CSE 20 Lecture 9
Boolean Algebra: Theorems and Transformations

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Theorems & Proofs: 4 Postulates

P1: $a+b = b+a$, $a \cdot b=b \cdot a$ (commutative)

P2: $a+bc = (a+b)\cdot(a+c)$ (distributive)

$a\cdot(b+c) = a\cdot b + a\cdot c$

P3: $a+0=a$, $a\cdot 1 = a$ (identity)

=> $a\cdot 0=0$, $a+1=1$ (annulment)

P4: $a+a'=1$, $a\cdot a'= 0$ (complement)

Theorem 6 (Involution Laws):

For every element a in B, $(a')' = a$

Proof: a is one complement of a'.

The complement of a' is unique.

Thus $a = (a')'$.

Theorem 7 (Absorption Law): For every pair a,b in B, $a\cdot(a+b) = a$; $a + a\cdot b = a$.

Proof: $a(a+b)$

$= (a+0)(a+b)$ (P3)

$= a+0\cdot b$ (P2)

$= a + 0$ (Annulment)

$= a$ (P3)

Theorems and Proofs

Theorem 8: For every pair a, b in B

$a + a' \cdot b = a + b$; $a \cdot (a' + b) = a \cdot b$

Proof: $a + a' \cdot b$

$= (a + a') \cdot (a + b)$ (P2)

$= (1) \cdot (a + b)$ (P4)

$= a + b$ (P3)

Theorem 9: De Morgan's Law

Theorem: For every pair a, b in set B:

$(a+b)' = a'b'$, and $(ab)' = a'+b'$.

Proof: We show that a+b and a'b' are complementary.

In other words, we show that both of the following are true (P4):

$(a+b)+(a'b') = 1$, $(a+b)(a'b') = 0$.

Theorem 9: De Morgan's Law (cont.)

Proof (Continue):

(a+b)+(a'b')

=(a+b+a')(a+b+b') (P2)

=(1+b)(a+1) (P4)

=1 (Theorem 3)

(a+b)(a'b')

=(a'b')(a+b) (P1)

=a'b'a+a'b'b (P2)

=0*b'+a'*0 (P4)

=0+0 (Theorem 3)

=0 (P3)

3. Theorems: Switching Algebra vs. Multiple Valued Boolean Algebra

- Boolean Algebra is termed Switching Algebra when B = {0, 1}
- When |B| > 2, the system is multiple valued.
  - Example: M = {(0, 1, 2, 3), #, &}

#	0	1	2	3
0	0	1	2	3
1	1	1	3	3
2	2	3	2	3
3	3	3	3	3

&	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	2	2
3	0	1	2	3

iClicker: M = {(0, 1, 2, 3), #, &}

A. Boolean algebra can have only two elements {0, 1}.
B. The identity elements are 0 and 3:
   – a # 0 = a
   – a & 3 = a
C. The complement of 1 is 2
D. Two of the above
E. None of the above.

#	0	1	2	3
0	0	1	2	3
1	1	1	3	3
2	2	3	2	3
3	3	3	3	3

&	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	2	2
3	0	1	2	3

Example: M = {(0, 1, 2, 3), #, &}

P1: Commutative Laws

- a # b = b # a
- a & b = b & a

P2: Distributive Laws

- a # (b & c) = (a # b) & (a # c)
- a & (b # c) = (a & b) # (a & c)

P3: Identity Elements

- a # 0 = a
- a & 3 = a

P4: Complement Laws

- a # a’ = 3
- a & a’ = 0

#	0	1	2	3
0	0	1	2	3
1	1	1	3	3
2	2	3	2	3
3	3	3	3	3

&	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	2	2
3	0	1	2	3

Boolean Transform

- Given a Boolean expression, we reduce the expression (#literals, #terms) using laws and theorems of Boolean algebra.
- When B={0,1}, we can use tables to visualize the operation.
  - The approach follows Shannon's expansion.
  - The tables are organized in two dimension space and called Karnaugh maps.

4. Boolean Transformations

Show that a'b'+ab+a'b = a'+b

Proof 1: a'b'+ab+a'b = a'b'+(a+a')b   P2

= a'b' + b   P4

= a' + b   Theorem 8

Proof 2: a'b'+ab+a'b

= a'b'+ab+a'b+a'b   Theorem 5

= a'b' + a'b +ab+a'b   P1

= a'(b'+b) + (a+a')b   P2

= a'*1 +1*b   P4

= a' + b   P3

Boolean Transformation

$(a'b'+c)(a+b)(b'+ac)'$

$= (a'b'+c)(a+b)(b(ac)')$ (DeMorgan's)

$= (a'b'+c)(a+b)b(a'+c')$ (DeMorgan's)

$= (a'b'+c)b(a'+c')$ (Absorption)

$= (a'b'b+bc)(a'+c')$ (P2)

$= (0+bc)(a'+c')$ (P4)

$= bc(a'+c')$ (P3)

$= a'bc+bcc'$ (P2)

$= a'bc+0$ (P4)

$= a'bc$ (P3)