CSE 20 Lecture 12
Induction

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Induction Outlines

- Introduction
- Theorem
- Examples: The complexity calculation
  - Tower of Hanoi
  - Merge Sort
  - Fibonacci

Induction: Introduction

Definition: The process of estimating the validity of observations of part of a class of facts as evidence for a proposition about the whole class. (Webster 2. logic b.)

We verify the statement

starting from the smallest cases, and

incrementing to larger cases with

assumption that the smaller cases are true.

Key: We need to be able to grow from the smallest cases.

Induction: Theorem

Let P(n) be an assertion defined on the integer n.

1. Base case: $P(n_0)$ is true for an integer n_0.
2. Assumption: We assume that for a $k \geq n_0$, P(n) is true for all n with $k \geq n \geq n_0$.
3. Incremental case: We can show P(k+1) is true for any $k \geq n_0$ using assumption in 2.

Then P(n) is true for every $n \geq n_0$.

Proof: By contradiction. Suppose P(n) is false for some $n \geq n_0$. Let m be the least such n. We thus have

- $m > n_0$ (from condition 1).
- P(m-1) is true (from assumption).

From condition 3, we have P(m) to be true which contradicts the assumption.

Induction: Example

The Tower of Hanoi: The sequence of the number of moves: $f_1=1$, $f_n=2f_{n-1}+1$, can be expressed as $f_n= 2^n-1$.

Proof: By induction.

Base case: The base case is true because $f_1 = 2^1-1=1$

Assumption: We assume $f_n= 2^n-1$ for n in $1\leq n\leq k$ where $k\geq 1$.

Incremental: We show that $f_{k+1}=2^{k+1}-1$ for any $k\geq 1$.

$f_{k+1}=2f_k+1$ (Complexity of recursion)

$= 2\times(2^k-1)+1$ (From assumption)

$=2^{k+1}-1$ (Arithmetic operation)

Based on induction theorem $f_n= 2^n-1$ for all integer $n\geq 1$

Induction: Example

Merge Sort Complexity: $f_0=0$, $f_n=2f_{n-1}+2^n$.

We can express $f_n=n2^n$ for $n\geq0$.

Proof: By induction.

Base case: The base case is true because $f_0 = 0\times2^0=0$

Assumption: We assume $f_n= n2^n$ for n in $0\leq n\leq k$ where $k\geq0$.

Incremental: We show that $f_{k+1}=(k+1)2^{k+1}$ for any $k\geq0$.

$f_{k+1}=2f_k+2^{k+1}$ (Complexity of recursion)

$= 2\times(k2^k)+2^{k+1}$ (From assumption)

$=(k+1)2^{k+1}$ (Arithmetic operation).

From induction theorem, $f_n=n2^n$ for all integer $n\geq0$.

Induction: Example

Complexity in terms of the number of elements:

Merge Sort Complexity: $f_0=0$, $f_n=2f_{n-1}+2^n$.

We can express $f_n=n2^n$ for $n\geq 0$.

Note that we have $m=2^n$ elements to sort.

In other words, the complexity to sort m elements takes $m\log_2 m$ comparison operations.

Induction: Example

Fibonacci sequence: $f_0=0, f_1=1$, $f_n=f_{n-1}+f_{n-2}$, can be expressed as $f_n= \sqrt{5}/5\{[(1+\sqrt{5})/2]^n - [(1-\sqrt{5})/2]^n\}$

Proof: By induction.

Base case: $f_0 = \sqrt{5}/5\{[(1+\sqrt{5})/2]^0 - [(1-\sqrt{5})/2]^0\} = 0$

$f_1 = \sqrt{5}/5\{[(1+\sqrt{5})/2] - [(1-\sqrt{5})/2]\} = 1$ (Why f_1?)

Assumption: The expression is correct for n in $1 \le n \le k$ where $k \ge 1$.

Incremental: $f_{k+1}=f_k+f_{k-1}$

$= \sqrt{5}/5\{[(1+\sqrt{5})/2]^k - [(1-\sqrt{5})/2]^k\} + \sqrt{5}/5\{[(1+\sqrt{5})/2]^{k-1} - [(1-\sqrt{5})/2]^{k-1}\}$ (From Assumption)

$= \sqrt{5}/5\{[(1+\sqrt{5})/2]^{k+1} - [(1-\sqrt{5})/2]^{k+1}\}$ (Arithmetic operation)

Induction: Fibonacci exm. iClicker

A. The f_1 case in the proof is necessary.

B. The f_0 case already covers the base case.

C. Fibonacci sequence is supposed to be integers. Thus, we should remove the square root of 5 in the formula.

D. None of the above.

Induction: Example

Exercise: Show that

$\sqrt{5}/5\{[(1+\sqrt{5})/2]^{k} - [(1-\sqrt{5})/2]^{k}\} + \sqrt{5}/5\{[(1+\sqrt{5})/2]^{k-1} - [(1-\sqrt{5})/2]^{k-1}\} = \sqrt{5}/5\{[(1+\sqrt{5})/2]^{k+1} - [(1-\sqrt{5})/2]^{k+1}\}$

Hint: Derive that

$$[(1+\sqrt{5})/2]^{k} + [(1+\sqrt{5})/2]^{k-1} = [(1+\sqrt{5})/2]^{k+1}$$

And $[(1-\sqrt{5})/2]^{k} + [(1-\sqrt{5})/2]^{k-1} = [(1-\sqrt{5})/2]^{k+1}$

Induction: Example

Statement: All horses are the same color.

Proof: By induction.

Base case: A horse is the same color.

Assumption: Assume that n horses are the same color for all $1 \le n \le k$ where $k \ge 1$.

Incremental: We show that k+1 horses are the same color. We separate the horses into two groups x and y with $|x| \le k$, $|y| \le k$, $|x|+|y|=k+2$. Thus x and y overlap by one.

From assumption, x are the same color, y are the same color.

Because x and y overlaps, x and y are the same color.

Induction: iClicker

From the proof that all horses are the same color, we can conclude the following:

- A. All horses are the same color.
- B. The induction theorem has flaws.
- C. The induction theorem is correct, but the argument in the proof has flaws.
- D. The assumption in the proof is wrong.
- E. None of the above.

Induction: iClicker

From the proof that all horses are the same color, we can conclude the following:

- A. The base case has flaws.
- B. The assumption has flaws.
- C. The incremental case cannot grow from base case.
- D. The incremental case has flaws when we have 3 or more horses.
- E. None of the above.