CSE252C – Object Recognition – Assignment #2
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`http://www-cse.ucsd.edu/classes/sp11/cse252c`
Target Due Date: Monday April. 25, 2011.

1. Dimensionality reduction on the MNIST database.
   (a) Generate a random projection matrix $G \in \mathbb{R}^{d' \times d}$ with entries $G_{ij} \sim \mathcal{N}(0, 1/d')$. Use $d' = 49$, which represents a factor of 16 smaller than the full dimensionality ($d = 784$). Compute the mean squared difference between the entries of $G^\top G$ and a $d \times d$ identity matrix. It should be close to $1/d'$.
   (b) Compute the ROC curve as in Homework #1 problem 3 using L_2 distances on $G\boldsymbol{x}^i$ in place of $\boldsymbol{x}^i$. How does the new EER compare to the old one?
   (c) Repeat the preceding step with a different dimensionality reduction method of your choice, keeping d' fixed.

2. Mahalanobis distance.

   The *Mahalanobis distance* between $\boldsymbol{x}^i$ and $\boldsymbol{x}^j$ is given by $\Delta^2 = (\boldsymbol{x}^i - \boldsymbol{x}^j)^\top \Sigma^{-1} (\boldsymbol{x}^i - \boldsymbol{x}^j)$, where Σ is a $d \times d$ covariance matrix.
   (a) A covariance matrix Σ, by definition, is symmetric and positive definite, which means $\boldsymbol{a}^\top \Sigma \boldsymbol{a} > 0$ for all $\boldsymbol{a} \in \mathbb{R}^d$. Show that a necessary and sufficient condition for Σ to be positive definite is that all of its eigenvalues are positive.
   (b) Δ^2 is equivalent to the squared Euclidean distance between $\boldsymbol{y}^i$ and $\boldsymbol{y}^j$, where $\boldsymbol{y}$ is a linearly transformed version of $\boldsymbol{x}$. What is that transformation?
   (c) Give an example of an application for which Mahalanobis distance is appropriate (e.g., compared to L_2 distance) and explain intuitively what Σ^{-1} captures in this case.

3. Properties of Chi Squared distance.

   Recall that the χ^2 distance is given by $\chi^2_{ij} = \frac{1}{2} \sum_{k=1}^{d} (x^i_k - x^j_k)^2 / (x^i_k + x^j_k)$ where the $\boldsymbol{x}$'s are normalized histogram vectors. Prove or disprove the following statements:
   (a) $\chi^2_{ij} \in [0, 1]$.
   (b) The matrix $Q \in \mathbb{R}^{n \times n}$ with entries $Q_{ij} = \sum_{k=1}^{d} x^i_k x^j_k / (x^i_k + x^j_k)$ is positive definite.
   (c) χ^2_{ij} is a metric.

4. Gabor Functions.

   The expression for the (unnormalized) isotropic 2D Gabor function is given by a Gaussian times a complex exponential
   $$h(\boldsymbol{x}) = e^{-\|\boldsymbol{x}\|^2/2\sigma^2} e^{j2\pi \boldsymbol{u}_o^\top \boldsymbol{x}}$$
   where $\boldsymbol{x} = (x, y)^\top$ and $\boldsymbol{u}_o = (u_o, v_o)^\top$, and it serves as an oriented bandpass filter. The even and odd Gabor functions are equal to the real and imaginary parts of h, respectively.
   (a) Compute four examples of even and/or odd 2D Gabor functions on the interval $\boldsymbol{x} \in [-14, 13] \times [-14, 13]$ using parameters chosen in the following ranges: $\sigma \in [1, 3]$ and $\boldsymbol{u}_o \in [0, 0.3] \times [0, 0.3]$. For each example, display the function as an image and as a surface plot.
   (b) Apply the above set of filters to two different MNIST digits and display the results. Select a few of the filtered images to explain what the filter responses are responding to in the input images.