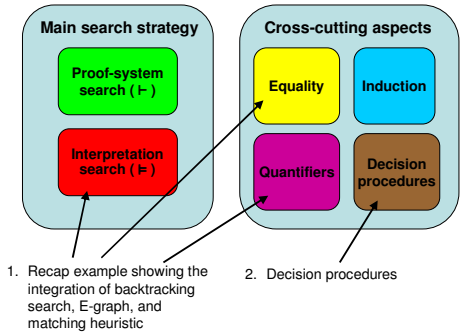
Plan for today



A recap example

define fact *hasConstValue(X:Var,C:Const)*
with meaning ⟦ $X == C$ ⟧

if $currStmt == [X = C]$
then $hasConstValue(X,C)@out$

if $hasConstValue(X,C)@in \wedge$
$currStmt == [Y = X]$
then $mustPointTo(Y,C)@out$

if $hasConstantValue(Y,C)@in \wedge$
$currStmt == [X = Y]$
then transform to $[X = C]$

VC for the trans rule

- Show that if statements x := y and x := c are executed in a state where y == c, then the resulting states are the same.

if $hasConstantValue(Y,C)@in \wedge$
$currStmt == [X = Y]$
then transform to $[X = C]$

VC for the trans rule

- Show that if statements x := y and x := c are executed in a state where y == c, then the resulting states are the same.
- Show:

$\forall\ x,y,c,\sigma\ .\quad \sigma[y] = c\ \Rightarrow$
$\forall\ v\ .\ step(x := y,\ \sigma)[v] =$
$step(x := c,\ \sigma)[v]$

Background axioms

- If a := k gets stepped in store σ, the resulting store is σ with "a" updated to "k".
- If a := b gets stepped in store σ, the resulting store is σ with "a" updated to the value of "b".
- Show:

$\forall\ x,y,c,\sigma\ .\quad \sigma[y] = c\ \Rightarrow$
$\forall\ v\ .\ step(x := y,\ \sigma)[v] =$
$step(x := c,\ \sigma)[v]$

Background axioms

- Axioms:

$\forall\ a,k,\sigma\ .\ step(a := k,\ \sigma) = store(\sigma,\ a,\ k)$

$\forall\ a,b,\sigma\ .\ step(a := b,\ \sigma) = store(\sigma,\ a,\ \sigma[b])$

- Show:

$\forall\ x,y,c,\sigma\ .\quad \sigma[y] = c\ \Rightarrow$
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Expand

- Axioms:

  $\forall\ a,k,\sigma\ .\ step(a := k,\ \sigma) = store(\sigma,\ a,\ k)$

  $\forall\ a,b,\sigma\ .\ step(a := b,\ \sigma) = store(\sigma,\ a,\ \sigma[b])$

- Show:

  $\forall\ x,y,c,\sigma\ .\quad \sigma[y] = c\ \Rightarrow$

  $\forall\ v\ .\ step(x := y,\ \sigma)[v] =$

  $step(x := c,\ \sigma)[v]$

Expand

$\forall\ x,y,c,\sigma\ .\ \neg\sigma[y] = c\ \forall$

$\forall\ v\ .\ step(x := y,\ \sigma)[v] =$

$step(x := c,\ \sigma)[v]$

Skolemize

$\neg\,\sigma[y] = c\ \vee$

$step(x := y,\ \sigma)[v] =$

$step(x := c,\ \sigma)[v]$

Skolemize

$\neg\ \sigma[y] = c \vee$

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Refutation

$\neg\ \sigma[y] = c \vee$

$step(x := y,\ \sigma)[v] = step(x := c,\ \sigma)[v]$

Negate formula and show that the negation is unsatisfiable

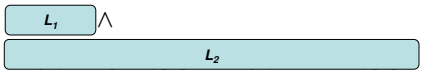
Refutation

$\sigma[y] = c \wedge$

$step(x := y,\ \sigma)[v] \neq step(x := c,\ \sigma)[v]$

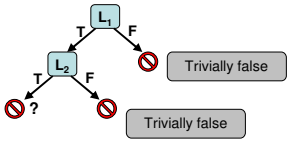
Negate formula and show that the negation is unsatisfiable

Exhaustive interpretation search



Exhaustive interpretation search

$L_1 \wedge$
L_2



Exhaustive interpretation search

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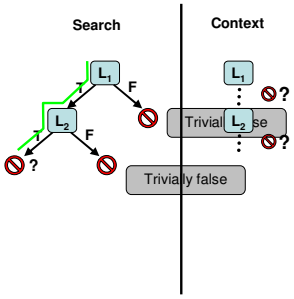
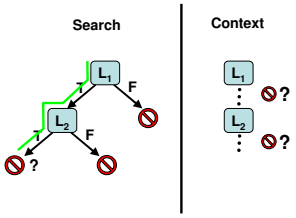


Two ways to refute:
- Formula becomes trivially false
- Set of assumed literals is inconsistent

Exhaustive interpretation search

$L_1 \wedge$
L_2



$L_1 \triangleq \sigma[y] = c$
$L_2 \triangleq step(x := y, \sigma)[v] \neq step(x := c, \sigma)[v]$

Equality using E-graph

$L_1 \triangleq \sigma[y] = c$
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Equality using E-graph

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$\sigma[y]$ -------- c

Equality using E-graph

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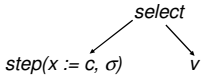

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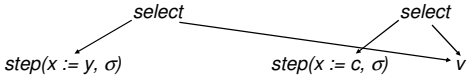

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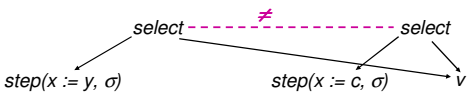
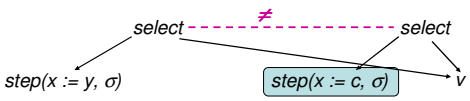


Matching

$\forall\ a,k,\sigma\ .\ step(a := k, \sigma) = store(\sigma, a, k)$

- Pick a trigger
- If trigger appears in E-graph, instantiate quantifier body

$step(x := c, \sigma) = store(\sigma, x, c)$



Matching

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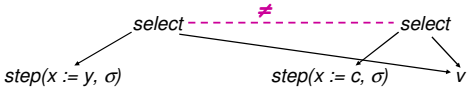

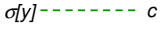

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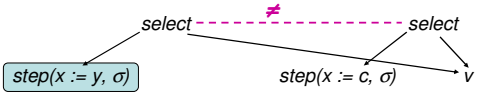
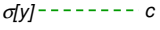


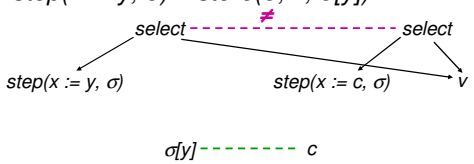


Matching

In summary, add:

step(x := c, σ) = store(σ, x, c)

step(x := y, σ) = store(σ, x, σ[y])



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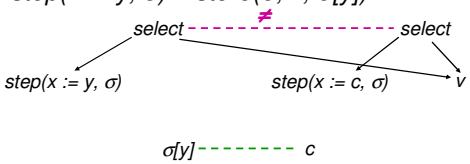

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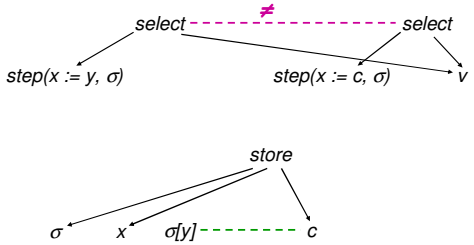

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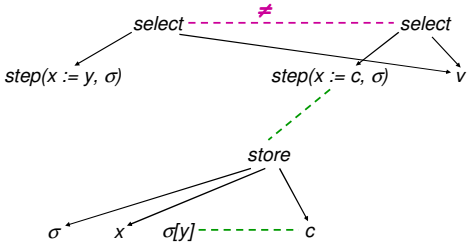

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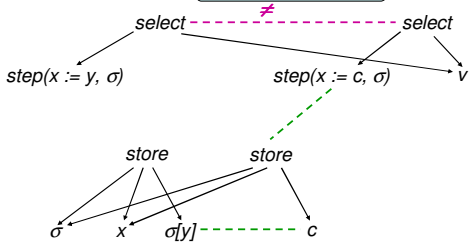

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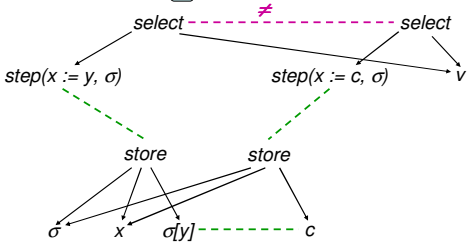
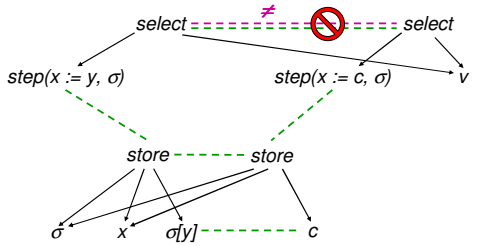
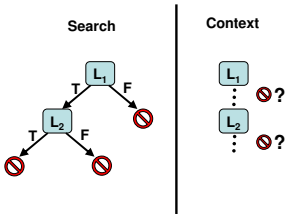


Compute congruence closure



Exhaustive Interpretation search

$L_1 \wedge$
L_2



$L_1 \triangleq \sigma[y] = c$
$L_2 \triangleq step(x := y, \sigma)[v] \neq step(x := c, \sigma)[v]$

Decision procedures

Decision procedures

- Decision procedures are complete algorithms for determining the validity of a formula in a given logic
- Decision procedures exist for many logics
  - EUF
  - Theory of lists
  - Theory of arrays
  - Theory of linear arithmetic over reals or integers
  - Theory of bit-vectors
  - …

Decision procedures

- Decision procedures can be used as standalone provers
- But we are more concerned with how decision procedures can be used within the context of a "heuristic" theorem prover
  - A heuristic theorem prover is a theorem prover for an undecidable logic that uses heuristics to guide its search
  - We use the term "heuristic" to avoid confusion between the larger heuristic prover and the decision procedures that are being integrated into this larger prover

Decision procedures

- Why incorporate decision procedures into a heuristic prover?
- Because once the search reaches a formula in a decidable subset of the original logic, the strategies of the heuristic prover may be inefficient and incomplete

Issues

- There are two issues to consider when incorporating decision procedures into a heuristic prover
  - Communication between decision procedures and the heuristic prover
  - Communication between decision procedures

In Simplify--

- Communication between decision procedures
  - Don't have to deal with this, because Simplify-- has only one decision procedure, namely EUF

In Simplify--

- Communication form heuristic prover to decision procedures
- Communication from decision procedures to the heuristic prover

In Simplify--

- Communication form heuristic prover to decision procedures
  - Push equalities into the E-graph incrementally
    - Does not require the decision procedure to expose its internal details
- Communication from decision procedures to the heuristic prover
  - Matching heuristic looks into E-graph
    - Motivation is to improve the heuristic of the prover
    - For efficiency, expose details of the decision procedure's data structures
  - Explicating proofs used to guide the backtracking search
    - Motivation is efficiency

Issues again

- Communication between decision procedure and the heuristic prover
  - We've seen how this works in Simplify--
- Communication between decision procedures
  - This is what's next

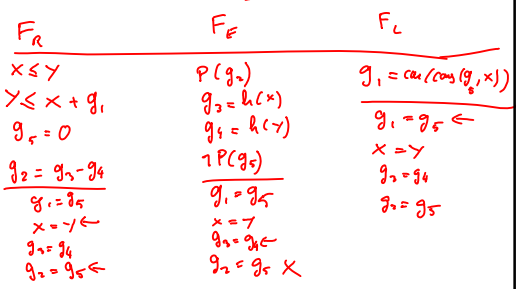
Combining decision procedures

- Efficient decision procedures exist for many decidable logics, but some formulas do not belong to any of these logics
- Instead, they belong to a combination of these logics
- For example:

if $currStmt == [X = Y]$
then $geq(X,Y)@out$

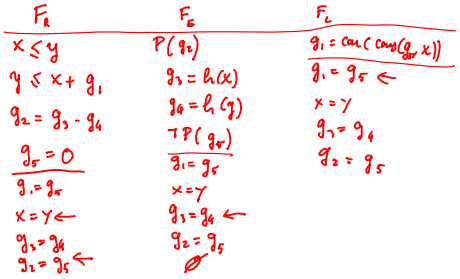
Nelson-Oppen example

- $x \le y \wedge y \le x + car(cons(0,x)) \wedge P(h(x) - h(y)) \wedge \neg P(0)$



Nelson-Oppen example

- $(x \le y) \wedge (y \le x + car(cons(0,x))) \wedge (P(h(x) - h(y))) \wedge (\neg P(0))$



Correctness

- If a contradiction is found, return UNSAT
  - This is clearly sound, if each decision procedure is sound
- If there are no more equalities to be found by any of the decision procedures, return SAT
  - Is this complete? Have the decision procedures exchanged enough info?
  - Each decision procedure has found its own satisfying assignments, but how do we know that these satisfying assignments are compatible (ie: don't contradict each other)

Convex theories

- A theory is convex if whenever a satisfiable conjunction of literals entails a disjunction of equalities of variables, then it entails one of the equalities
- Example:
  - Theory of linear arithmetic with equalities
- For convex theories:
  - If no equalities can be found, then it is impossible for there to be a disjunction of equalities that can be found; therefore, no missed equalities

Nonconvex theories

- Example:
  - Reals under multiplication
    - $xy = 0 \wedge z = 0$ entails $x = z \vee y = z$
  - Integers under + and $\le$
    - $x = 1 \wedge y = 2 \wedge 1 \le z \wedge z \le 2$ entails $x = z \vee y = z$
  - Theory of sets
  - Theory of arrays
- For such theories, must perform a case split when a disjunction of equalities is entailed
  - Try each disjunct recusively.
  - If any one returns SAT, return SAT
  - If all disjuncts return UNSAT, return UNSAT

Algorithm

- Given a formula F that is a conjunction of literals over theories S and T, returns whether F is SAT or UNSAT
  1. Assign conjunctions to F_S and F_T so that F_S is a conjunction of S-literals and F_T is a conjunction of T-literals
  2. If either F_S or F_T is unsatisfiable, return UNSAT
  3. If either F_S or F_T entails some equality between variables not entailed by the other, then add the equality as a new conjunct to the one that does not entail it. Goto step 2.
  4. If either F_S or F_T entails a disjunction $x_1 \vee \ldots x_k$ of equalities between variables, then for each i from 1 to k, apply the procedure recursively to $F_S \wedge F_T \wedge x_i$. If any recursive call returns SAT, return SAT. Otherwise return UNSAT.
  5. Return SAT

Adding Nelson-Oppen to Simplify--

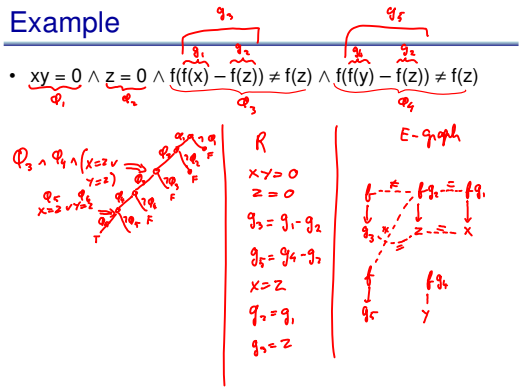
- Each decision procedure keeps track of its own information
- Decision procedure for theory T exports a function assert(F), where F is a literal in T
- While performing the backtracking search, if a literal is asserted, add that literal (using assert) to the decision procedure for the theory the literal belongs to
  - If the literal belongs to a combination of theories, split the literal into a conjunction of literals, each one belonging to only one theory

Adding Nelson-Oppen to Simplify--

- Calling assert on a decision procedure may cause a whole bunch of equalities to be propagated, all of which are added to the E-graph
- Case splitting falls naturally out of the backtracking search algorithm
  - If a disjunction of equalities is implied in one of the decision procedures, then add the disjunction as a new clause in the current formula

Example

- $xy = 0 \wedge z = 0 \wedge f(f(x) - f(z)) \neq f(z) \wedge f(f(y) - f(z)) \neq f(z)$

Example

- $xy = 0 \wedge z = 0 \wedge f(f(x) - f(z)) \neq f(z) \wedge f(f(y) - f(z)) \neq f(z)$