CSE 20 Lecture 13: Analysis of Recursive Functions

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3. Analysis

- 3.1 Introduction
- 3.2 Homogeneous Linear Recursion

3.1 Introduction

Derive the bound of functions or recursions
Estimate CPU time and memory allocation
Eg.
PageRank calculation
  Allocation of memory, CPU time,
  Resource optimization
MRI imaging
  Real time?
VLSI design
  Design automation flow to meet the deadline for tape out?
Further Study
  Algorithm, Complexity

3.1 Introduction

- Derive the bound of functions or recursions
- Estimate CPU time and memory allocation
- Example on Fibonacci Sequence: Estimate f_n.
  - Index: 0 1 2 3 4 5 6 7 8 9
  - f_n: 0 1 1 2 3 5 8 13 21 34

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$f_0 = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^0 - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^0 = 0$$

$$f_1 = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right) - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right) = 1$$

Example: Fibonacci Sequence

0	1	2	3	4	5	6	7	8	9
0	1	1	2	3	5	8	13	21	34

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$\approx \frac{1}{2.236}(1.618^n - 0.618^n)$$

$$\approx \frac{1}{2.236} \bullet 1.618^n$$

$$f_9 \approx \frac{1}{2.236} \bullet 1.618^9 = 33.98$$

Example: Fibonacci Sequence, iClicker

Suppose $g_0=0$ and $g_1=2$, $g_n=g_{n-1}+g_{n-2}$ what is g_n

- A. $g_n=f_n$
- B. $g_n=2f_n$
- C. $g_n=3f_n$
- D. None of the above

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

3.2 Homogeneous Linear Recursion

- (1) Arithmetic Recursion

  $a, a+d, a+2d, \ldots, a+kd$

- (2) Geometric Recursion

  $a, ar, ar^2, \ldots, ar^k$

- (3) Linear Recursion

  $a_n = e_1 a_{n-1} + e_2 a_{n-2} + \ldots + e_k a_{n-k} + f(n)$

Linear Recursion and Homogeneous Linear Recursion

- Linear Recursion: There are no powers or products of a_j's
- Homogenous Linear Recursion: A linear recursion with f(n)=0.

Solving Linear Recursion

Input: Formula $$a_n = e_1 a_{n-1} + e_2 a_{n-2} + \ldots + e_k a_{n-k}$$

and k initial values $$a_0, a_1, \ldots, a_{k-1}$$

Output: a_n as a function of n

1. Set characteristic polynomial:

$$x^k = e_1 x^{k-1} + e_2 x^{k-2} + \ldots + e_k$$

2. Find the root of the characteristic polynomial (assuming r_i are distinct)

$$(r_1, r_2, \ldots, r_k)$$

3. Express $$a_n = c_1 r_1^{\ n} + c_2 r_2^{\ n} + \ldots + c_k r_k^{\ n}$$

4. Determine c_i from k initial values

Solving Linear Recursion

Input: Formula $a_n = e_1 a_{n-1} + e_2 a_{n-2} + \ldots + e_k a_{n-k}$

and k initial values $a_0, a_1, \ldots, a_{k-1}$

1. Set characteristic polynomial:

Rewrite the formula with n=k

$$a_k = e_1 a_{k-1} + e_2 a_{k-2} + \ldots + e_k a_0$$

Replace a_i with x^i

$$x^k = e_1 x^{k-1} + e_2 x^{k-2} + \ldots + e_k$$

Solving Linear Recursion

Input: Formula $a_n = e_1 a_{n-1} + e_2 a_{n-2} + \ldots + e_k a_{n-k}$

and k initial values $a_0, a_1, \ldots, a_{k-1}$

2. Find the root of the polynomial $x^k = e_1 x^{k-1} + e_2 x^{k-2} + \ldots + e_k$

Or,

$$x^k - e_1 x^{k-1} - e_2 x^{k-2} - \ldots - e_k = 0$$

$$(r_1, r_2, \ldots, r_k)$$

Solving Linear Recursion

Input: Formula $a_n = e_1 a_{n-1} + e_2 a_{n-2} + \ldots + e_k a_{n-k}$

and k initial values $a_0, a_1, \ldots, a_{k-1}$

3. Express (assuming that the roots are distinct.)

$$a_n = c_1 r_1^{\,n} + c_2 r_2^{\,n} + \ldots + c_k r_k^{\,n}$$

4. Determine c_i

from k initial values $a_0 = c_1 + c_2 + \ldots + c_k$

$$a_1 = c_1 r_1 + c_2 r_2 + \ldots + c_k r_k$$

$$\ldots = \ldots$$

$$a_{k-1} = c_1 r_1^{\,k-1} + c_2 r_2^{\,k-1} + \ldots + c_k r_k^{\,k-1}$$

Solving Linear Recursion

Input: Formula $a_n = e_1 a_{n-1} + e_2 a_{n-2} + \ldots + e_k a_{n-k}$

and k initial values $a_0, a_1, \ldots, a_{k-1}$

3. Set (when the roots are not distinct.) $(r_1, r_2, \ldots, r_u)$

$$a_n = \sum_{i=1}^{u} g_i(r_i)$$

$$g_i(r_i) = \sum_{j=1}^{w_i} c_{ij} n^{j-1} r_i^n$$

where r_i is a root of multiplicity w_i

Example on Fibonacci sequence

Input: initial values $a_0=0$ and $a_1=1$; and recursion formula $a_n=a_{n-1}+a_{n-2}$.

Rewrite recursion: $a_n-a_{n-1}-a_{n-2}=0$.

Or $a_2-a_1-a_0=0$.

1. Characteristic polynomial: $x^2-x-1=0$.

2. Roots of the polynomial:

$$r_1=\frac{1+\sqrt{5}}{2}, \quad r_2=\frac{1-\sqrt{5}}{2}$$

3. Set: $a_n=c_1r_1^n+c_2r_2^n$.

Example on Fibonacci sequence

Input: initial values $a_0=0$ and $a_1=1$; and recursion formula $a_n=a_{n-1}+a_{n-2}$.

4. Determine c_i from k initial values

$a_0=c_1r_1^0+c_2r_2^0$ => $c_1+c_2=0$

$a_1=c_1r_1^1+c_2r_2^1$ => $c_1r_1+c_2r_2=1$.

Thus, we have $a_n=c_1r_1^n+c_2r_2^n$, where

$$c_1=\frac{1}{\sqrt{5}}, c_2=-\frac{1}{\sqrt{5}} \quad r_1=\frac{1+\sqrt{5}}{2}, \quad r_2=\frac{1-\sqrt{5}}{2}$$

Example 2

Given:

Initial values a_0 = 1 and a_1 = 1

Recursion: a_n = a_{n-1} + $2a_{n-2}$

Rewrite Recursion: a_n- a_{n-1} - $2a_{n-2}$ = 0

Characteristic Polynomial: x^2 - *x* – 2 = 0

Characteristic Roots: r_1 = 2 and r_2 = -1

Step 1: a_n = $c_1 r_1$ + $c_2 r_2$

Use initial values n = 0, n =1 for a_n

a_0 = $c_1(\ 2)\ ^0$+ $c_2(\ -1)\ ^0$

a_1 = $c_1(\ 2)\ ^1$+ $c_2(\ -1)\ ^1$

Example 2 (cont)

Two initial values

$a_0=c_1+c_2$: $c_1+c_2=1$

$a_1=2c_1+(-1)c_2$: $2c_1-c_2=1$

Thus, we have

$c_1=2/3$, $c_2=1/3$.

Since $a_n=c_1r_1^n+c_2r_2^n$,

we have $a_n=2/3\times2^n+1/3\times(-1)^n$,

Example 3

For the case that root r is a root of multiplicity w:

$$a_n = c_1 r^n + c_2 n r^n + . . . + c_w n^{w-1} r^n$$

Given:

Initial values a_0 = 1 and a_1 = 1

Recursion: a_n = -2a_{n-1} - a_{n-2}

Rewrite Recursion: a_n+ 2a_{n-1} + a_{n-2} = 0

Characteristic Polynomial: x^2 + *2x* + 1 = 0

Characteristic Roots: r_1 = r_2 = -1

Step 1: a_n = $c_1 r^n$ + $c_2 n r^n$

a_0 = $c_1(-1^0)$ + $c_2(0)(-1^0)$

a_1 = $c_1(-1^1)$ + $c_2(1)(-1^1)$

Step 2: Match initial values n = 0, n =1, Solve for c_i's

a_0 = c_1

c_1 = 0

a_1 = $c_1 1$ + $c_2(-1)$

1 = 0 + $c_2(-1)$ Substitute c_1 = 0, a_1 = 1

c_2 = − 1

Thus

a_n = $c_2 r^n$ = $n(-1)^{n+1}$