CSE 20 - Spring 2014
Quiz 2

Quiz 2b Solutions

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1. If a and b are two rational numbers then prove that $a+b$ is also rational.
2. If a is a rational number and b is not a rational number prove that $a+b$ cannot be rational.

1. If a, b are rational, we can write $a = \frac{p}{q}, b = \frac{r}{s}$ for some integers p, q, r, s. Then, we can write $a+b$ as

$$a+b = \frac{ps+qr}{qs}$$

   Since both $ps+qr$ and qs are integers, then $a+b$ is also rational.

2. a is rational. So, we can write $a = \frac{p}{q}$ for some integers p, q. However, b is not rational. Now, assume that $a+b$ is rational and we'll arrive at a contradiction. If $a+b$ is rational, we can write $a+b = \frac{r}{s}$ for some integers r, s. So, we can write:

$$\begin{aligned} b &= (a+b) - a \\ &= \frac{r}{s} - \frac{p}{q} \\ &= \frac{qr-ps}{qs} \end{aligned}$$

   We have written b in the form $\frac{qr-ps}{qs}$ and $qr-ps, qs$ are integers. So, b must be rational. This is a contradiction.

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2 Convert the following statement into an expression in propositional logic:
On Sundays or on a rainy day Bob stays home and watches TV. Yesterday Bob stayed at his friends place and they watched TV. So it means yesterday it did not rain and yesterday was not Sunday.

Let us define boolean variables for the statements:

- It is a Sunday = p
- It is a rainy day = q
- Bob stays at home = r
- Bob watches TV = s

So, we can write the statements as:
"On Sundays or on a rainy day Bob stays home and watches TV." is $(p \vee q) \implies (r \wedge s)$.
"Yesterday Bob stayed at his friends place and they watched TV." is $(\neg r \wedge s)$
"yesterday it did not rain and yesterday was not Sunday" is $(\neg p \wedge \neg q)$.
Combining, we get two possible solutions:

- $((p \vee q) \implies (r \wedge s)) \implies ((\neg r \wedge s) \implies (\neg p \wedge \neg q))$
- $(((p \vee q) \implies (r \wedge s)) \wedge (\neg r \wedge s)) \implies (\neg p \wedge \neg q)$

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3 If a and b are two integers such that $(a^2 - b^2)$ is divisible by 6, then prove that $(a^2 - b^2)$ is also divisible by 12.

There are many ways to solve this question but we present only one simple way here. Students are encouraged to discuss different solutions they have with TAs/Tutors/Instructor.

Let us make a few key observations which will help us cut down number of cases:

Property 1 If $(a^2 - b^2)$ is divisible by 6, then both a, b are even or both a, b are odd.

To see this, notice that $(a^2 - b^2) = (a + b) \times (a - b)$. So, if one of a, b is even and other odd, then both $a + b$ and $a - b$ are odd and their product will be odd. So, if $(a^2 - b^2)$ is odd, 6 cannot divide it.

Property 2 If 6 divides $(a^2 - b^2)$, then $(a^2 - b^2)$ is of the form $12k$ or $12k + 6$ for some integer k.

Since $(a^2 - b^2)$ is divisible by 6, then $(a^2 - b^2) = 6n$ for some integer n. This n can be even or odd and substituting $n = 2k$ or $n = 2k + 1$ gives $(a^2 - b^2) = 12k$ or $12k + 6$.

Let us prove that $(a^2 - b^2)$ mod 12 = 0. It suffices to prove $(a^2 - b^2)$ is not of the form $12k + 6$ and using Property 2, we can prove that 12 divides $(a^2 - b^2)$. Using Property 1, let us consider following cases for a, b

Case 1 Both a, b are even. Then, $a = 2m, b = 2l$ for some integers m, l. Then, $(a^2 - b^2) = 4(m^2 - l^2)$. Notice that 4 divides $(a^2 - b^2)$. But if $(a^2 - b^2) = 12k + 6$, $(a^2 - b^2)$ mod 4 = $12k + 6$ mod 4 = 6 mod 4 = 2. So, $(a^2 - b^2)$ cannot be of the form $12k + 6$

Case 2 Both a, b are odd. Then $a = 2m + 1, b = 2l + 1$ for some integers m, l. Then, Then, $(a^2 - b^2) = (a + b) \times (a - b) = 4(m + l + 1)(m - l)$. So, $(a^2 - b^2)$ mod 4 = 0.

But, if $(a^2 - b^2) = 12k + 6$, $(a^2 - b^2)$ mod 4 = $12k + 6$ mod 4 = 6 mod 4 = 2. So, $(a^2 - b^2)$ cannot be of the form $12k + 6$

This proves that $(a^2 - b^2)$ should be of the form $12k$ for some integer k. So, it is divisible by 12. ∎