

CSE 20 Spring 2009
Exam 2

1. Is the following

$$(p \wedge q) \vee \sim p$$

a tautology, a contradiction, or neither? Show your work or your reasoning.

Comment: an expression is a tautology if it always evaluates to 1, no matter what combination of 0 and 1 values you give to its variables. For example

$$p \vee \sim p \vee q$$

is always 1 no matter what the values of p and q are.

An expression is a contradiction if it always evaluates to 0, no matter what combinations of 0 and 1 values you give to its variables. For example

$$r \wedge t \wedge p \wedge \sim p$$

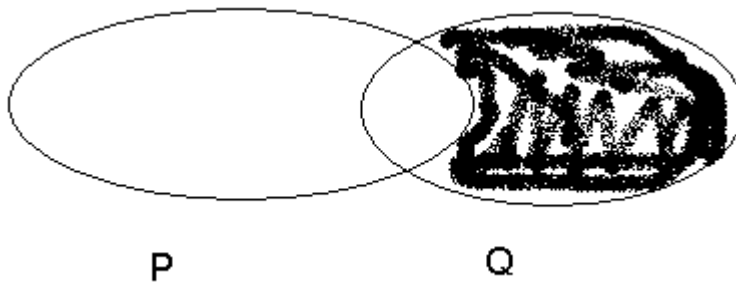
is always 0, no matter what the values of p and q are.

The above expression can be either 1 or 0, depending on what values q and p have, so it is neither a tautology nor a contradiction. For example if

$p = 1$ and $q = 1$, the expression evaluates to 1

But if $p = 1$ and $q = 0$, the expression evaluates to 0

2. Consider the following Venn diagram with two sets P and Q .



The shaded part contains all of Q except that part that is also in P .

a) Using the set operations

$\cap$ (intersect), $\cup$ (union), and $\sim$ (complement), give a set expression that describes the elements in the shaded area above

Recall the union of two sets is the set with the elements that are in either, or in both.

The intersection is the set with elements that are only in both of them.

For the shaded area, we want elements that are in Q but not in P. This means they are in Q but outside of P. This can be written as Q and not P, which is equal to $Q \cap \sim P$.

A slightly different way of putting this is that we want the elements that are in Q but not in the part of Q that overlaps with P, i.e. in Q but not in $P \cap Q$. This will be $Q \cap \sim(P \cap Q)$

b) Sets and Logic

let $p(x)$ stand for x is an element of the set P, and $q(x)$ for x is an element of the set Q.

Give a logical expression using the propositional logic operators

$\wedge$ $\vee$ and $\sim$

that describes the elements in the shaded area above

Reasoning as above, gives us $q(x) \wedge \sim p(x)$ or, alternatively,

$q(x) \wedge \sim(p(x) \wedge q(x))$.

3. "If the program is running then there is a least 250K of RAM." Which of the following is equivalent to this statement? Write yes or no for each.

___Y___ a) The program will only run if there is at least 250K of RAM.

___Y___ b) A necessary condition for the program to run is that there is at least 250K of RAM.

___N___ c) A sufficient condition for the program to run is that there is at least 250K of RAM.

Comment. This is in classical "if then" format, so it is an implication.

Let p = "the program is running", and q = "there is at least 250K of RAM".

Then we have here $p \Rightarrow q$.

a) suppose we say "a only if b". This statement would be false in the case where a was true but b was false. This means it is equivalent to $a \Rightarrow b$.

In this example a is "the program will run" and b is "there is at least 250K of RAM" which matches p and q, so that it is simply $p \Rightarrow q$ and means the same thing

Another way to look at this statement means that a is only true when b is true. In the case where a is false, the statement "a only if b" is true "by default".

This is also exactly $p \Rightarrow q$, because for this to be true, p cannot be true unless q is true. Also, the expression is true in the case where p is false.

b) in this case we have "there is at least 250K of RAM" is necessary for "the program to run". Using our above definitions for p and q, it says that q is necessary for p. This corresponds exactly to the implication

$$p \Rightarrow q$$

c) in this case we have "q is sufficient for p". But this corresponds exactly to the implication

$$q \Rightarrow p.$$

which is not the same as $p \Rightarrow q$ so the answer is No.

So putting these all together, the answer is Y, Y, N for the three parts

4. Construct a truth table for

$$p \wedge (p \vee \sim q) \Rightarrow q$$

This is made a lot easier if we simplify first

Recall the absorption rule

$a \wedge (a \vee b)$ is equivalent to a.

So this means all we need to do is write the truth table for

$$p \Rightarrow q$$

Of course, doing the truth table for the whole expression is fine. Remember the $\text{exp1} \Rightarrow \text{exp2}$ is equivalent to

$$\sim \text{exp1} \vee \text{exp2}$$

which tells us what the last column in the table has to look like.

5. Start with the predicate logic statement

$$\forall n \in \mathbb{N}, n^2 \text{ even} \Rightarrow n \text{ even}$$

Comment: This says that for all natural numbers (i.e. non-negative integers) if the square of n is even then n must be even. (This happens to be a true statement.)

Which of the following statements says the same thing? Enter yes or not for each one.

___N___ a) Every integer has an even square and is even

This is not the same. A clue might be that it is not even true, because not all integers are even. However we can still ask whether one statement is equivalent to another, even if we do not know if it is true.

The predicate logic equivalent of this statement would be
 $\forall n \in \mathbb{N}, n^2 \text{ even} \wedge n \text{ even}$

$p \wedge q$ is the translation of the statement here, but the original is $p \Rightarrow q$ which has a different truth table.

___Y___ b) If a given integer has an even square then that integer is even

This simply says the same thing.

___Y___ c) If the square of an integer is even then it is even

So does this.

6. For the predicate logic statement

$$\forall n \in \mathbb{N}, n^2 \text{ even} \Rightarrow n \text{ even}$$

Use predicate logic notation to write the converse and the contrapositive of this statement.

a) converse

$q \Rightarrow p$ is the converse of $p \Rightarrow q$.

so, reversing the implication we get:

$$\forall n \in \mathbb{N}, n \text{ even} \Rightarrow n^2 \text{ even}$$

b) contrapositive

$\sim q \Rightarrow \sim p$ is the contrapositive of $p \Rightarrow q$

So this gives us:

$$\forall n \in \mathbb{N}, \sim(n \text{ even}) \Rightarrow \sim(n^2 \text{ even})$$

or

$$\forall n \in \mathbb{N}, n \text{ odd} \Rightarrow n^2 \text{ odd}$$

In the contrapositive, we want the equivalent statement but starting with the negation of the consequent, . The above is exactly equivalent to the original statement

6. Consider the predicate logic statement

$$\forall p \in \text{Computer-Programs}, \text{correct}(p) \Rightarrow \text{compiles-with-no-warnings}(p)$$

Use the existential operator $\exists$ to write the negation of this statement in similar predicate logic form.

The answer is:

$$\exists p \in \text{Computer-Programs}, \sim(\text{correct}(p) \Rightarrow \text{compiles-with-no-warnings}(p))$$

Comment: To negate a universal quantification we simply change it to the existential form and negate the predicate.

In this example, if you wanted to go a little further, you might try to simplify the negated (compound) predicate as follows.

$$\begin{aligned} &\sim(\text{correct}(p) \Rightarrow \text{compiles-with-no-warnings}(p)) \\ &\sim(\sim \text{correct}(p) \vee \text{compiles-with-no-warnings}(p)) \\ &\text{correct}(p) \wedge \sim(\text{compiles-with-no-warnings}(p)) \end{aligned}$$

which gives us

$$\exists p \in \text{Computer-Programs}, \text{correct}(p) \wedge \sim(\text{compiles-with-no-warnings}(p))$$

Comment: Most mistakes by students involved the failure to correctly negate the predicate. People left the implication and then negated the left side, the right side or both.

When manipulating an implication $p \Rightarrow q$, it is sometimes easier to simply rewrite it in its " $\sim p \vee q$ " form and then apply the changes.

7. Infinite sets.

Consider the set P of all primes. Use predicate logic notation to write the statement that the set of all primes is infinite. Use the convention that we used in class to indicate the infinite.

The convention we used is that if we have a set where every time we pick some element x , we can find an element of the set that is bigger than x , then the set must be infinite.

We could do this in several ways. One would be to say that for every prime x there exists a prime y such that y is larger than x . i.e.

$$\forall x \in P, (\exists y \in P, y > x)$$

Alternatively we can reference the set N of natural numbers, and to say that the set of primes P is infinite, we would say for every integer n , there is a prime p that is bigger than n .

$$\forall n \in N, (\exists p \in N, p \in P \wedge p \geq n)$$

Since the set of integers is infinite, then the set of primes must be.

Note that some people tried to equate this with the theorem we went over in class in which we wrote down the (unproved) conjecture that there is an infinite number of prime pairs. The prime pairs example is different from this one, although they use a similar convention for saying a set is infinite.

8. Consider the following two proposed identities. Are they correct? Write yes or no for each. If your answer is no for one of them, then construct a set D with at most 3 elements that serves as a counter example, i.e. an example for which the proposed identity does not hold.

a) Yes

$$\exists x \in D, (x < -10 \vee x > 10)$$

$$\Leftrightarrow ((\exists x \in D, x < -10) \vee (\exists x \in D, x > 10))$$

The left hand side says that there exists an element x of D such that x is either less than -10 or bigger than 10. This means that either there is an element x of D which is less than -10 or that there is an element x of D that is bigger than 10. But this is just the right hand side. So they are equivalent.

b) No

$$\forall x \in D (x < -10 \vee x > 10) \Leftrightarrow ((\forall x \in D, x < -10) \vee (\forall x \in D, x > 10))$$

The problem here is that the left hand side says that for all elements x of D , either x is less than -10 or x is greater than 10. This allows the case where some elements of D are less than -10 and some elements of D are greater than 10. But the right hand side says that all elements of D are less than 10 or all elements are greater than 10 - they have to be all less than or all greater than, but not both.

Consider the set $D = \{-11, 11\}$.

The counter example must have one side of the biconditional $\Leftrightarrow$ that is true and the other side false. For the expression above in this part whenever the right hand side is true, the left hand side will also be true, so the biconditional is "half-true". But in the counter example the left hand side is true but the right hand side is false.

Name: _____

7

A counterexample for this problem must contain at least one element < -10 and one element > 10 , and no elements between -10 and 10 . There must be none in between because then neither side would be true. In a counter example that has these properties, the left hand side will be satisfied but not the right.