CSE20 Lecture 6: Number Systems 5. Residual Numbers (cont) & 6. Cryptography

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Residual Numbers
(NT-1 and Shaum's Chapter 11)

- Introduction
- Definition
- Operations
- Inverse Conversion

Inverse Conversion

Number x

Mod Operation

Moduli (m_1, m_2, …, m_k)

Residual number

(x_1, x_2, …, x_k)

+, -, x operations

for each x_i under m_i

Results

Chinese Remainder Theorem

Chinese Remainder Theorem

Given a residual number (r_1, r_2, …, r_k) with moduli (m_1, m_2, …, m_k), where all m_i are mutually prime, set $M= m_1 \times m_2 \times \ldots \times m_k$, and $M_i=M/m_i$.

1. Find S_i that $(M_i \times S_i)\%m_i = 1$ (S_i an inverse of M_i in mod m_i)
2. The corresponding number

$$x = (\textstyle\sum_{i=1,k}(M_i\ S_i\ r_i))\%M.$$

Example

Given $(m_1,m_2,m_3)=(2,3,7)$, $M=2\times3\times7=42$, we have

$M_1=m_2\times m_3=3\times7=21$ $(M_1S_1)\%m_1=(21S_1)\%2=1$

$M_2=m_1\times m_3=2\times7=14$ $(M_2S_2)\%m_2=(14S_2)\%3=1$

$M_3=m_1\times m_2=2\times3=6$ $(M_3S_3)\%m_3=(6S_3)\%7=1$

Thus, $(S_1, S_2, S_3) = (1,2,6)$

For a residual number $(0,2,1)$:

$x=(M_1S_1r_1 + M_2S_2r_2 + M_3S_3r_3)\%M$

$=(21\times1\times0 + 14\times2\times2 + 6\times6\times1)\%42$

$= (0 + 56 + 36)\%42 = 92\%42 = 8$

Example

For a residual number (1,2,5):

- $x=(M_1S_1r_1 + M_2S_2r_2 + M_3S_3r_3)\%M$

  $= (21\times1\times1 + 14\times2\times2 + 6\times6\times5)\%42$

  $= (21 + 56 + 180)\%42$

  $= 257\%42 = 5$

Example: iClicker

Given $(m_1,m_2,m_3)=(2,3,5)$, $M=2\times3\times5=30$, we have

$M_1=m_2\times m_3=3\times5=15$ $(M_1 S_1)\%m_1=(15S_1)\%2=1$

$M_2=m_1\times m_3=2\times5=10$ $(M_2 S_2)\%m_2=(10S_2)\%3=1$

$M_3=m_1\times m_2=2\times3=6$ $(M_3 S_3)\%m_3=(6S_3)\%5=1$

Thus, (S_1, S_2, S_3) is

A. (1, 1, 1)

B. (1, 2, 1)

C. (2, 1, 2)

D. None of the above

Example: iClicker

Given $(m_1,m_2,m_3)=(2,3,5)$, $M=2\times3\times5=30$, we have

$M_1=m_2\times m_3=3\times5=15$ $\quad$ $(M_1S_1)\%m_1=(15S_1)\%2=1$

$M_2=m_1\times m_3=2\times5=10$ $\quad$ $(M_2S_2)\%m_2=(10S_2)\%3=1$

$M_3=m_1\times m_2=2\times3=6$ $\quad$ $(M_3S_3)\%m_3=(6S_3)\%5=1$

For a residual number $(x_1,x_2,x_3)=(1,2,3)$, the corresponding number x is

A. 5

B. 19

C. 23

D. None of the above

Proof of Chinese Remainder Theorem

Let $A = \sum_{i=1,k}(M_i\ S_i\ r_i)$, we show that

1. $A\%m_v = r_v$ and 2. $x = A\%M$ is unique.

1. $A\%m_v = [\sum_{i=1,k}(M_i\ S_i\ r_i)\ ]\%\ m_v$

   $= [\Sigma(M_i S_i r_i)\ \%\ m_v]\%m_v = (M_v S_v r_v)\%m_v$

   $= [(M_v S_v)\%m_v \times r_v\%m_v\ ]\%m_v = r_v\%m_v = r_v$

2. Proof was shown in lecture 5.

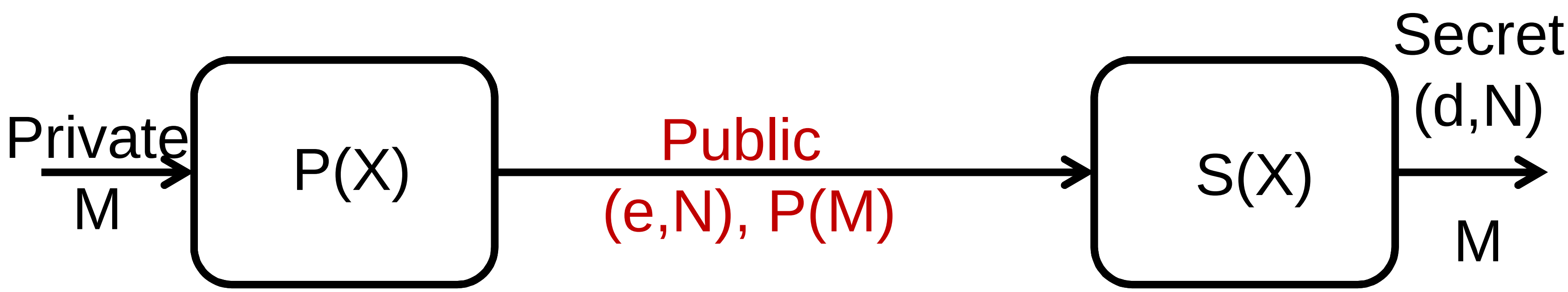
6. Cryptography

1. Introduction
2. RSA Protocol
3. Remarks

6.1 Cryptography: Introduction

- Application of residual number systems
- Number theory (skip)
- Show the basic concept and process
- Many variations

6.2 RSA Protocol



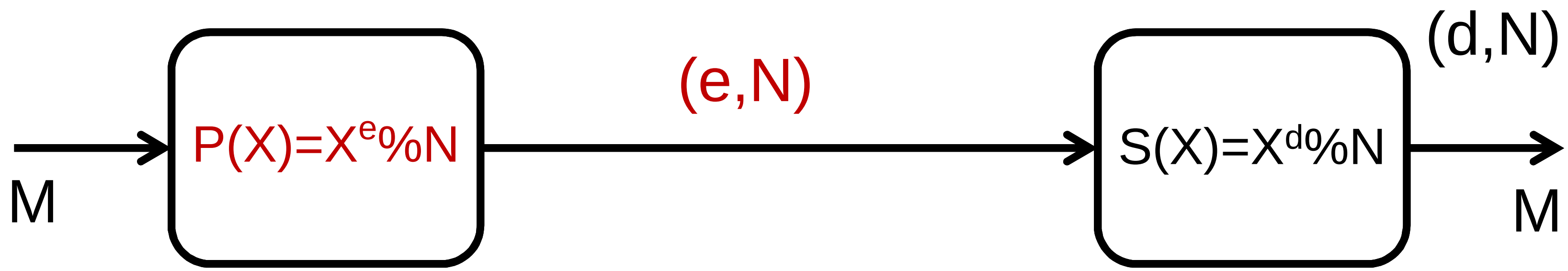
- Function $P(X)=X^e\%N$ is public.
- Function $S(X)=X^d\%N$ is secret.
- Message M is private, but P(M) is observed by all.
- Desired feature: S(P(M))=M.

Example: (e,N)=(7,55), (d,N)=(23,55)

$M=12 \Rightarrow P(12)=12^7\%55=23 \Rightarrow S(23)=23^{23}\%55=12$

$M=8 \Rightarrow P(8)=8^7\%55=2 \Rightarrow S(2)=2^{23}\%55=?$

6.2 RSA Protocol



1. N=pq where p & q are primes and kept secret.
2. e is mutually prime to f(N)=(p-1)(q-1)
3. d is the inverse of e mod f(N), i.e. (ed)%f(N)=1

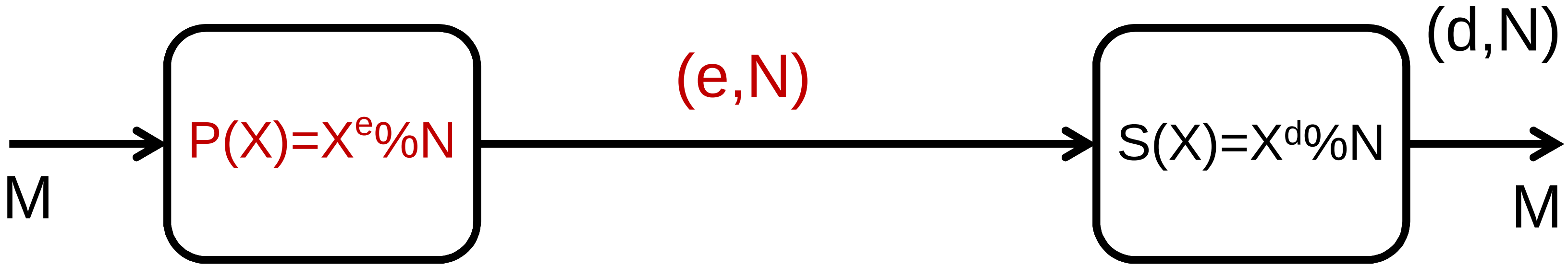
Theorem: S(P(M))=P(S(M))=M for 0<=M<N

Note that $S(P(M))=M^{ed}\%N$

Theorem: $M^{f(N)}\%N=1$ for 0<=M<N

Assumption: p & q are hard to find. Consequently, it is difficult to derive d.

6.2 RSA Protocol



1. N=pq where p & q are primes and kept secret.
2. e is mutually prime to f(N)=(p-1)(q-1)
3. d is the inverse of e mod f(N), i.e. (ed)%f(N)=1

Example: N=pq=3x11=33, f(N)=(3-1)(11-1)=20

Let e=3, then d=7 (3x7%20=1).

$$M=9 \Rightarrow P(9)=9^3\%33=3 \Rightarrow S(3)=3^7\%33=?$$

6.3 Remark

- Residual number system is used in cryptography.
- RSA protocol uses public key for coding P(X) and secret key to decode S(X).
- Use wide words (>1000 bits) so that the solution is computationally expensive without the knowledge of the function S(X).