4.1 Recursive function: analysis

1. A frog knows 5 jumping styles (A, B, C, D, E). A, B jump forward by 1 foot, and C, D, E jump forward by 2 feet. Let a_i denote the number of ways to jump over a total distance of i feet.
(a) What is a_1, a_2, a_3?
(b) Derive the recursive formula of a_n?
(c) Find the solution of the recursion.
2. Find the solution of the following recurrence:

$$\begin{aligned} a_n &= -a_{n-1} + a_{n-2} + a_{n-3} \\ a_0 &= 0 \\ a_1 &= 0 \\ a_2 &= 1 \end{aligned}$$

3. Consider the following homogeneous linear recurrence relation:
$a_n = 3ra_{n-1} - 3r^2 a_{n-2} + r^3 a_{n-3}$. Show that $a_n = c_1 r^n + c_2 n r^n + c_3 n^2 r^n$ satisfies the recurrence relation, where c_1, c_2, and c_3 are constant coefficients.

Solution:

A frog knows 5 jumping styles, named A, B, C, D, and E. Both A and B jump forward by one foot, while C, D, and E jump forward by two feet. Let a_i be the total number of ways to jump a total distance of i feet.

1. What are a_1, a_2, and a_3?
2. Derive the recursive formulation of a_i.
3. Solve the recursion.

To start with, we note that there are only two ways to jump one foot, so

$$a_1 = 2.$$

On the other hand, we have three ways to jump two feet using only one jump. We could also jump two feet by jumping one foot twice: all of the $2^2 = 4$

combinations AA, AB, BA, and BB will work. Thus, we have the number of ways to jump two feet is

$$a_2 = 3 + 2^2 = 3 + 4 = 7.$$

Now we need to figure out how to jump three feet. Consider the very last jump: it's a jump of either one foot or two feet. So let's first consider if the last jump is one foot long. Then we have $a_2 = 7$ ways to jump the initial two feet, and we can finish off the jumps in one of two ways (A and B), so we have $2 \times a_2 = 2 \times 7 = 14$ ways to jump three feet if we finish by jumping one foot. On the other hand, the last jump may be two feet long. In this case, we have $a_1 = 2$ ways to jump the first foot, and we can finish off the jumps in one of three ways (C, D, and E), so we have $3 \times a_1 = 3 \times 2 = 6$ ways to jump three feet if we finish by jumping two feet. Finally, we add all the ways to finish, both by making a final jump of two feet and one foot, to get

$$a_3 = 14 + 6 = 20.$$

To figure out the recusion, we just generalize the argument we used for a_3: if we want to jump i feet, we can either jump $i - 1$ feet and finish with A or B, or we can jump $i - 2$ feet and finish with C, D, or E. Thus, we have

$$a_i = 2a_{i-1} + 3a_{i-2}.$$

Finally, we have to solve the recurrence. The characteristic polynomial of the recurrence relation above is

$$x^2 - 2x - 3 = (x - 3)(x + 1).$$

This polynomial has the roots

$$\begin{aligned} r_1 &= 3 \\ r_2 &= -1 \end{aligned}$$

The general form of the solution to this recurrence is thus

$$a_i = c_1 r_1^i + c_2 r_2^i = 3^i c_1 + (-1)^i c_2.$$

We have only to solve for c_1 and c_2 using our initial conditions. We have the two equations

$$\begin{aligned} a_1 &= 3^1 c_1 + (-1)^1 c_2 &= 3c_1 + (-1)c_2 &= 2 \\ a_2 &= 3^2 c_1 + (-1)^2 c_2 &= 9c_1 + c_2 &= 7 \end{aligned}$$

Adding these equations together, we get

$$a_1 + a_2 = 12c_1 = 9$$

and therefore

$$c_1 = \frac{3}{4}.$$

Plugging this into the equation for a_1 yields

$$c_2 = \frac{1}{4}.$$

Thus, the solution to the recurrence is

$$a_i = 3^i \frac{3}{4} + (-1)^i \frac{1}{4} = \frac{1}{4}(3^{i+1} + (-1)^i).$$

Question 2

Find the solution of the following recurrence:

$$\begin{aligned} a_n &= -a_{n-1} + a_{n-2} + a_{n-3} \\ a_0 &= 0 \\ a_1 &= 0 \\ a_2 &= 1 \end{aligned}$$

We first find the characteristic polynomial of the recurrence relation, which is

$$x^3 + x^2 - x - 1 = (x-1)(x+1)^2.$$

This polynomial has the roots

$$\begin{aligned} r_1 &= 1 \\ r_2 = r_3 &= -1 \end{aligned}$$

Note that we have a root with multiplicity two (r_2 and r_3). Thus, the general form of the solution will be

$$a_n = c_1 r_1^n + c_2 r_2^n + c_3 n r_3^n = c_1 + (-1)^n c_2 + (-1)^n n c_3.$$

Again, we solve for the constants using our initial conditions. We have the equations:

$$\begin{aligned} a_0 &= c_1 + (-1)^0 c_2 + (-1)^0 0 c_3 &&= c_1 + c_2 &&= 0 \\ a_1 &= c_1 + (-1)^1 c_2 + (-1)^1 1 c_3 &&= c_1 + (-1)c_2 + (-1)c_3 &&= 0 \\ a_2 &= c_1 + (-1)^2 c_2 + (-1)^2 2 c_3 &&= c_1 + c_2 + 2c_3 &&= 1 \end{aligned}$$

Solving this system of linear equations yields the constants

$$\begin{aligned} c_1 &= \tfrac{1}{4} \\ c_2 &= -\tfrac{1}{4} \\ c_3 &= \tfrac{1}{2} \end{aligned}$$

Thus, the solution to the recurrence is

$$a_n = \frac{1}{4} + (-1)^n(\frac{1}{2}n - \frac{1}{4}).$$

Question 3

Consider the homogeneous linear recurrence relation

$$a_n = 3ra_{n-1} - 3r^2a_{n-2} + r^3a_{n-3}.$$

Show that

$$p(n) = c_1r^n + c_2nr^n + c_3n^2r^n = r^n(c_1 + c_2n + c_3n^2)$$

satisfies the recurrence relation, where c_1, c_2, and c_3 are constant coefficients.

We have to show that, if we plug in $p(n)$ in place of each a_n on the right hand side of the recurrence, we get $p(n)$ back out.

$$\begin{aligned}
& 3rp(n-1) - 3r^2p(n-2) + r^3p(n-3) \\
= \; & 3r(r^{n-1}(c_1 + c_2(n-1) + c_3(n-1)^2)) \\
& -3r^2(r^{n-2}(c_1 + c_2(n-2) + c_3(n-2)^2)) \\
& +r^3(r^{n-3}(c_1 + c_2(n-3) + c_3(n-3)^2)) \\
= \; & 3r^n(c_1 + c_2(n-1) + c_3(n-1)^2) \\
& -3r^n(c_1 + c_2(n-2) + c_3(n-2)^2) \\
& +r^n(c_1 + c_2(n-3) + c_3(n-3)^2) \\
= \; & r^n[3(c_1 + c_2(n-1) + c_3(n-1)^2) \\
& -3(c_1 + c_2(n-2) + c_3(n-2)^2) \\
& +(c_1 + c_2(n-3) + c_3(n-3)^2)] \\
= \; & r^n[3c_1 + (3n-3)c_2 + (3n^2 - 6n + 3)c_3 \\
& -3c_1 - (3n-6)c_2 - (3n^2 - 12n + 12)c_3 \\
& +c_1 + (n-3)c_2 + (n^2 - 6n + 9)c_3] \\
= \; & r^n(c_1 + nc_2 + n^2c_3) \\
= \; & p(n)
\end{aligned}$$