

Week 5

1. Recurrence Relations & Master's Theorem

2. k^{th} -smallest element

I Recurrence Relations

• Master's Theorem : (pg. 49)

If $T(n) = aT(\lceil n/b \rceil) + O(n^d)$, for $a > 0$
 $b > 1$ and $d \geq 0$, then

$$T(n) = \begin{cases} O(n^d) & \text{if } d > \log_b a \\ O(n^d \log n) & \text{if } d = \log_b a \\ O(n^{\log_b a}) & \text{if } d < \log_b a \end{cases}$$

Ex:

a) $T(n) = 8T(n/2) + n^2$

$$d=2 < 3 = \log_2 8 = \log_b a \Rightarrow T(n) = O(n^{\log_b a}) = O(n^3)$$

b) $T(n) = 4T(n/2) + n^2$

$$2 = \log_2 4 = 2 \Rightarrow T(n) = O(n^2 \log n)$$

c) $T(n) = 2T(n/2) + n^2$

$$1 < \log_2 2 = 1 \Rightarrow T(n) = O(n^2)$$

d) (Tricky use of Master's Thm)

$$T(n) = T(\sqrt[3]{n}) + 1$$

Let $n = 2^m$ so

$$\begin{aligned} T(2^m) &= T((2^m)^{1/3}) + 1 \\ &= T(2^{m/3}) + 1 \end{aligned}$$

Let $R(m) = T(2^m)$,

$$R(m) = R(m/3) + 1$$

Master's :

$$d = 0 = 0 = \log_3 1 \Rightarrow R(m) = O(\log m)$$

$$m = \log n \quad \text{so} \quad T(n) = T(2^m) = R(m) = O(\log m)$$

$\Rightarrow T(n) = O(\log \log n)$

2 k^{th} Smallest Element

Input: Two sorted arrays, A, B of length m and n resp. ; k

Output: k^{th} Smallest element in $A \cup B$.

Picture:

A
1
2
3

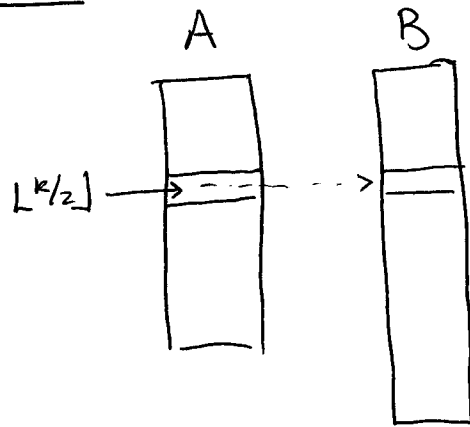
B
0
10
20
30
40

k=3 returns 2

k=4 returns 3

k=5 returns 10

picture :



$$\text{Let } V_a = A[L^{k/2}] \\ V_b = B[L^{k/2}]$$

- If $V_a < V_b$, then the k^{th} smallest element cannot be in $A[1 \dots L^{k/2}]$
- ~~Else~~ Else, k^{th} smallest cannot be in $B[1 \dots L^{k/2}]$

Idea: Compare V_a, V_b and throw away $\lfloor k/2 \rfloor$ elements k^{th} smallest can't be and repeat finding $\lceil k/2 \rceil$ smallest element in remaining lists.

Alg: FindKthElem(A, B, a, b, k)
// Initially call with $a=b=1$

$O(1)$ 1. Base: If $k=1$ return $\min\{A[a], B[b]\}$

$O(1)$ 2. Set $V_a = A[a + L^{k/2}]$ and $V_b = B[b + L^{k/2}]$

3. If $V_a < V_b$:

$T(k/2)$ FindKthElem($A, B, a + L^{k/2}, b, \lceil k/2 \rceil$)

Else

$T(k/2)$ FindKthElem($A, B, a, b + L^{k/2}, \lceil k/2 \rceil$)

RT:

$$T(k) = T(k/2) + O(1) \Rightarrow O(\log k) \text{ runtime.}$$

Correctness: If v is the k^{th} smallest elem. of $A \cup B$,
then my alg. returns v .

Pf: (By induction)

Base: If $k=1$ then step 1 returns the smaller
of the heads of A, B .

Induction: Assume my alg returns the k^{th} sm. elem.
for all $k \leq n$. Show it for n .

We compare $\lfloor \frac{n}{2} \rfloor$ elements and remove the $\lfloor \frac{n}{2} \rfloor$
that are for sure smaller and then find
the smallest $\lceil \frac{k}{2} \rceil$ item in $A[\lfloor \frac{k}{2} \rfloor \dots m]$
and $B[1 \dots m]$ which by induction hyp. is
correctly found. $\square$