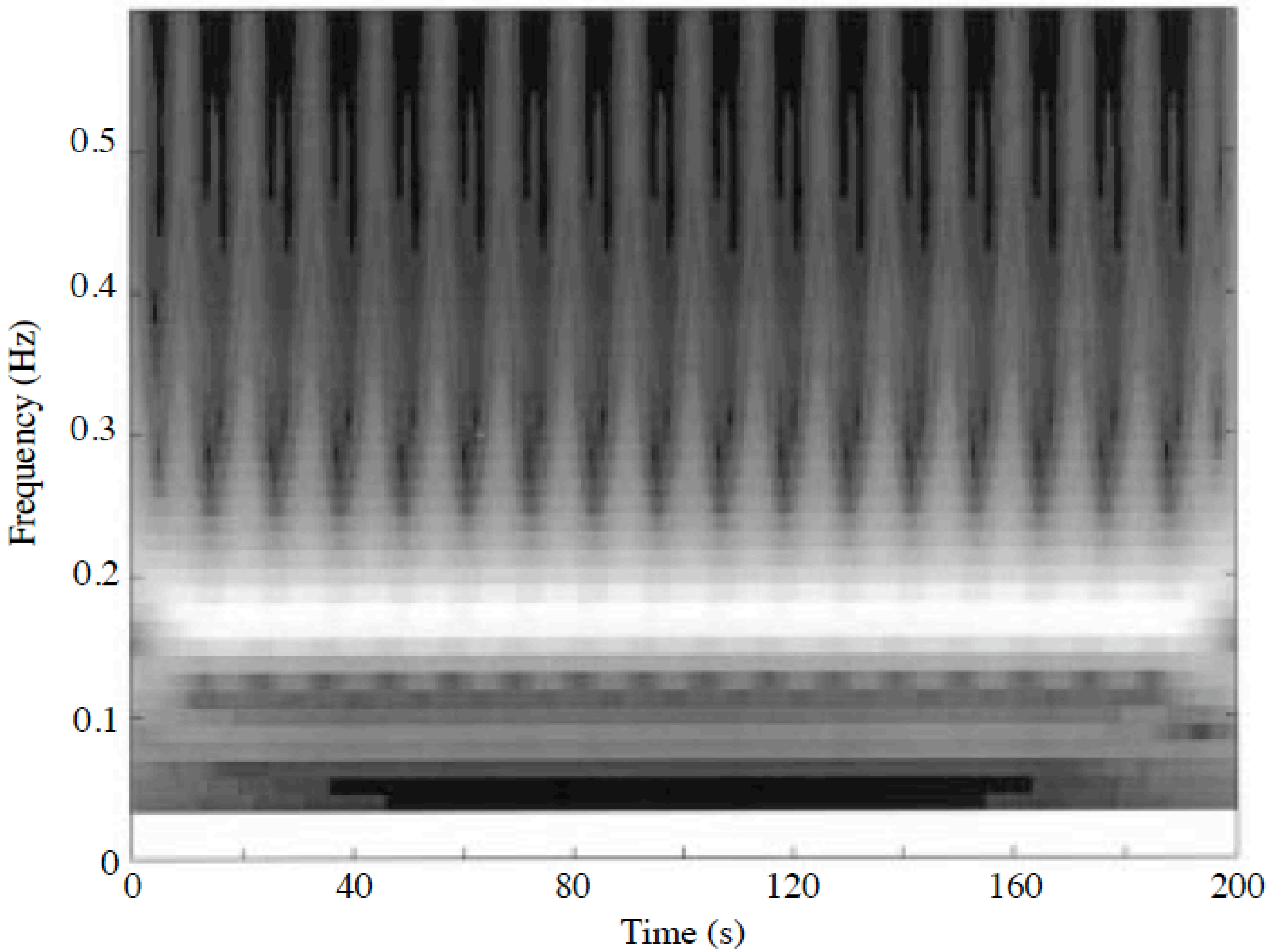


Figure 47. The Morlet wavelet spectrum shows only harmonics and some indication of interwave frequency modulation. There are no details for the intrawave frequency modulations.

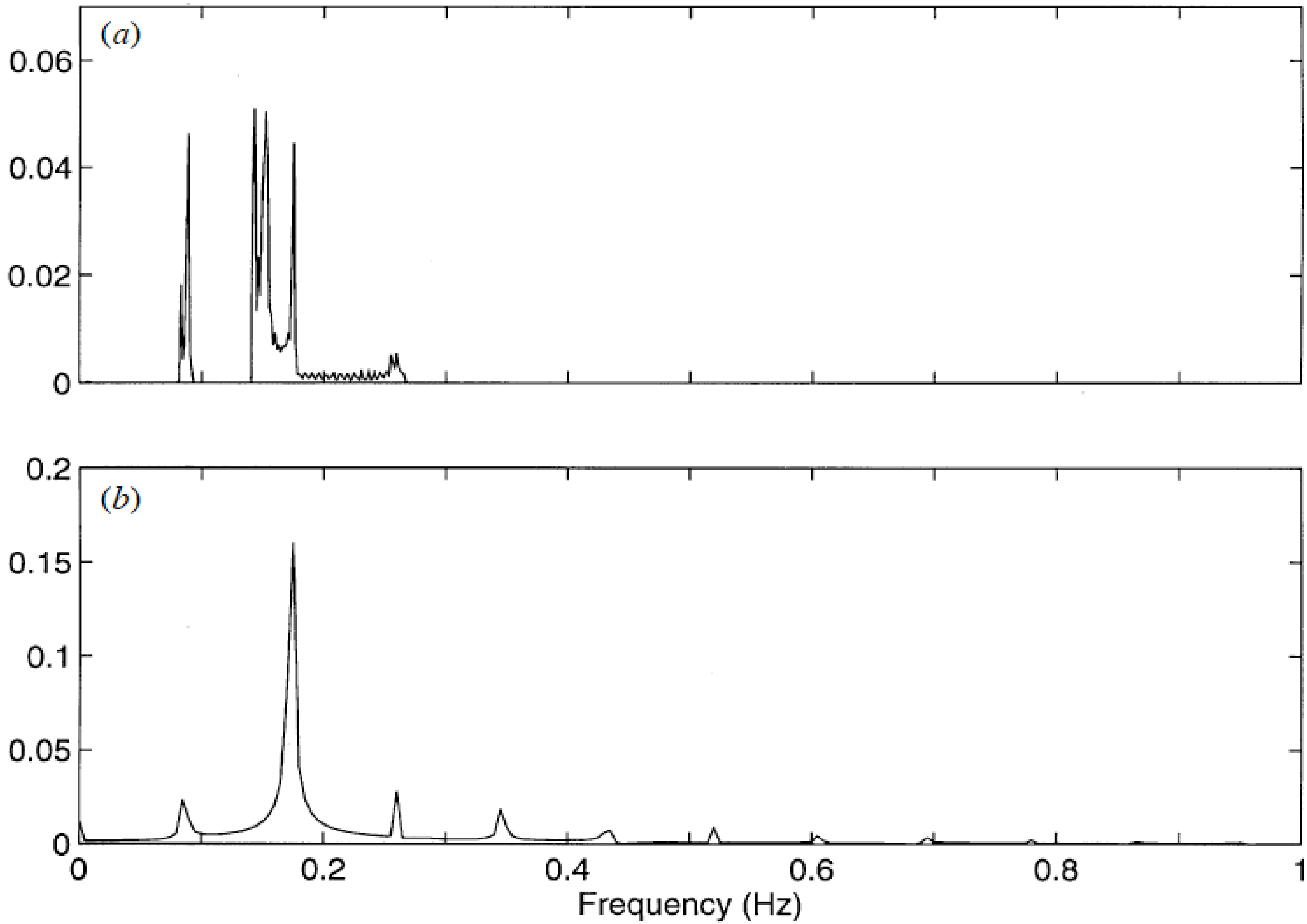


Figure 48. Comparison between the marginal spectrum and the Fourier spectrum for the Rössler equation: (*a*) the marginal spectrum showing the low-frequency oscillation near 0.1 Hz, and the high bimodal frequency distribution around 0.2 Hz generated by the intrawave frequency modulation; (*b*) the Fourier spectrum contains only harmonics with no indication of the two time scales involved in the period doubling. The main peak is the weighted mean of the physical