Solution and Grading Policy for Midterm Two

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1 Problem One

- $(8\%3, 8\%7, 8\%8) = (2, 1, 0)$ (3 points, 1 for each);
- $(17\%3, 17\%7, 17\%8) = (2, 3, 1)$ (3 points, 1 for each);
- $(2 \times 2)\%3 = 1$, $(1 \times 3)\%7 = 3$, $(0 \times 1)\%8 = 0$ (3 points, 1 for each);
- $(r_1, r_2, r_3) = (1, 3, 0)$ (1 point);

2 Problem Two

- $M = 4 \times 5 \times 7 = 140$ (1 point);
- $M_1 = 5 \times 7 = 35$ (1 point);
- $(35s_1)\%4 = 1 \Rightarrow s_1 = 3$ (3 points);
- $M_2 = 4 \times 7 = 28$ (1 points);
- $(28s_2)\%5 = 1 \Rightarrow s_2 = 2$ (3 points);
- $M_3 = 4 \times 5 = 20$ (1 points);
- $(20s_3)\%7 = 1 \Rightarrow s_3 = 6$ (3 points);
- $x = (35 \times 3 \times 1 + 28 \times 2 \times 2 + 20 \times 6 \times 3)\%140 = 17$ (2 points);

3 Problem Three

Refer to slide 9 of lecture 7, broken into 5 parts of 2 points each:

- A set of elements B with two operations "OR" $(+, \vee)$ and "AND" $(*, \wedge)$ satisfying the following four laws.
- P1. Commutative Laws: $a + b = b + a$, $a * b = b * a$
- P2. Distributive Laws: $a + (b * c) = (a + b) * (a + c)$, $a * (b + c) = (a * b) + (a * c)$
- P3. Identity Elements: $a + 0 = a$, $a * 1 = a$
- P4. Complement Laws: $a + a' = 1$, $a * a' = 0$

Policy: Full credit for mentioning all the items. Partial credit (2 points for each item above) otherwise. Some credit was also given for other correct statements relating to boolean algebra where relevant.

4 Problem Four

Full credit if proven correctly using boolean algebra. Partial credit if proven correctly using other methods (e.g. truth table, etc). One of the possible solution using boolean algebra:

If $(ab)' = a' + b'$, it means $a' + b'$ is the complement of ab. It means it must satisfy the following properties of complements:

$$\begin{aligned} ab + (a' + b') &= 1 \\ ab(a' + b') &= 0 \end{aligned}$$

The first property:

$$\begin{aligned} & ab + (a' + b') \\ &= (ab + a') + b' && [Associative] \\ &= (b + a') + b' && [Th.8] \\ &= a' + (b + b') && [Associative, Commutative] \\ &= a' + 1 && [Complementlaw] \\ &= 1 \end{aligned}$$

The second property:

$$\begin{aligned} & ab(a' + b') \\ &= aa'b + abb' && [Distributive, Commutative] \\ &= 0 + 0 && [ComplementLaw] \\ &= 0 \end{aligned}$$

Policy: If this method was used, full credit was given for proving both the properties. If only one property was proved, half credit was given. If only $(a+b) = a'b'$ was proved, but duality was not mentioned, then partial credit was given accordingly. Any other forms of proof that correctly used boolean algebra for a complete proof was also given full credit.

5 Problem Five

$$\begin{aligned} & a'bc + ab' + bc' \\ &= (a' + ab' + bc')(b + ab' + bc')(c + ab' + bc') \\ &= (a' + b' + bc')(b + ab' + bc')(c + ab' + bc') \\ &= (a' + b' + c')(b + a + bc')(c + ab' + bc') \\ &= (a' + b' + c')(a + b)(ab' + c + b) \\ &= (a' + b' + c')(a + b)(a + b + c) \\ &= (a' + b' + c')(a + b) \end{aligned}$$

Policy: 15 points if correct in all steps, otherwise 0. If there are trivial mistakes, get partial credits.

6 Problem Six

$$
\begin{aligned}
& a'b' + b'c + ac + bc + a'bc' \\
&= a'b' + (b' + b)c + ac + a'bc' \\
&= a'b' + c + ac + a'bc' \\
&= a'b' + c + a'bc' \\
&= a'b' + c + a'b \\
&= a'(b' + b) + c \\
&= a' + c
\end{aligned}
$$

Policy: 15 points if correct in all steps, otherwise 0. If there are trivial mistakes, get partial credits.

7 Problem Seven

- (a): $a_1 = 2$, $a_2 = 5$, $a_3 = 12$ (3 points, 1 for each);
- (b): $a_n = 2a_{n-1} + a_{n-2}$ (6 points);
- (c): Assume $a_n = c_1 r_1^n + c_2 r_2^n$. The characteristic polynomial for (b) is $x^2 - 2x - 1 = 0$ and we can solve it to get the values of r_1 and r_2 showed below:

  $$r_1 = 1 + \sqrt{2},\ r_2 = 1 - \sqrt{2} \qquad \text{(2 points, 1 for each)}$$

  By plugging $n = 0, 1$, we have the following linear equations for c_1 and c_2:

  $$
  \begin{aligned}
  c_1 + c_2 &= 1 \\
  (1 + \sqrt{2})c_1 + (1 - \sqrt{2})c_2 &= 2
  \end{aligned}
  $$

  which determines the values of c_1 and c_2:

  $$c_1 = \frac{2 + \sqrt{2}}{4},\ c_2 = \frac{2 - \sqrt{2}}{4} \qquad \text{(4 points, 2 for each)}$$

8 Problem Eight

Proof: By plugging $a_{n-1} = c_1 r^{n-1} + c_2(n-1)r^{n-1}$ and $a_{n-2} = c_1 r^{n-2} + c_2(n-2)r^{n-2}$ into the recurrence relation, we have

$$
\begin{aligned}
& 2ra_{n-1} - r^2 a_{n-2} \\
&= 2(c_1 r^n + c_2(n-1)r^n) - (c_1 r^n + c_2(n-2)r^n) \\
&= (2c_1 r^n - c_1 r^n) + (2(n-1) - (n-2))c_2 r^n \\
&= c_1 r^n + nc_2 r^n \\
&= a_n
\end{aligned}
$$

Policy: Full points if all deduction is correct. Five points if characteristic polynomial is used to obtain $r_1 = r_2 = r$. Partial credit if only a_{n-1}, a_{n-2} are expressed.