

CSE 252B: Computer Vision II

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LECTURE 0 Guest Lecture

0.1. Brief bio

Ben Ochoa received the B.S., M.S., and Ph.D. degrees in electrical engineering from the University of California, San Diego, in 1999, 2003, and 2007, respectively. Since 2008, he has been the vice president of research and development at 2d3. For more information, see <http://vision.ucsd.edu/~bochoa/>.

0.2. Review of linear algebra

0.2.1. Matrix Rank

The rank of a matrix is the number of linearly independent rows or columns of the matrix. Given a matrix $A \in \mathbb{R}^{m \times n}$, its rank $r \leq \min(m, n)$ is calculated as the number of nonzero singular values of A . Note that $\text{rank}(A) = 0$ implies A is the 0 matrix.

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0.2.2. Singular Value Decomposition (SVD)

The SVD of $A \in \mathbb{R}^{m \times n}$ is defined as

$$(0.1) \quad A = UDV^T ,$$

where $U \in \mathbb{R}^{m \times m}$, $D = \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_{\min(m,n)}) \in \mathbb{R}^{m \times n}$, and $V \in \mathbb{R}^{n \times n}$. U and V are orthogonal matrices, and $\sigma_1, \sigma_2, \dots, \sigma_{\min(m,n)}$ are called the singular values of A . Typically, and assumed in this lecture, the singular values are ordered in descending order.

0.2.3. Null Spaces of A

Right null space	Left null space
$AX = 0$	$YA = 0$

where $A \in \mathbb{R}^{m \times n}$. Here, $X \in \mathbb{R}^{n \times (n-r)}$ is called the (right) null space of A and $Y \in \mathbb{R}^{(m-r) \times m}$ is called the left null space of A , where $r = \text{rank}(A)$.

The right and left null spaces of a matrix A can be calculated using SVD as follows. From $A = UDV^T$, the right null space $X = \text{null}(A)$ is the matrix composed of the columns of V that correspond to singular values that are equal to zero. Similarly, the left null space Y of A is the matrix composed of the rows of U^T that correspond to singular values equal to zero.

For example, this can be used to compute the epipoles from the fundamental matrix F . The SVD of $F = UDV^T$ where $F, U, D, V \in \mathbb{R}^{3 \times 3}$ and $\text{rank}(F) = 2$. For the epipole in first image (\mathbf{e}_1) we have

$$(0.2) \quad F\mathbf{e}_1 = \mathbf{0} ,$$

so \mathbf{e}_1 is the last column of V . For the epipole in second image (\mathbf{e}_2) we have

$$(0.3) \quad F^T\mathbf{e}_2 = \mathbf{0}$$

$$(0.4) \quad \mathbf{e}_2^T F = \mathbf{0}^T ,$$

hence \mathbf{e}_2^T is the last row of U^T (or \mathbf{e}_2 is the last column of U).

0.3. Configurations of some geometric primitives

0.3.1. Configurations of points and lines in 2D

Consider the homogeneous coordinates of a point $\mathbf{x} = (x, y, w)^T$ and line $\mathbf{l} = (a, b, c)^T$. If \mathbf{x} lies on \mathbf{l} , then

$$(0.5) \quad \mathbf{x}^T \mathbf{l} = 0$$

$$(0.6) \quad \mathbf{l}^T \mathbf{x} = 0 .$$

Also, recall that homogeneous coordinates are only determined up to scale.

0.3.1.1. *2D Points*. Given a set of n points, we can arrange them into a single matrix as

$$(0.7) \quad A = \begin{bmatrix} \mathbf{x}^{1T} \\ \mathbf{x}^{2T} \\ \vdots \\ \mathbf{x}^{nT} \end{bmatrix} \in \mathbb{R}^{n \times 3}.$$

The rank and null space of A determine the configuration of the set of points. The following table shows the different possible configurations and the corresponding rank and null space of A .

Illustration	Configuration	rank(A)	Columns of null(A)
	general position	3	<i>empty</i>
	collinear	2	line that points lie on
	coincident	1	two lines intersecting at points

0.3.1.2. *2D Lines*. Similarly, given a set of n lines, we can arrange them into a single matrix as

$$(0.8) \quad A = \begin{bmatrix} \mathbf{l}^{1T} \\ \mathbf{l}^{2T} \\ \vdots \\ \mathbf{l}^{nT} \end{bmatrix} \in \mathbb{R}^{n \times 3}.$$

The rank and null space of A determine the configuration of the set of lines. The following table shows the different possible configurations and the corresponding rank and null space of A .

Illustration	Configuration	rank(A)	Columns of null(A)
	general position	3	<i>empty</i>
	pencils of lines	2	point at intersection of lines
	coincident	1	two points on lines

0.3.2. Configurations of points and planes in 3D

Consider a point $\mathbf{X} = (X, Y, Z, T)^T$ and plane $P = (a, b, c, d)^T$. If \mathbf{X} lies on P then

$$(0.9) \quad \mathbf{X}^T P = 0$$

$$(0.10) \quad P^T \mathbf{X} = 0$$

0.3.2.1. *3D Points*. Given a set of n points, we can arrange them into a single matrix as

$$(0.11) \quad A = \begin{bmatrix} \mathbf{X}^{1T} \\ \mathbf{X}^{2T} \\ \vdots \\ \mathbf{X}^{nT} \end{bmatrix} \in \mathbb{R}^{n \times 4} .$$

The rank and null space of A determine the configuration of the set of points. The following table shows the different possible configurations and the corresponding rank and null space of A .

Illustration	Configuration	rank(A)	Columns of null(A)
See figure 1	general position	4	<i>empty</i>
See figure 2	coplanar	3	plane that points lie on
See figure 3	collinear	2	two planes intersecting at line that points lie on
See figure 4	coincident	1	three planes intersecting at points

0.3.2.2. *3D Planes*. Similarly, given a set of n planes, we can arrange them into a single matrix as

$$(0.12) \quad A = \begin{bmatrix} P^{1T} \\ P^{2T} \\ \vdots \\ P^{nT} \end{bmatrix} \in \mathbb{R}^{n \times 4} .$$

The rank and null space of A determine the configuration of the set of planes. The following table shows the different possible configurations and the corresponding rank and null space of A .

Illustration	Configuration	rank(A)	Columns of null(A)
See figure 5	general position	4	<i>empty</i>
See figure 4	bundle of planes	3	point at intersection of planes
See figure 3	sheaf of planes	2	two points on 3D line at intersection of planes
See figure 2	coincident	1	three points on planes

0.4. An application of the left null space

The (right) null space has many applications in computer vision, but do not overlook the left null space. For example, consider two non-coincident 2D lines \mathbf{l}^1 and \mathbf{l}^2 . The point \mathbf{x} at the intersection of \mathbf{l}^1 and \mathbf{l}^2 can be calculated

using the cross product by $\mathbf{x} = \mathbf{l}^1 \times \mathbf{l}^2$. Alternatively, \mathbf{x} can be calculated by taking the null space of a matrix whose rows are the lines,

$$(0.13) \quad \begin{bmatrix} \mathbf{l}^{1T} \\ \mathbf{l}^{2T} \end{bmatrix} \mathbf{x} = \mathbf{0} .$$

However, this equation can also be used to determine two lines \mathbf{l}^1 and \mathbf{l}^2 that intersect at \mathbf{x} . The two lines are simply the rows of the left null space of \mathbf{x} .

As shown previously, the left null space of \mathbf{x} can be calculated using SVD. $\mathbf{x} = UDV^T$, where $U = [\mathbf{u}^1 \mid \mathbf{u}^2 \mid \mathbf{u}^3] \in \mathbb{R}^{3 \times 3}$, $D = (\sigma_1, 0, 0)^T$, and V^T is a scalar. As such, the left null space of \mathbf{x} is $[\mathbf{u}^2 \mid \mathbf{u}^3]^T = [\mathbf{l}^1 \mid \mathbf{l}^2]^T$.

0.4.1. Householder matrix

Although SVD can be used to calculate the left null space of a vector $\mathbf{x} \in \mathbb{R}^{n \times 1}$, a much less computationally expensive method involves determining an orthogonal matrix $H \in \mathbb{R}^{n \times n}$ such that

$$(0.14) \quad H\mathbf{x} = \begin{bmatrix} \pm\|\mathbf{x}\| \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Such a matrix H is called a Householder matrix and is calculated by

$$(0.15) \quad H = I - 2 \frac{\mathbf{v}\mathbf{v}^T}{\mathbf{v}^T\mathbf{v}}$$

where

$$(0.16) \quad \mathbf{v} = \mathbf{x} + \text{sign}(x_1) \|\mathbf{x}\| \mathbf{e}_1 .$$

Here $\mathbf{e}_1 = (1, 0, 0, \dots, 0)^T$, not the epipole in the first image, and \mathbf{x} is a general vector, not a 2D point in homogeneous coordinates.

$$(0.17) \quad H\mathbf{x} = \begin{bmatrix} \pm\|\mathbf{x}\| \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$(0.18) \quad \begin{bmatrix} \mathbf{a}^T \\ [\mathbf{x}]^\perp \end{bmatrix} \mathbf{x} = \begin{bmatrix} \pm\|\mathbf{x}\| \\ \mathbf{0} \end{bmatrix} ,$$

where \mathbf{a}^T is the first row of H and $[\mathbf{x}]^\perp$ is the matrix containing the remaining rows of H . From this, we see that $\mathbf{a}^T \mathbf{x} = \pm\|\mathbf{x}\|$ and

$$(0.19) \quad [\mathbf{x}]^\perp \mathbf{x} = \mathbf{0} .$$

That is, $[\mathbf{x}]^\perp$ is the left null space of \mathbf{x} , where $[\mathbf{x}]^\perp$ is the matrix H with its first row omitted.

0.4.2. Points, lines, and planes that lie on and pass through

The left null space of points and lines in 2D and points and planes in 3D provides some useful results.

0.4.2.1. *Two lines through a 2D point.*

$$(0.20) \quad [\mathbf{x}]^\perp \mathbf{x} = \mathbf{0}$$

$$(0.21) \quad \begin{bmatrix} \mathbf{l}^{1T} \\ \mathbf{l}^{2T} \end{bmatrix} \mathbf{x} = \mathbf{0}$$

0.4.2.2. *Two points on a 2D line.*

$$(0.22) \quad [\mathbf{l}]^\perp \mathbf{l} = \mathbf{0}$$

$$(0.23) \quad \begin{bmatrix} \mathbf{x}^{1T} \\ \mathbf{x}^{2T} \end{bmatrix} \mathbf{l} = \mathbf{0}$$

0.4.2.3. *Three planes through a 3D point.*

$$(0.24) \quad [\mathbf{X}]^\perp \mathbf{X} = \mathbf{0}$$

$$(0.25) \quad \begin{bmatrix} P^{1T} \\ P^{2T} \\ P^{3T} \end{bmatrix} \mathbf{X} = \mathbf{0}$$

0.4.2.4. *Three points on a 3D plane.*

$$(0.26) \quad [P]^\perp P = \mathbf{0}$$

$$(0.27) \quad \begin{bmatrix} \mathbf{X}^{1T} \\ \mathbf{X}^{2T} \\ \mathbf{X}^{3T} \end{bmatrix} P = \mathbf{0}$$

0.5. Direct linear transformation algorithm

Using the methods from this lecture, we can reformulate some of the equations that describe different mappings in computer vision. This is useful when applying the direct linear transformation (DLT) algorithm to estimate these mappings.

0.5.1. Planar homography

The following notation will be used for the planar homography matrix H .

$$(0.28) \quad H = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$$

$$(0.29) \quad H = \begin{bmatrix} \mathbf{h}^{1T} \\ \mathbf{h}^{2T} \\ \mathbf{h}^{3T} \end{bmatrix},$$

where $\mathbf{h}^{iT} = (h_{i1}, h_{i2}, h_{i3})$, i.e., \mathbf{h}^{iT} is the i th row of H . Further, let us denote

$$(0.30) \quad \mathbf{h} = (h_{11}, h_{12}, h_{13}, h_{21}, h_{22}, h_{23}, h_{31}, h_{32}, h_{33})^T$$

$$(0.31) \quad \mathbf{h} = \begin{bmatrix} \mathbf{h}^1 \\ \mathbf{h}^2 \\ \mathbf{h}^3 \end{bmatrix}$$

$$(0.32) \quad \mathbf{h} = \text{vec}(H^T).$$

0.5.1.1. *2D points.*

$$(0.33) \quad \mathbf{x}_2^i \sim H \mathbf{x}_1^i$$

$$(0.34) \quad [\mathbf{x}_2^i]^\perp H \mathbf{x}_1^i = \mathbf{0}$$

$$(0.35) \quad \begin{bmatrix} \mathbf{l}_2^{i1T} \\ \mathbf{l}_2^{i2T} \end{bmatrix} \begin{bmatrix} \mathbf{h}^{1T} \\ \mathbf{h}^{2T} \\ \mathbf{h}^{3T} \end{bmatrix} \mathbf{x}_1^i = \mathbf{0}$$

$$(0.36) \quad \begin{bmatrix} a_2^{i1} & b_2^{i1} & c_2^{i1} \\ a_2^{i2} & b_2^{i2} & c_2^{i2} \end{bmatrix} \begin{bmatrix} \mathbf{h}^{1T} \mathbf{x}_1^i \\ \mathbf{h}^{2T} \mathbf{x}_1^i \\ \mathbf{h}^{3T} \mathbf{x}_1^i \end{bmatrix} = \mathbf{0}$$

$$(0.37) \quad \begin{bmatrix} a_2^{i1} \mathbf{h}^{1T} \mathbf{x}_1^i + b_2^{i1} \mathbf{h}^{2T} \mathbf{x}_1^i + c_2^{i1} \mathbf{h}^{3T} \mathbf{x}_1^i \\ a_2^{i2} \mathbf{h}^{1T} \mathbf{x}_1^i + b_2^{i2} \mathbf{h}^{2T} \mathbf{x}_1^i + c_2^{i2} \mathbf{h}^{3T} \mathbf{x}_1^i \end{bmatrix} = \mathbf{0}$$

$$(0.38) \quad \begin{bmatrix} a_2^{i1} \mathbf{x}_1^{iT} & b_2^{i1} \mathbf{x}_1^{iT} & c_2^{i1} \mathbf{x}_1^{iT} \\ a_2^{i2} \mathbf{x}_1^{iT} & b_2^{i2} \mathbf{x}_1^{iT} & c_2^{i2} \mathbf{x}_1^{iT} \end{bmatrix} \begin{bmatrix} \mathbf{h}^1 \\ \mathbf{h}^2 \\ \mathbf{h}^3 \end{bmatrix} = \mathbf{0}$$

$$(0.39) \quad \begin{bmatrix} \mathbf{l}_2^{i1T} \otimes \mathbf{x}_1^{iT} \\ \mathbf{l}_2^{i2T} \otimes \mathbf{x}_1^{iT} \end{bmatrix} \mathbf{h} = \mathbf{0}$$

$$(0.40) \quad \left(\begin{bmatrix} \mathbf{l}_2^{i1T} \\ \mathbf{l}_2^{i2T} \end{bmatrix} \otimes \mathbf{x}_1^{iT} \right) \mathbf{h} = \mathbf{0}$$

$$(0.41) \quad \left([\mathbf{x}_2^i]^\perp \otimes \mathbf{x}_1^{iT} \right) \mathbf{h} = \mathbf{0}$$

The last equation is each of the two rows in design matrix used for estimating \mathbf{h} .

0.5.1.2. *2D lines.*

$$(0.42) \quad \mathbf{l}_2^j \sim H^{-T} \mathbf{l}_1^j$$

$$(0.43) \quad \mathbf{l}_1^j \sim H^T \mathbf{l}_2^j$$

$$(0.44) \quad [\mathbf{l}_1^j]^\perp H^T \mathbf{l}_2^j = \mathbf{0}$$

$$(0.45) \quad \left(\mathbf{l}_2^{jT} \otimes [\mathbf{l}_1^j]^\perp \right) \mathbf{h} = \mathbf{0}$$

0.5.1.3. *2D points and 2D lines.* Sets of corresponding points and lines can be combined to estimate the planar homography.

$$(0.46) \quad \begin{bmatrix} \left([\mathbf{x}_2^i]^\perp \otimes \mathbf{x}_1^{iT} \right) \\ \left(\mathbf{l}_2^{iT} \otimes [\mathbf{l}_1^i]^\perp \right) \end{bmatrix} \mathbf{h} = \mathbf{0}$$

0.5.2. 3D projective transformation

The equations for points and hyperplanes in 2D scale to 3D. In 3D, $H \in \mathbb{R}^{4 \times 4}$ and $\mathbf{h} = \text{vec}(H^T) \in \mathbb{R}^{16 \times 1}$. Each point of plane correspondence generates three rows in the design matrix.

0.5.2.1. *3D points.*

$$(0.47) \quad \mathbf{X}_2^i \sim H \mathbf{X}_1^i$$

$$(0.48) \quad [\mathbf{X}_2^i]^\perp H \mathbf{X}_1^i = \mathbf{0}$$

$$(0.49) \quad \left([\mathbf{X}_2^i]^\perp \otimes \mathbf{X}_1^{iT} \right) \mathbf{h} = \mathbf{0}$$

0.5.2.2. *3D planes.*

$$(0.50) \quad P_2^j \sim H^{-T} P_1^j$$

$$(0.51) \quad P_1^j \sim H^T P_2^j$$

$$(0.52) \quad [P_1^j]^\perp H^T P_2^j = \mathbf{0}$$

$$(0.53) \quad \left(P_2^{jT} \otimes [P_1^j]^\perp \right) \mathbf{h} = \mathbf{0}$$

0.5.2.3. *3D points and 3D planes.*

$$(0.54) \quad \begin{bmatrix} \left([\mathbf{X}_2^i]^\perp \otimes \mathbf{X}_1^{iT} \right) \\ \left(P_2^{iT} \otimes [P_1^i]^\perp \right) \end{bmatrix} \mathbf{h} = \mathbf{0}$$

0.6. Additional figures

These figures are illustration for the various configurations of 3D points and planes. Some of them were generated using the Householder matrix method. For example, 3D coincident points or bundle of planes (figure 4) was generated using the following matlab code:

```
% random 3d point (homogeneous coordinates)
x = [randn(3,1); 1];
v = x + sign(x(1))*norm(x)*[1;0;0;0];
h = eye(4) - 2*(v*v')/(v'*v);

plot3(x(1),x(2),x(3),'.r','MarkerSize',40);
hold on;
drawplane(h(2,:));
drawplane(h(3,:));
drawplane(h(4,:));
```

where drawplane is the following function

```
function drawplane(p)
% p is [a b c d]' where
% a*x + b*y + c*z + d = 0 defines the plane
x=-5:.5:5;
[X,Y] = meshgrid(x);
a=p(1); b=p(2); c=p(3); d=p(4);
Z=(-d- a * X - b * Y)/c;
mesh(X,Y,Z);
shading flat;
```

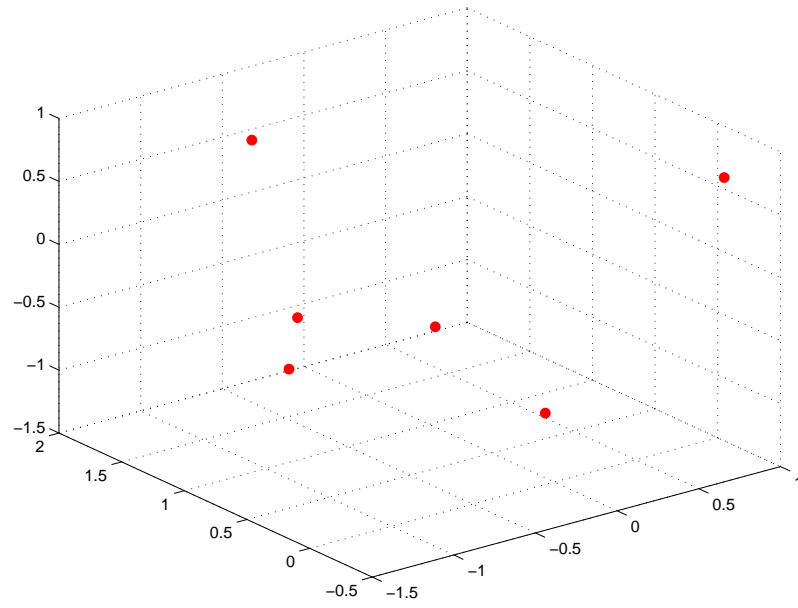


Figure 1. 3D points in general position

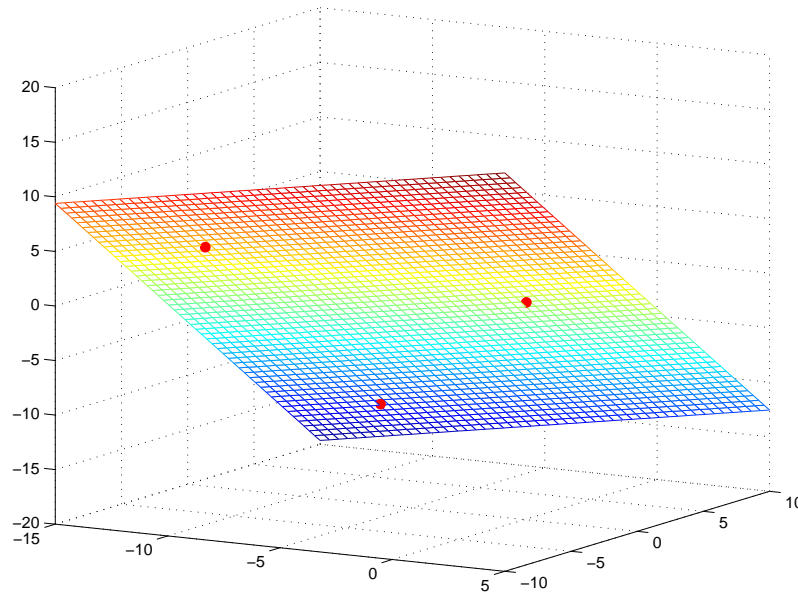


Figure 2. 3D points in coplanar position, also coincident planes.

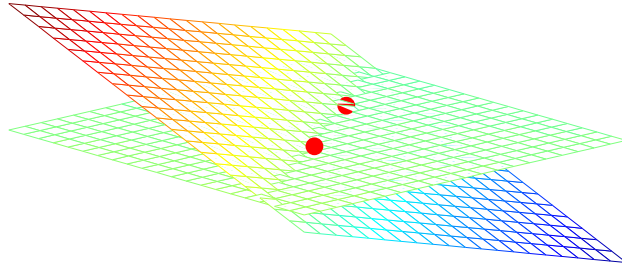


Figure 3. 3D points in collinear position, also a sheaf of planes.

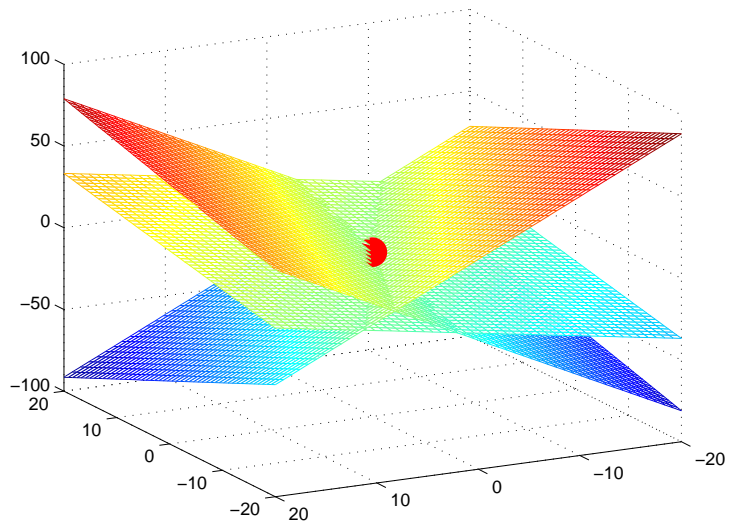


Figure 4. 3D coincident points, also a bundle of planes.

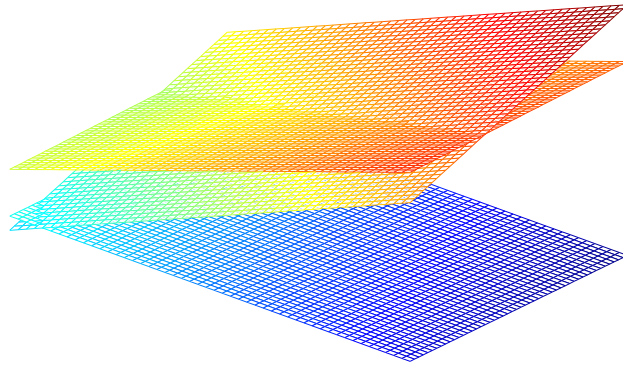


Figure 5. Planes in general position