CSE 20

Lecture 3: Representing integers in different bases

Representation of integers

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So, $x = [x_k x_{k-1} \ldots x_1 x_0]_b$

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   If an algorithm $\mathcal{A}_1$ write $x = [x_k x_{k-1} \ldots x_0]_b$ and another algorithm $\mathcal{A}_2$ write $x = [y_k y_{k-1} \ldots y_0]_b$, then is the two representations same.

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   In other words, is $y_i = x_i$ for all $0 \leq i \leq k$?

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There is no unique representation in base b-"sum"-representation.

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Attempt 2: Lets assume the earth is flat. Then when a ship came from the horizon the whole ship would appear at the same time.
But that does not happen - first the mast is seen then the whole ship. So a contradiction.
Hence initial assumption that earth is flat does not hold.

Proof of Uniqueness

Let us assume:

$$N = x_0 * b^0 + x_1 * b^1 + \cdots + x_k * b^k, \tag{1}$$

$$N = y_0 * b^0 + y_1 * b^1 + \cdots + y_k * b^k. \tag{2}$$

And there exists ($\exists$) i such that $x_i \neq y_i$?

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So we have:

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Thus,
$$(x_0 - y_0) * b^0 + \dots (x_{i-1} - y_{i-1}) * b^{i-1} \leq (b-1)\left(b^0 + \cdots + b^{i-1}\right).$$

So $(b-1)\left(b^0 + b^1 + \cdots + b^{i-1}\right)$ must be $\geq b^i$

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And this cannot be $\geq b^i$, so a contradiction.

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We assumed not uniquiness:

$$N = x_0 * b^0 + x_1 * b^1 + \cdots + x_k * b^k, \tag{3}$$

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And proved that this leads to a contradiction.

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So our assumption was wrong. The representation is unique.