

Knapsack Problem:

$$\text{MAX} \quad \sum v_i x_i$$

$$\text{SUB} \quad \sum w_i x_i = b$$

$$x_i \geq 0, \text{ integers}$$

Coin Changing Problem:

$$\text{MIN} \quad \sum w_i x_i$$

$$\text{SUB} \quad \sum v_i x_i = b$$

$$x_i \geq 0, \text{ integers}$$

Assumptions:

$$(i) \quad \begin{array}{ll} w_i > 0 \\ v_i \geq 0, & \text{integers} \end{array}$$

$$(ii) \quad 1 = v_1 < v_2 < \dots < v_n$$

$$(iii) \quad \frac{w_1}{v_1} > \frac{w_2}{v_2} > \dots > \frac{w_n}{v_n}$$

i.e. the largest (nth) coin is the best,
since it has the least weight per unit
value.

$G_k(y)$: The weight achieved by the Greedy Algorithm when the change is y and only the first k kinds of coins are used.

$F_k(y)$: The minimum weight formed by the optimum solution when the change is y and only the first k kinds of coins are considered.

Example 1

$$v_1 = 1, \quad v_2 = 5, \quad v_3 = 14, \quad v_4 = 18$$

$$w_i = 1 \quad \text{for all } i$$

We want to minimize the number of coins, stamps, etc.

$$G_1(y) = w_1 \left\lfloor \frac{y}{1} \right\rfloor = w_1 y = F_1(y)$$

$$\begin{aligned} G_2(y) &= w_2 \left\lfloor \frac{y}{5} \right\rfloor + w_1 [y(\bmod 5)] \\ &= F_2(y) \end{aligned}$$

$$G_3(y) = w_3 \left\lfloor \frac{y}{14} \right\rfloor + G_2[y(\bmod 14)]$$

$$G_4(y) = w_4 \left\lfloor \frac{y}{18} \right\rfloor + G_3[y(\bmod 18)]$$

Theorem 1:

suppose

$$G_k(y) = F_k(y)$$

Let

$$v_{k+1} > v_k$$

and

$$v_{k+1} = p_k v_k - \delta_k$$

$$0 \leq \delta_k < v_{k+1}$$

Then the following are equivalent

$$(a) \quad G_{k+1}(y) \leq G_k(y) \quad \text{for all } y$$

$$(b) \quad G_{k+1}(y) = F_{k+1}(y) \quad \text{for all } y$$

$$(c) \quad G_{k+1}(p_k v_k) = F_{k+1}(p_k v_k)$$

$$(d) \quad w_{k+1} + G_k(\delta_k) \leq p_k w_k$$

Back to Example 1:

$$v_3 = 14 = 3 \times 5 - 1 = 3 \times v_2 - 1$$

$$\begin{aligned} \Rightarrow \quad p_2 &= 3, \quad \delta_2 = 1 \\ 1 + G_2(1) &= 2 \leq 3 \times 1 = p_2 w_2 \\ w_3 + G_2(\delta_2) &\leq p_2 w_2 \end{aligned}$$

$\therefore$ Greedy Algorithm works in this case

$$v_4 = 18 = 2 \times 14 - 10 = 3 \times v_3 - 10$$

$$\Rightarrow \quad p_3 = 2, \quad \delta_3 = 10$$

$$w_4 + G_3(\delta_3) \leq p_3 w_3 ?$$