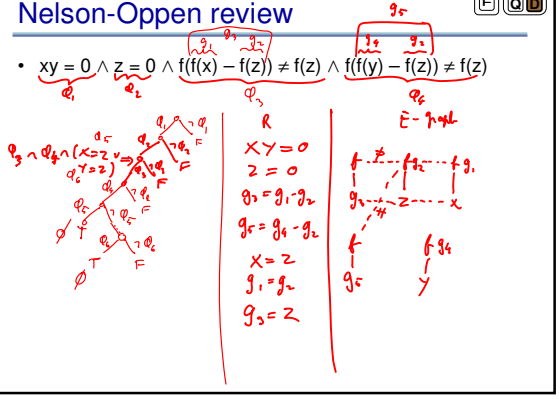
Nelson-Oppen review

- $xy = 0 \wedge z = 0 \wedge f(f(x) - f(z)) \neq f(z) \wedge f(f(y) - f(z)) \neq f(z)$



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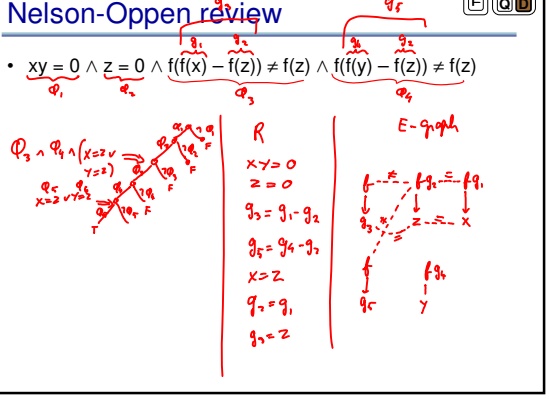

Drawback of Nelson-Oppen

Drawback of Nelson-Oppen

- Theory must be convex, otherwise must backtrack
- Some large overheads:
  - Each decision procedure must maintain its own equalities
  - There are a quadratic number of equalities that can be propagated

Shostak's approach

- Alternate approach to combining theories that addresses some of the performance drawbacks of Nelson-Oppen
  - Published in 1984 in JACM, but the original formulation was later found to be flawed in several ways
  - Long line of work to correct these mistakes
  - Culminating in "Deconstructing Shostak" by Ruess and Shankar, which gives sound and complete version of Shostak
- Unpublished manuscript by Crocker from 1988 showing that Shostak is 10 times faster than Nelson-Oppen
- Recent paper by Barrett, Dill and Stump in 2002 shows that Shostak can be seen as a special case of Nelson-Oppen

Shostak's approach

- Shostak is used in a variety of theorem provers, including PVS and SVC
- We will cover the intuition behind Shostak's approach, but we won't see the details

The key idea in Shostak

- Keep one congruence closure data-structure S for all theories
- Each individual decision procedure finds new equalities based on the ones that are already in S
- As individual decision procedures find equalities, add them to S

Adding equalities to S

- Straightforward to encode equalities over uninterpreted function symbols in S
  - Since S is a congruence-closure data structure, since congruence closure was originally intended for exactly these kinds of equalities
- Interpreted functions symbols require more care
  - For example, an equality y + 1 = x + 2 cannot be processed by simply putting y + 1 and x + 2 in the same equivalence class, since the original equality in fact entails a multitude of equalities, such as y = x + 1, y – 1 = x, y -2 = x -1, etc.

Impose two restrictions

1. Theories must be solvable: any set of equalities in the theory must have an equivalent solved form
   - Equalities are in solved form if the left hand side of the equalities are only variables and the right-hand sides are expressions that don't reference any of the left-hand side variables

x + y = z + 3
x – y = 3z + 1

$2x = 4z + 4 \quad x = 2z + 2$

$y = z + 3 - x$

....

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   - Equalities are in solved form if the left hand side of the equalities are only variables and the right-hand sides are expressions that don't reference any of the left-hand side variables

x + y = z + 3 → $y = z + 3 - x$
x – y = 3z + 1 $= -z + 1$

$2x = 4z + 4$
$x = 2z + 2$

Solved Form
$x = 2z + 2$
$y = -z + 1$
Not unique

Impose two restrictions

1. Theories must be solvable: any set of equalities in the theory must have an equivalent solved form
   - Equalities are in solved form if the left hand side of the equalities are only variables and the right-hand sides are expressions that don't reference any of the left-hand side variables
   - Will use this to substitute solved variables in all terms

Impose two restrictions

2. Theory must be canonizable
   - There is a canonizer function σ such that if a = b, then σ(a) is syntactically equal to σ(b)
   - Canonizer for linear arithmetic: transform terms into ordered monomials
   - σ(a + 3c + 4b + 3 + 2a + 4) = 3a + 4b + 3c + 7
   - The intuition is that by canonizing all terms, we can then use syntactic equality to determine semantic equality

Putting it all together

- $f(x-1)-1 = x+1 \wedge f(y)+1 = y-1 \wedge y+1 = x$

$f(x-1) = x+2 \Rightarrow f(y) = y+3$

$f(y) = y-2 \qquad y-2 = y+3 \Rightarrow 0 = 5$

$x = y+1$

Putting it all together

- $f(x-1)-1 = x+1 \wedge f(y)+1 = y-1 \wedge y+1 = x$

$f(x-1) = x+2 \Rightarrow f(y) = y+3$

$f(y) = y-2 \Rightarrow y+3 = y-2 \Rightarrow 0 = 1$ ⊘

$x = y+1$

ACL2 decision procedures

- ACL2 architecture
  - Given a goal, ACL2 has a set of strategies it can apply
  - For example: rewriting, simplification, induction
  - Applying a strategy produces sub-goals from the given goal
  - Each sub-goal needs to be proven recursively

Adding linear arithmetic

- First attempt was to just use the decision procedures directly as a strategy
- Not found to be useful, because it was rarely the case that the goal would reduce to TRUE using linear arithmetic
- Rather, they found they needed to add linear arithmetic in the rewrite system
- A rewrite rule: $A \Rightarrow T_1 = T_2$
  - To trigger, need to establish A
  - They often needed linear arithmetic to establish A

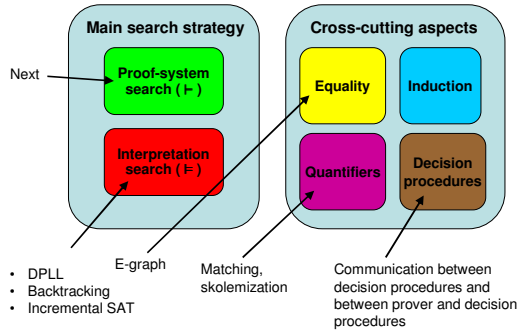
Keep a linear arith DB

- A rewrite rule: $A \Rightarrow T_1 = T_2$
- To establish A, add $\neg A$ to the current database of linear equalities and inequalities
- If an inconsistency is reached, we know A holds
  - We can perform the rewrite
  - Remove $\neg A$ from the database, and add A
- As in Simplify, the arith DB is used for matching heuristic to instantiate quantifiers

Decision procedures summary

- Communication between decision procedures
  - Nelson-Oppen (simplify), Shostak (PVS, SVC)
- Communication from heuristic prover to decision procedures
  - assert formulas (most theorem provers)
- Communication from decision procedures to heuristic theorem prover
  - yes/no answers (all theorem provers)
  - terms to use for matching (Simplify, ACL2)
  - proofs to prune search space (Verifun)

So far



The two statements

$\Gamma \models \phi$ $\quad$ $\Gamma \vdash \phi$

set of formulas, one formula

"entails, or models" $\quad$ "is provable from "

In all worlds where the formulas in Γ hold, ϕ holds

ϕ is provable from assumptions Γ

Semantic $\quad$ Syntactic

Link between ⊨ and ⊢

- Soundness: $\Gamma \vdash \phi$ implies $\Gamma \models \phi$
- Completeness: $\Gamma \models \phi$ implies $\Gamma \vdash \phi$

- Virtually all inference systems are sound
- Therefore, to establish $\Gamma \models \phi$, all one needs to do is find a derivation of $\Gamma \vdash \phi$

Goal: find a proof

- Need two things:
  - A proof system
  - A seach strategy
- These two are heavily intertwined
- Let's start by looking at some proof systems

Hilbert-style systems

- Many axioms and usually just one inference rule, modus ponens

Axiom (schemas) $\quad$ Inference rule

1. $X \Rightarrow (Y \Rightarrow X)$
2. $(X \Rightarrow (Y \Rightarrow Z)) \Rightarrow ((X \Rightarrow Y) \Rightarrow (X \Rightarrow Z))$
3. $F \Rightarrow X$
4. $X \Rightarrow T$
5. $\neg \neg X \Rightarrow X$
6. $X \Rightarrow (\neg X \Rightarrow Y)$

$$\frac{A \quad A \Rightarrow B}{B}\ \text{MP}$$

Coming up with a complete set of axiom schemas is not trivial

Example proof

1. $X \Rightarrow (Y \Rightarrow X)$
2. $(X \Rightarrow (Y \Rightarrow Z)) \Rightarrow ((X \Rightarrow Y) \Rightarrow (X \Rightarrow Z))$
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- Show: $P \Rightarrow P$

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$$\frac{A \quad A \Rightarrow B}{B}\ \text{MP}$$

- Show: $P \Rightarrow P$
  - Instantiate 2 with X being P, Y being $P \Rightarrow P$, and Z being P:
    - $(P \Rightarrow ((P \Rightarrow P) \Rightarrow P)) \Rightarrow ((P \Rightarrow (P \Rightarrow P)) \Rightarrow (P \Rightarrow P))$
  - Instantiate 1, taking X to be P and Y to be $P \Rightarrow P$:
    - $P \Rightarrow ((P \Rightarrow P) \Rightarrow P)$
  - Instantiate 1 with X and Y to be P:
    - $P \Rightarrow (P \Rightarrow P)$

Example proof

1. $X \Rightarrow (Y \Rightarrow X)$
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    - $(P \Rightarrow ((P \Rightarrow P) \Rightarrow P)) \Rightarrow ((P \Rightarrow (P \Rightarrow P)) \Rightarrow (P \Rightarrow P))$ *
  - Instantiate 1, taking X to be P and Y to be $P \Rightarrow P$:
    - $P \Rightarrow ((P \Rightarrow P) \Rightarrow P)$ **
  - Instantiate 1 with X and Y to be P:
    - $P \Rightarrow (P \Rightarrow P)$ ***
  - Apply MP on * and **:
    - $(P \Rightarrow (P \Rightarrow P)) \Rightarrow (P \Rightarrow P)$ ****
  - Apply MP on *** and ****:
    - $P \Rightarrow P$

Hilbert-style systems

- Does not mimic the way humans do proofs
- To prove $A \Rightarrow B$ in a Hilbert-style system, must find the right way instantiate axioms and then apply MP to get $A \Rightarrow B$
- How does a human prove $A \Rightarrow B$?

Hilbert-style systems

- Does not mimic the way humans do proofs
- To prove $A \Rightarrow B$ in a Hilbert-style system, must find the right way instantiate axioms and then apply MP to get $A \Rightarrow B$
- How does a human prove $A \Rightarrow B$?
- Assume A, and show B
- In this context, showing $P \Rightarrow P$ is very easy

Natural deduction

- The system of natural deduction was developed by Gentzen in 1935 out of dissatisfaction with Hilbert-style axiomatic systems, which did not closely mirror the way humans usually perform proofs
- Gentzen wanted to create a system that mimics the “natural” way in which humans think

Natural deduction rule for $A \Rightarrow B$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B}$$

Natural deduction rule for A ⇒ B

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B} \Rightarrow I$$

- This is called an introduction rule, since it introduces the ⇒ connective

Natural deduction rule for A ⇒ B

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- This is called an introduction rule, since it introduces the ⇒ connective
- Each connective also has an elimination rule

Natural deduction rule for A ⇒ B

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B} \Rightarrow I \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash A \Rightarrow B}{\Gamma \vdash B} \Rightarrow E$$

- This is called an introduction rule, since it introduces the ⇒ connective
- Each connective also has an elimination rule

Natural deduction

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge I \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} \wedge E_1 \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B} \wedge E_2$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \vee I_1 \qquad \frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C \quad \Gamma \vdash A \vee B}{\Gamma \vdash C} \vee E$$

Natural deduction

$$\frac{\Gamma \vdash A \quad \Gamma \vdash \neg A}{\Gamma \vdash F} FI \qquad \frac{\Gamma \vdash F}{\Gamma \vdash A} FE$$

$$\frac{}{\Gamma \vdash T} TI \qquad \frac{\Gamma \vdash T}{\Gamma \vdash A} TE$$

$$\frac{\Gamma, A \vdash F}{\Gamma \vdash \neg A} \neg I \qquad \frac{\Gamma \vdash \neg A \quad \Gamma \vdash A}{\Gamma \vdash F} \neg E$$

Natural deduction

$$\frac{}{\Gamma, A \vdash A} \text{Assume}$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge I \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} \wedge E_1 \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B} \wedge E_2$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \vee I_1 \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B} \vee I_2 \qquad \frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C \quad \Gamma \vdash A \vee B}{\Gamma \vdash C} \vee E$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B} \Rightarrow I \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash A \Rightarrow B}{\Gamma \vdash B} \Rightarrow E$$

Natural deduction

$$\frac{\Gamma, A \vdash \mathbf{F}}{\Gamma \vdash \neg A}\ \neg \mathrm{I} \qquad \frac{\Gamma \vdash \neg \neg A}{\Gamma \vdash A}\ \neg \mathrm{E}$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash \neg A}{\Gamma \vdash \mathbf{F}}\ \mathbf{F}\mathrm{I} \qquad \frac{\Gamma \vdash \mathbf{F}}{\Gamma \vdash A}\ \mathbf{F}\mathrm{E}$$

$$\frac{}{\Gamma \vdash \mathbf{T}}\ \mathbf{T}\mathrm{I} \qquad \text{No } \mathbf{T} \text{ elimination}$$

Note: one can get rid of the **F**E without losing expressiveness. Can someone see why?

Once we have a proof system

- Once we have a proof system, the goal is to devise a search algorithm to find a proof
- Search algorithm sound: proofs that it finds are correct
- Search algorithm complete: if there is a proof, the algorithm will find it
- These soundness and completeness properties relate the search algorithm to the proof system, and should not be confused with soundness and completeness of the proof system

Two main strategies

- Given a formula to prove:
  - One can start from axioms and apply inference rules forward, until a derivation of the given formula is found
  - One can start from the formula to prove (the goal) and apply inference rules backward to find sub-goals until all sub-goals are axioms
- The forward version is sometimes called forward chaining, the backward version backward chaining

Forward search

- Keep a knowledge base, which is the set of formulas that have been proved so far
- Given goal to prove:
  - Start with empty knowledge base
  - While goal not in knowledge base:
    - Instantiate an axiom or an inference rule to deduce a new formula
    - Add the new formula to the knowledge base
    - If the goal is in the knowledge base, return VALID
- No need to backtrack

Forward search -- refutation

- Start with knowledge base being the negation of the goal
- While enlarging the knowledge base, if **F** becomes part of the knowledge, then return VALID

Backward search

- Given goal to prove:
  - If the goal is T then return VALID
  - Otherwise:
    - Let S be the set of inference rules that can be applied backward
    - Pick some subset S' of S that we want to consider
    - For each inference rule in S':
      - Apply the inference rule backward on the goal to produce n sub-goals (axioms produce sub-goals of T)
      - Run the search recursively on each sub-goal
      - If all recursive calls return VALID, return VALID
    - Return INVALID

Note: This is a depth-first search. Can have other search orders, like breadth first, iterative deepening

Proofs

- One can easily adapt these algorithms to keep track of the proof tree, so that a proof can be produced if the goal is valid
- Contrast this with our backtracking search in the semantic domain, where generating a proof is not as simple
- On the other hand, what about when the proof fails?
  - much easier to get counter-example in interpretation search than in proof-system search

Non-determinism

- Whatever the direction of the search, one of the biggest problems is that there are a lot of choices to make. This is called non-determinism.
  - There may be many inference rules that are applicable
  - Even for one rule, there may be multiple instantiations
  - For example, applying $\vee$ E backward requires one to determine A and B

$$\frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C \quad \Gamma \vdash A \vee B}{\Gamma \vdash C}\ \vee E$$

Two kinds of non-determinism

- Don't care non-determinism (also called conjunctive non-determinism)
  - All choices will lead to a successful search, so we "don't care" which one we take
  - Only consideration for making the choice is efficiency
- Don't know non-determinism (also called disjunctive non-determinism)
  - Some of the choices will lead to a successful search, but we "don't know" which one a priori
  - In order to deal with this kind of non-determinism, try all choices using some traversal order (depth-first, breadth-first, iterative deepening

Next lecture

- We'll see how to reduce non-determinism
- We'll learn about tactics and tacticals, one of the important techniques used in proof system searches
- We'll learn about some proof systems that are more suited for automated reasoning, like the sequent calculus and resolution