CSE20 Final Exam (June 11, 2012) Solution

1. (number systems: one's complement) Show the operation of $17+(-14)$ in one's complement of binary number system. Assume that each binary number is represented with 10 bits. (10 points)

$$\begin{aligned} 17 &= 10001_2 \\ &= 0000010001_2 \\ 14 &= 1110_2 \\ &= 0000001110_2 \\ -14 &= 1111111111_2 - 0000001110_2 \\ &= 1111110001_2 \\ 17-14 &= 0000010001_2 + 1111110001_2 \\ &= 0000000010_2 + 1 \\ &= 0000000011_2 \end{aligned} \tag{1}$$

2. (number systems: two's complement) We have defined and learned the idea of two's complement for n-bit binary numbers.

2.1. Define the complement (corresponding to two's) using an n-digit system with base 8. (5 points)

$$\begin{aligned} -x &\Rightarrow 8^n - x \\ Range &= [-4\times 8^{n-1}, 4\times 8^{n-1}-1] \end{aligned} \tag{2}$$

2.2. Show the arithmetic of $x-y$ where $x=11_8$ and $y=17_8$ in the complement representations (corresponding to two's) using a 5-digit system with base 8. (5 points)

$$\begin{aligned} x &= 11_8 \\ &= 00011_8 \\ y &= 00017_8 \\ -y &= 77777_8 - 00017_8 + 1_8 \\ &= 77761_8 \\ x-y &= 00011_8 + 77761_8 \\ &= 77772_8 \end{aligned} \tag{3}$$

3. (Boolean algebra: proof of consensus theorem) Prove the following equality using Boolean algebra laws and theorems.

3.1 Prove the consensus theorem: $ab + a'c = ab + a'c + bc$. (5 points)

$$\begin{aligned}
ab + a'c + bc &= ab + a'c + 1 * bc && (identity) \\
&= ab + a'c + (a + a')bc && (complement) \\
&= ab + a'c + abc + a'bc && (distributive) \\
&= ab + abc + a'c + a'bc && (commutative) \\
&= ab(1 + c) + a'c(1 + b) && (distributive) \\
&= ab + a'c && (boundness)
\end{aligned} \tag{4}$$

3.2 Prove the Boolean equality $(a + b)(a' + c) = (a + b)(a' + c)(b + c)$. (5 points)

$$\begin{aligned}
(a + b)(a' + c)(b + c) &= (aa' + ac + a'b + bc)(b + c) && (distributive) \\
&= (ac + a'b + bc)(b + c) && (complmenet) \\
&= abc + acc + a'bb + a'bc + bbc + bcc && (distributive) \\
&= abc + ac + a'b + a'bc + bc && (idempotent) \\
&= ac(b + 1) + a'b(c + 1) + bc && (distributive) \\
&= ac + a'b + bc && (boundness) \\
&= 0 + ac + a'b + bc && (identity) \\
&= a'a + ac + a'b + bc && (complement) \\
&= a(a' + c) + b(a' + c) && (distributive) \\
&= (a + b)(a' + c) && (distributive)
\end{aligned} \tag{5}$$

4. (Boolean algebra) Express Boolean function $E(a,b,c) = (b + ac')'(ab' + c)$ in sum-of-products form using Boolean algebra laws and theorems. Express in the minimal expression. (10 points)

$$\begin{aligned}
E(a,b,c) &= (b+ac')'(ab'+c) && (DeMorgen)\\
&= (b')(ac')'(ab'+c) && (DeMorgen)\\
&= b'(a'+c)(ab'+c) && (distributive)\\
&= (a'b'+b'c)(ab'+c) && (distributive)\\
&= a'b'ab'+a'b'c+ab'b'c+b'cc && (complement)\\
&= a'b'c+ab'b'c+b'cc && (idempotent)\\
&= a'b'c+ab'c+b'c && (distributive)\\
&= (a'+a+1)b'c && (complement)\\
&= b'c
\end{aligned} \tag{6}$$

5. (Boolean algebra: product of sums) Express Boolean function $E(a,b,c) = a'bc + b'[(a+b')(a+c)]'$ in product-of-sums form using Boolean algebra laws and theorems. Express in the minimal expression. (10 points)

$$\begin{aligned}
E(a,b,c) &= a'bc+b'[(a+b')(a+c)]' && (DeMorgen)\\
&= a'bc+b'[(a+b')'+(a+c)'] && (DeMorgen)\\
&= a'bc+b'(a'b+a'c') && (distributive)\\
&= a'bc+a'bb'+a'b'c' && (complement)\\
&= a'bc+a'b'c' && (distributive)\\
&= a'(bc+b'c') && (distributive)\\
&= a'(b+b'c')(c+b'c') && (theorem8)\\
&= a'(b+c')(b'+c)
\end{aligned} \tag{7}$$

6. (recursive function: permutation) Suppose all the permutations on the set of $\{1,2,3,4,5,6\}$ are listed in lexicographic order from 0 to $6!-1$.
6.1 What is the RANK (order) in the list for 453261? (10 points)

$$\begin{aligned} RANK(453261) &= 3\times 5! + 3\times 4! + 2\times 3! + 1\times 2! + 1\times 1! \\ &= 3\times 120 + 3\times 24 + 2\times 6 + 1\times 2 + 1\times 1 \\ &= 360 + 72 + 12 + 2 + 1 \\ &= 447 \end{aligned} \tag{8}$$

6.2 What permutation will have the RANK 165? (5 points)

Assume the permutation to be $a_6a_5a_4a_3a_2a_1$, local rank to be r_i for ith level. Originally we start from level 6 and have $r_6 = 165$.

$$\begin{aligned} q_6 &= r_6/5! && = 1 \\ r_5 &= r_6 - q_6 \times 5! && = 45 \\ a_6 &= \{1,\mathbf{2},3,4,5,6\} && = 2 \\ q_5 &= r_5/4! && = 1 \\ a_5 &= \{1,\mathbf{3},4,5,6\} && = 3 \\ r_4 &= r_5 - q_5 \times 4! && = 21 \\ q_4 &= r_4/3! && = 3 \\ a_4 &= \{1,4,5,\mathbf{6}\} && = 6 \\ r_3 &= r_4 - q_4 \times 3! && = 3 \\ q_3 &= r_3/2! && = 1 \\ a_3 &= \{1,\mathbf{4},5\} && = 4 \\ r_2 &= r_3 - q_3 \times 2! && = 1 \\ q_2 &= r_2/1! && = 1 \\ a_2 &= \{1,\mathbf{5}\} && = 5 \\ r_1 &= r_2 - q_2 \times 1! && = 0 \\ q_1 &= r_1/0! && = 0 \\ a_1 &= \{\mathbf{1}\} && = 1 \end{aligned} \tag{9}$$

So we have the permutation for RANK 165 as

$$a_6a_5a_4a_3a_2a_1 = 236451$$

7. (recursive function: induction) Use induction to prove the following identity for any positive integer n: $1\times 2+2\times 3+\ldots+(n-1)\times n=n(n-1)(n+1)/3$. (10 points)

From the question we have the following.

$$\begin{aligned} a_n &= n(n-1) \\ sum(n) &= \Sigma_{i=1}^{n} a_i \\ f(n) &= n(n-1)(n+1)/3 \end{aligned} \tag{10}$$

And we need to prove that

$$\forall n \in \{1,2,\ldots,+\infty\}\text{ , } sum(n) = f(n)$$

Base case:

$$\begin{aligned} sum(1) &= 1\times(1-1) = 0 \\ f(1) &= 1\times(1-1)\times(1+1)/3 = 0 \\ \Rightarrow sum(1) &= f(1) \end{aligned} \tag{11}$$

Assumption:

$$\exists k > 0 \text{ s.t. } sum(k) = \Sigma_{i=1}^{k} a_i = f(k) = k(k-1)(k+1)/3$$

Incremental case:

$$\begin{aligned} sum(k+1) &= \Sigma_{i=1}^{k+1} a_i \\ &= \Sigma_{i=1}^{k} a_i + a_{k+1} \\ &= sum(k) + a_{k+1} \\ &= k(k-1)(k+1)/3 + k(k+1) \\ &= k(k+1)\left((k-1)/3+1\right) \\ &= k(k+1)(k+2)/3 \\ &= [(k+1)][(k+1)-1][(k+1)+1]/3 \\ &= f(k+1) \end{aligned} \tag{12}$$

8. (recursive function: induction) Prove by induction that any postage of at least 8 cents can be obtained using 3 cents and 5 cents stamps. (10 points)

Define the combination of 3 cents and 5 cents for n cents as

$$n = c_3(n) \times 3 + c_5(n) \times 5$$

where $c_3(n)$ and $c_5(n)$ are positive integers or zero.
Define the feasibility as a boolean variable a_n.

$$a_n = \text{true} \;\Leftrightarrow \exists\, c_3(n) \;\&\; c_5(n) \text{ s.t. } n = c_3(n) \times 3 + c_5(n) \times 5$$

We want to prove

$$a_n = \text{true } \forall n \geq 8$$

Base case:

$$\begin{aligned} 8 &= 1 \times 3 + 1 \times 5 && \Rightarrow c_3(8) = 1 && \& \; c_5(8) = 1 && \Rightarrow a_8 = true \\ 9 &= 3 \times 3 + 0 \times 5 && \Rightarrow c_3(9) = 3 && \& \; c_5(9) = 0 && \Rightarrow a_9 = true \\ 10 &= 0 \times 3 + 2 \times 5 && \Rightarrow c_3(10) = 0 && \& \; c_5(10) = 2 && \Rightarrow a_{10} = true \end{aligned} \tag{13}$$

Assumption:

$$\forall n \in [8, k] \text{ , } \exists\, c_3(n) \;\&\; c_5(n) \text{ s.t.}$$

$$n = c_3(n) \times 3 + c_5(n) \times 5 \;\&\; a_n = true$$

Incremental case:

$$\begin{aligned} k+1 &= (k+1-3) + 3 \\ &= (k-2) + 3 \\ &= c_3(k-2) \times 3 + c_5(k-2) \times 5 + 3 \\ &= (c_3(k-2) + 1) \times 3 + c_5(k-2) \times 5 \\ \Rightarrow \exists\, c_3(k+1) &= c_3(k-2) + 1 \\ c_5(k+1) &= c_5(k-2) \\ \Rightarrow a_{k+1} &= true \end{aligned} \tag{14}$$

9. (recursive function) A frog knows 3 jumping styles (A, B, C). With style A the frog jumps forward by 1 feet, and with styles, B, C, the frog jumps forward by 2 feet. Let a_i denote the number of ways to jump over a total distance of i feet.

9.1. Write the values of a_1, a_2, a_3? (5 points)

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 1 \times 1 + 2 &&= 3 \\ a_3 &= 1 \times a_2 + 2 \times a_1 &&= 5 \end{aligned} \tag{15}$$

9.2. Derive the recursive formula of a_n? (5 points)

$$a_n = a_{n-1} + 2 \times a_{n-2}$$

9.3. Find the solution of the recursion. (5 points)

$$\begin{aligned} x^2 &= x + 2 \\ &= x^2 - x - 2 \\ &= (x-2)(x+1) \\ \Rightarrow r_1 &= 2 \\ r_2 &= -1 \\ \Rightarrow a_n &= c_1(r_1)^n + c_2(r_2)^n \\ &= c_1 2^n + c_2(-1)^n \\ a_1 &= 2c_1 - c_2 = 1 \\ a_2 &= 4c_1 + c_2 = 3 \\ \Rightarrow c_1 &= \frac{2}{3} \\ c_2 &= \frac{1}{3} \\ \Rightarrow a_n &= \frac{2}{3} \times 2^n + \frac{1}{3} \times (-1)^n \end{aligned} \tag{16}$$