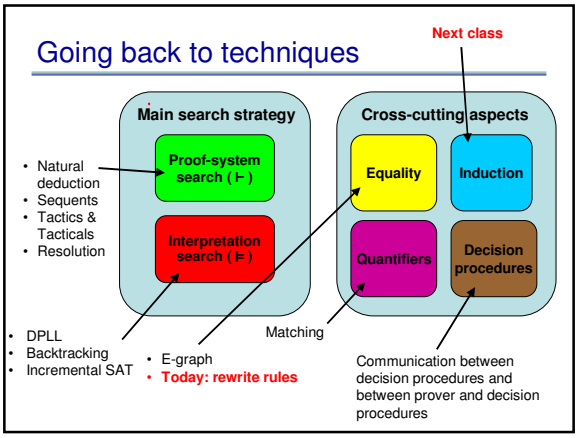
Going back to techniques



Rewrite rules

Equality

- We've seen one way of dealing with equality in this class: the E-graph
- Another way to deal with equalities is to treat them as rewrite rules
- An equality a = b can be used either as the rewrite rule a → b or b → a

Plan

- First, we'll see how rewrite rules are used in a few system
  - We'll see some of the issues that come up
- Then we'll dig into the details
  - Formal foundations: rewrite systems
- This is a huge area of work and research
  - We'll see the snowflake on the tip of the iceberg

Rewrite rules in resolution

- Show that the following is unsat:
  - { P(1), ¬ P(f(0)), f(0) = 1 }

¬P(1) f(0) → 1 1 → f(0)

Rewrite rules in resolution

- Show that the following is unsat:
  - { P(1), ¬ P(f(0)), f(0) = 1 }
- This technique is called demodulation

Resolution: another example

- Show that the following is unsat:
  - { P(0), ¬ P(f(0)), f(0) = 0 }

0 → f(0)

f(0) → 0

Resolution: another example

- Show that the following is unsat:
  - { P(0), ¬ P(f(0)), f(0) = 0 }
- Which direction to pick
  - In the first example, can choose f(1) → 1 or 1 → f(1)
  - In the second example, pick f(0) → 0
- Termination issues are cropping up

Rewrite rules in ACL2

- Recall ACL2 architecture:
  - Given a goal formula, try to apply various techniques in turn. Each technique generates sub-goals. Recurse on sub-goals.
- Techniques:
  - Simplification
  - Instantiating known theorems
  - Term rewriting
  - As a last resort, induction

Rewrite rules in ACL2

- Rewrite rules in ACL2 are guarded
- A rewrite rule is a lemma of the form:
  - $h_1 \wedge \ldots \wedge h_n \Rightarrow a = b$
- Here again, must pick a direction for the equality
  - Try to rewrite more "complicated" terms to "simpler" terms

Rewrite rules in ACL2

- ACL2 also uses local equality assumptions for performing rewrites
- If lhs = rhs appears as a hypothesis in the goal, then replace lhs with rhs in rest of goal
- Boyer and Moore call this cross-fertilization
- More local than having rewrite lemmas

Performance issues

- Heuristics for termination
  - Decreasing measure (guarantees termination "statically")
  - Detect infinite loops and stop (can detect loops in the terms, or loops in the application of rules)
  - Or just have a timeout
- One can also cache the results of rewriting
  - Rewriting can be expensive

Moving to the formal details…

- We've seen some intuition for where to use rewrite rules
- Now, let's see some of the details
- Rewrite rules have been studied in the context of term rewrite systems, which to are just sets of rewrite rules

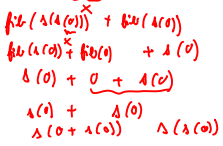
Term rewrite systems

- A term rewrite system is a pair (T, R) , where T is a set of terms and R is a set of rewrite rules of the form $l_1 \rightarrow l_2$
- T should actually be an algebra signature, but for our purposes, thinking of T as a set of terms is good enough
- Rewrite rules can have free variables, but variables on the rhs must be a subset of variables on the lhs

Example

$0 + y \rightarrow y$
$s(x) + y \rightarrow s(x + y)$
$fib(0) \rightarrow 0$
$fib(s(0)) \rightarrow s(0)$
$fib(s(s(x))) \rightarrow fib(s(x)) + fib(x)$

- Run on: fib(s(s(0)))



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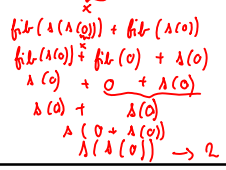


A single rewrite step

- We write $t_1 \rightarrow t_2$ to mean that term t_1 rewrites to term t_2
  - $s(0 + s(0)) \rightarrow s(s(0))$
- Notice that the rewrite rule can be applied to a sub-term!
- We need a way to formalize this

Contexts

- A context C[·] is a term with a "hole" in it:
  - $C[\cdot] = s(0 + \cdot)$
- The whole can be filled:
  - $C[s(0)] = s(0 + s(0))$
- When we write $t_1 = C[t_2]$, it means that term t_2 appears inside t_1, and C is the context inside which t_2 appears

A single rewrite step

- $t_1 \rightarrow t_2$ iff there exists a context C, and a rewrite rule $l_1 \rightarrow l_2$ such that:

A single rewrite step

- $t_1 \rightarrow t_2$ iff there exists a context C, and a rewrite rule $l_1 \rightarrow l_2$ such that:
  - $t_1 = C[l_1]$
  - $t_2 = C[l_2]$
- Can we show:
  - $s(0 + s(0)) \rightarrow s(s(0))$
  - from rewrite rule $s(x) + y \rightarrow s(x + y)$

A single rewrite step

- $t_1 \rightarrow t_2$ iff there exists a context C, a term t, a substitution θ, and a rewrite rule $l_1 \rightarrow l_2$ such that:

A single rewrite step

- $t_1 \rightarrow t_2$ iff there exists a context C, a term t, a substitution θ, and a rewrite rule $l_1 \rightarrow l_2$ such that:
  - $t_1 = C[t]$
  - $\theta = \text{unify}(l_1, t)$
  - $t_2 = C[\theta(l_2)]$

Transitive closure of rewrites

- $\rightarrow^*$ is the reflexive transitive closure of $\rightarrow$
- Another way to say the same: we write $t_1 \rightarrow^* t_2$ to mean that there exists a possibly empty sequence $s_0 \ldots s_n$ such that:
  - $t_1 \rightarrow s_0 \rightarrow \ldots \rightarrow s_n \rightarrow t_2$

Termination

- An term t is *in normal form* or is *irreducible* if there is no term t' such that $t \rightarrow t'$
- A rewrite system (T, R) is normalizing (also called weakly normalizing) if every $t \in T$ has a normal form
- A rewrite system (T,R) is terminating (also called strongly normalizing) if there is no term $t \in T$ that will rewrite forever
- What is the difference between strong and weak normalization?

Termination

- One technique for showing termination: assign a measure to each term, and show that rewrite rules strictly decrease the measure
- Example:

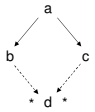
  $s(p(x)) \rightarrow x$
  $p(s(x)) \rightarrow x$
  $minus(0) \rightarrow 0$
  $minus(s(x)) \rightarrow p(minus(x))$

Termination

- Termination guarantees normalization
- However, it does not guarantee that there is a unique normal form for a given term
- We would like to have this additional uniqueness property
- Confluence is the additional property we need

Confluence

- Local confluence: for all terms a,b and c, if $a \rightarrow b$ and $a \rightarrow c$, then there exists a term d such that $b \rightarrow^* d$ and $c \rightarrow^* d$



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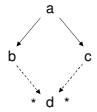
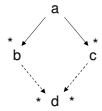


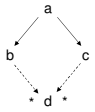
- Global confluence: for all terms a,b and c, if $a \rightarrow^* b$ and $a \rightarrow^* c$, then there exists a term d such that $b \rightarrow^* d$ and $c \rightarrow^* d$



Confluence

- Local confluence: for all terms a,b and c, if $a \rightarrow b$ and $a \rightarrow c$, then there exists a term d such that $b \rightarrow^* d$ and $c \rightarrow^* d$

  when there exists a d such that $b \rightarrow^* d$ and $c \rightarrow^* d$, we say that b and c are joinable



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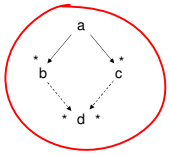
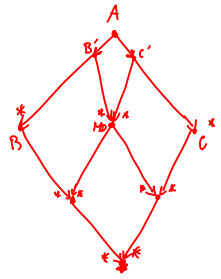


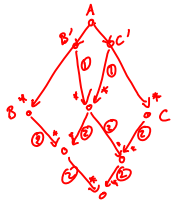
Relation between local and global

- Theorem: for a terminating system, local confluence implies global confluence
- Proof by picture…

Proof by picture



Proof by picture



Induction on longest rewrite sequence starting at A

- Base case: A irreducible ⇒ B, C & D are all A and we are done
- Inductive case

①: by local confluence
②: using ind assumption

Canonical systems

- A terminating and confluent system is canonical, meaning that each term has a unique canonical form
- Simple decision procedure for such systems
- To determine $t_1 = t_2$:

Canonical systems

- A terminating and confluent system is canonical, meaning that each term has a unique canonical form
- Simple decision procedure for such systems:
- To determine $t_1 = t_2$:

Find canonical form of t_1 & t_2 and compare canonical forms syntactically

Determining confluence

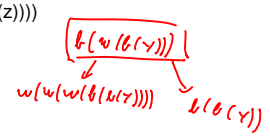
- We would like an algorithm for determining whether a terminating system is confluent
- To do this, we need to define the notion of critical pair

Critical pairs

- Let $l_1 \rightarrow r_1$ and $l_2 \rightarrow r_2$ be rewrite rules that have no variables in common (rename vars if needed)
- Suppose $l_1 = C[t]$ such that t is a non-trivial term (not a variable), and such that $\theta = \text{unify}(t, l_2)$
- Then $(\theta(C[r_2]), \theta(r_1))$ is a critical pair
- The intuition is that a critical pair represents a choice point: given a term $l_1 = C[l_2]$, we can either apply $l_1 \rightarrow r_1$ to get r_1, or we can apply $l_2 \rightarrow r_2$ to get $C[r_2]$

Critical pairs: example

b(w(x)) → w(w(w(b(x))))
w(b(y)) → b(y)
b(b(z)) → w(w(w(z)))



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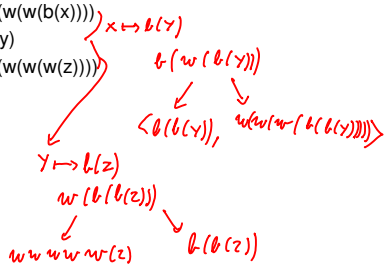

Critical pairs: another example

s(p(x)) → x
p(s(x)) → x
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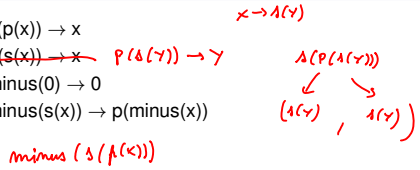

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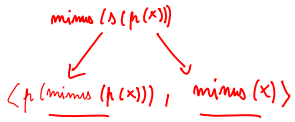

Critical pairs

- Theorem: A term rewrite system is locally confluent if and only if all its critical pairs are joinable
- Recall the meaning of joinable: b and c are joinable if there exists a d such that b $\rightarrow^*$ d and c $\rightarrow^*$ d
- Corollary: A terminating rewrite system is confluent if and only if all its critical pairs are joinable

Algorithm for deciding confluence

- Given a terminating rewrite system, find all critical pairs, and call this set CR

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- Given a terminating rewrite system, find all critical pairs, and call this set CR
- For each pair (s,t) ∈ CR:
  - Find all the normal forms of s and t
  - Note: s and t are guaranteed to have normal forms because the system is terminating.

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  - If sizeof(NS) > 1 or sizeof(NT) > 1 return "NOT CONFLUENT"
  - (more than one normal form implies NOT confluent since confluence would imply unique normal form)

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  - If sizeof(NS) > 1 or sizeof(NT) > 1 return "NOT CONFLUENT"
  - If NS and NT are disjoint, return "NOT CONFLUENT"
- Return "CONFLUENT"

What if the system is not confluent?

What if the system is not confluent?

- If we find a critical pair such that the normal forms are disjoint, add additional rewrite rules
- This is called the Knuth-Bendix completion procedure

Knuth-Bendix completion procedure

- Given a terminating rewrite system, find all critical pairs, and call this set CR
- While there exists a critical pair (s,t) ∈ CR such that the normal forms NS of s and the normals forms NT of t are disjoint:
  - For each s' in NS, and for each term t' in NT, add the rewrite rule s' → t'

Knuth-Bendix completion procedure

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  - For each s' in NS, and for each term t' in NT, add the rewrite rule s' → t'
    - Subtlety: should we add s' → t', or t' → s'?
    - Completion algorithm also takes a "reduction order" as an argument
    - Algorithm fails if s' and t' have the same reduction order

Knuth-Bendix completion procedure

- Three possible outcomes:
  - Terminates with success, yielding a terminating confluent rewrite system that is equivalent to the original rewrite system (in terms of equalities that are provable)
  - Terminate with a failure
  - Does not terminate

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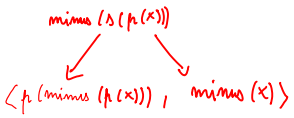

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