CSE 20: Lecture 8

Boolean Postulates and Theorems

CK Cheng

Outline

- Definitions
  - Interpretation of Set Operations
  - Interpretation of Logic Operations
- Theorems and Proofs
- Transformations

Logic

OR:

- x<10 OR x> 18
- We will go rain or shine.
  - Either one is good

AND:

- x< 10 AND x> 8
- CSE20 is fun and useful.
  - Both need to be true

Section 2: Interpretation of Boolean Algebra using Logic Operations

Logic Symbols, 0, 1; and AND, OR Gates.

a =1 => a is true ,

a =0 => a is false.

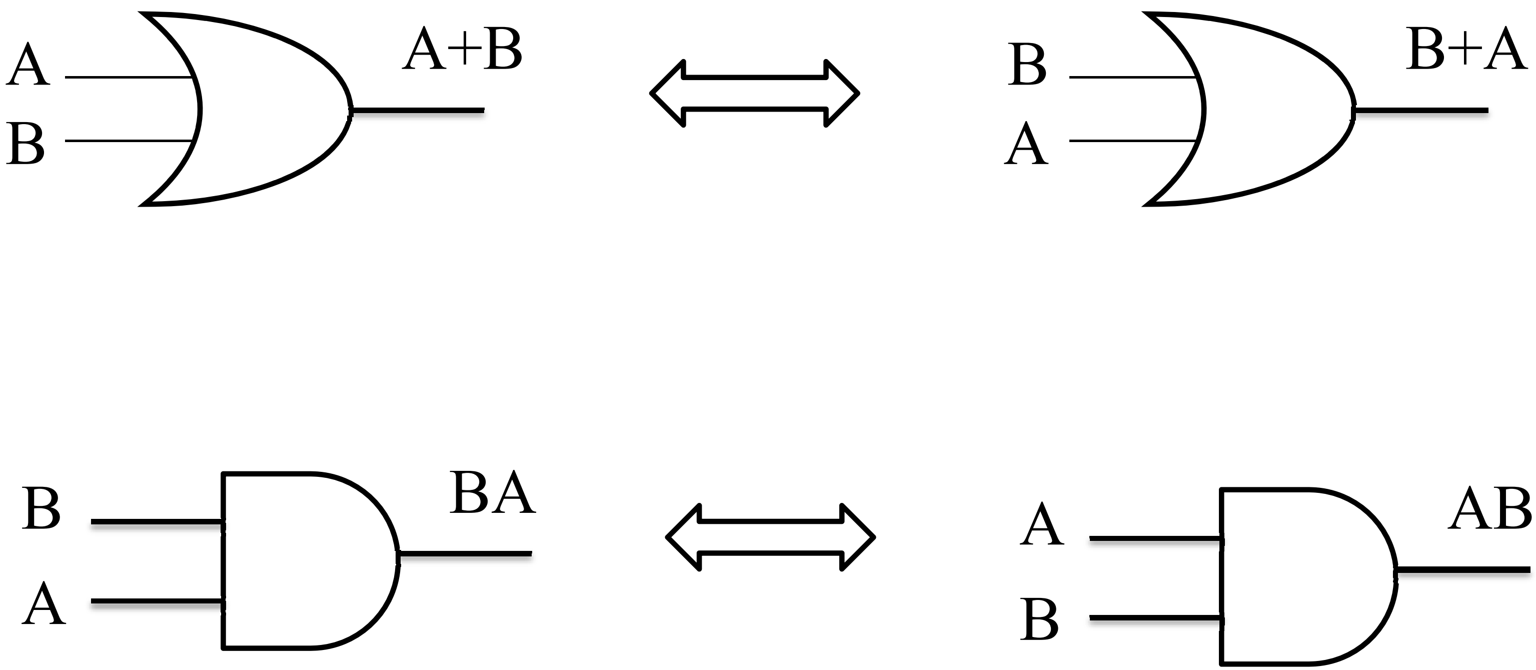
id	**a b**	***a OR b***
0	0 0	0
1	0 1	1
2	1 0	1
3	1 1	1

Id	**a b**	***a AND b***
0	0 0	0
1	0 1	0
2	1 0	0
3	1 1	1

Interpretation of P1 and P2 in Logic

P1: Commutative

- a is true OR b is true = b is true OR a is true
- a is true AND b is true = b is true AND a is true



Interpretation of P1 and P2 in Logic

P2: Distributive

- a is true OR (b is true AND c is true)
  = (a is true OR b is true) AND (a is true OR c is true)
- a is true AND (b is true OR c is true)
  = (a is true AND b is true) OR (a is true AND c is true)

Example:

We advance to the next game if
our score is higher OR (the competitor is absent AND we are present)

P2: Distributive Laws (truth table)

- $a \cdot (b+c) = (a \cdot b)+(a \cdot c)$
- $a+(b \cdot c) = (a+b) \cdot (a+c)$

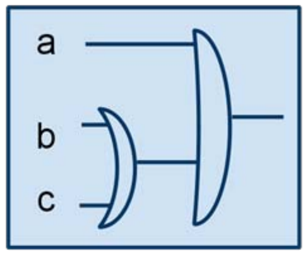
ID	a	b	c	b+c	a· (b+c)		a· b	a· c	(a· b)+(a· c)
0	0	0	0	0	0		0	0	0
1	0	0	1	1	0		0	0	0
2	0	1	0	1	0		0	0	0
3	0	1	1	1	0		0	0	0
4	1	0	0	0	0		0	0	0
5	1	0	1	1	1		0	1	1
6	1	1	0	1	1		1	0	1
7	1	1	1	1	1		1	1	1

P2: Distributive Laws: iClicker

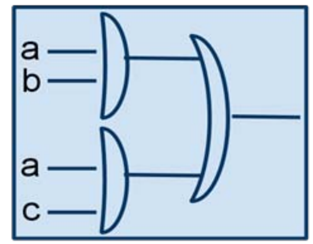
- $a+(b \cdot c) = (a+b) \cdot (a+c)$: A, B, C, D, or E (none)

id	a	b	c		A	B	C	D
0	0	0	0		0	0	0	0
1	0	0	1		0	1	0	1
2	0	1	0		0	1	1	1
3	0	1	1		1	1	1	1
4	1	0	0		1	0	0	1
5	1	0	1		1	1	0	1
6	1	1	0		1	1	1	1
7	1	1	1		1	1	1	1

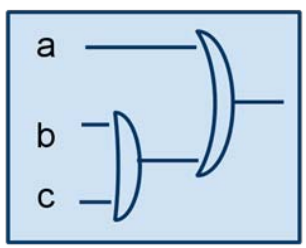
P2: Distributive Laws, cont.



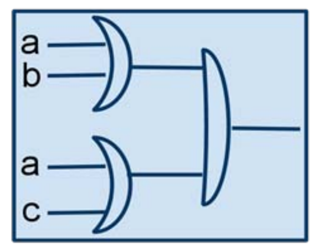
$a\cdot\ (b+c)$



$(a\cdot b)+(a\cdot c)$



$a+(b\cdot c)$



$(a+b)\cdot(a+c)$

Interpretation of P3 and P4 in Logic

P3: Identity 0: one false statement, 1: one true statement

- a is true OR one false statement = a is true
- a is true AND one true statement = a is true

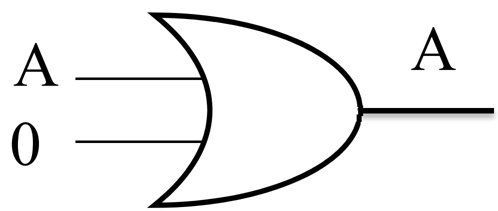
P4: Complement Negate the statement

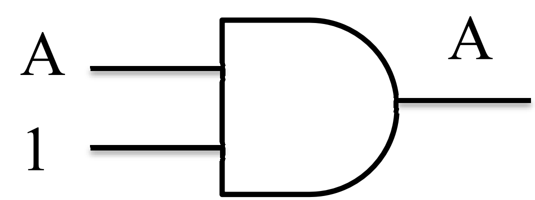
- a is true OR a is false = one true statement
- a is true AND a is false = one false statement

P3 Identity

$a+0 = a$, $a \cdot 1 = a$,

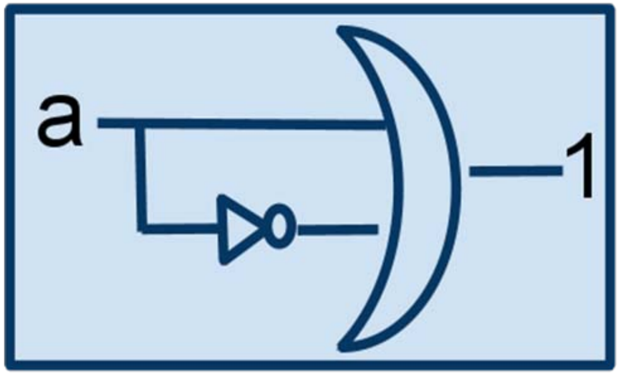
0 input to OR is passive 1 input to AND is passive



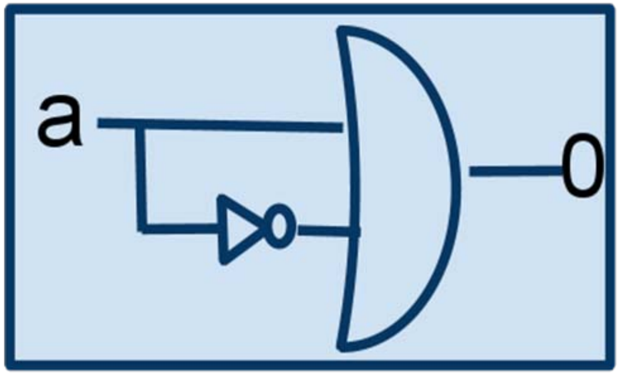


P4 Complement

a+a' = 1



a·a' = 0



2. Definition: iClicker

The statement of the 4 laws in the definition of Boolean algebra.

A. Artificial laws

B. Extraction of the operations in set and logic

C. Universal to all operations beyond set and logic

D. Necessary and sufficient set of the laws for all set operations.

E. All of the above.

3. Theorems and Proofs

Theorem 1: Principle of Duality

- Every algebraic identity that can be proven by Boolean algebra laws, remains valid if we swap all '+' and '·', 0 and 1.

Proof:

- Visible by inspection – all laws remain valid if we interchange all

  '+' and '·', 0 and 1

Theorem 2

Uniqueness of Complement: For every a in B, its complement a' is unique.

Proof: We prove by contradiction.

Suppose that a' is not unique, i.e. a_1', a_2' in B & $a_1' \neq a_2'$.

We have $a_1' = a_1' * 1$ (Postulate 3)

$= a_1' * (a + a_2')$ (Postulate 4)

$= (a_1' * a) + (a_1' * a_2')$ (Postulate 2)

$= 0 + (a_1' * a_2')$ (Postulate 4)

$= a_1'*a_2'$ (Postulate 3).

Likewise, we can also prove the same with a_2', i.e.

$$a_2' = a_1'*a_2'.$$

Consequently, we have $a_1' = a_2'$, which contradicts our initial assumption that $a_1' \neq a_2'$.

Theorem 3

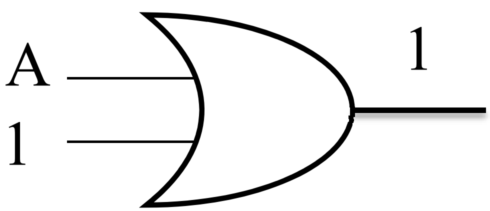
Boundedness: For every element a in B, $a+1=1$; $a*0=0$.

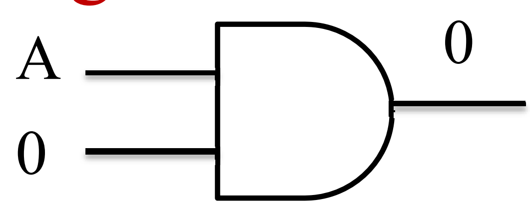
Proof: $a+1 = 1*(a+1)$ (Postulate 3)
$= (a + a')*(a+1)$ (Postulate 4)
$= a + a'*1$ (Postulate 2)
$= a + a'$ (Postulate 3)
$= 1$ (Postulate 4)

Comments:

'1' dominates as input in OR gates.

’0' dominates as input in AND gates.





Theorem 4

Statement:

- The complement of element 1 is 0 and vice versa, i.e.

$$0' = 1,\ 1' = 0.$$

Proof:

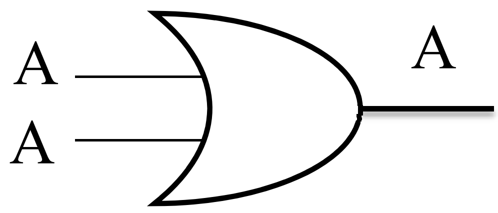
$$0 + 1 = 1 \text{ and } 0 * 1 = 0 \text{ (Postulate 3)}$$

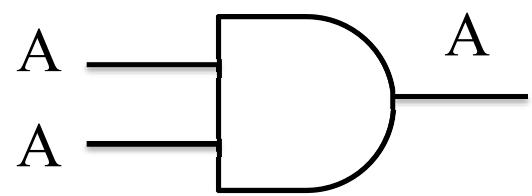
Thus 0’= 1, 1’= 0 (Postulate 4 and Theorem 2)

Theorem 5: Idempotent Laws

Statement: For every a in B,

$a + a = a$ and $a * a = a$.





Proof:

$a + a = (a + a) * 1$ (Postulate 3)

$= (a + a) * (a + a')$ (Postulate 4)

$= a + (a*a')$ (Postulate 2)

$= a + 0$ (Postulate 4)

$= a$ (Postulate 3)